Modern Physics. Phys 222

Artem G. Abanov

Contents

LECTURE 1
Introduction. Geometry.

- Contact info. Email. Zoom, if needed. Drop by before/after lectures.
- Office hour: Mondays, 12:30-1:30 pm, or by appointment (in-person or zoom)
- Canvas.
- Homework submissions. Each HW is 100pt. PDF SINGLE FILE. SUBMIT TO CANVAS. Check the file after submission.
- Coherent, understandable, readable.
- Homeworks 40% of the grade. Deadline: Every Friday, 10 am (including the first one). No late submissions. Collaborations!!!!! make study groups.
- What I consider cheating.
- Mistakes, etc.
- Homework solution on Friday lectures.
- Extra problems. Separate file. Send directly to me by email. DON NOT SUBMIT EXTRA PROBLEMS TO CANVAS!
- Book. LECTURE NOTES. Wikipedia, etc.
- Grading: mistakes in grading. Final grade.
- Exams: Exam 1 — 30%. Take home. Friday, July 7, due Monday, July 10, 10:00 am; Final — 30%. Wednesday, August 9, 10:30 am – 12:30 pm. MPHY 204
- Language.
- Lecture, questions, feedback. Going too fast, etc.
- Questions: profound vs. stupid.
- The Importance of Being Nerd.
- Course content and philosophy.

What do we know?

- Calculus (derivatives, integrals, partial derivatives, Taylor expansion, integration over a path, Fourier transformation.)
- Linear algebra (vectors, matrices, eigen values, eigen vectors.)
- Complex variables.
- Mechanics.
- Electrodynamics.
- Geometry.

Geometry

- What is the sum of all angles in a triangle? Why?
- What is distance?
- Metric tensor.
- A story of an ant on a sphere. Sum of the angles in a triangle. The number π.

What is a straight line?

- Length of a curve as a functional.
- Functional, variations, Extremum.
- Straight line in Euclidean space in Cartesian coordinates.

LECTURE 2
Mechanics.

- Home work solutions
- Calculus.

LECTURE 3
Geometry. Topology.

Physical world and its description.

- System of coordinates.
  - Scalars.
  - Vectors. Representation. Invariance.
    - Distinguish between vector and its representation.
    - Invariace of the scalar/dot product.
    - Invariance of the vector/cross product.
- Physics world is independent of the way we describe it.
  - Straight line in Euclidean space in Cartesian coordinates: $y = ax + b$.
  - Straight line in Euclidean space in Polar coordinates $r = \frac{a}{\cos(\phi - \phi_0)}$.

Description of space. Geometry

- What is a straight line?
- Metric tensor in Cartesian coordinate system: $(dl)^2 = (dx)^2 + (dy)^2$.
- Metric tensor in polar coordinates $(dl)^2 = (dr)^2 + r^2(d\phi)^2$.
- Change of variables in the integral: under the change $x = r(\phi)\cos(\phi)$, $y = r(\phi)\sin(\phi)$ we have

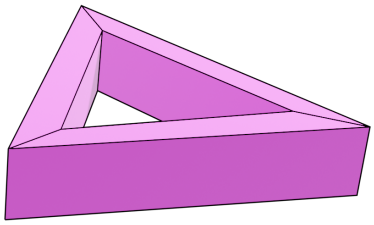
$$L = \int \sqrt{(dx)^2 + (dy)^2} = \int \sqrt{r^2 + (r')^2} d\phi.$$

- Metric tensor on a sphere $(dl)^2 = R^2(d\theta)^2 + R^2(d\phi)^2 \sin^2\theta$.
- "Straight" line on a sphere.
- What is our space?

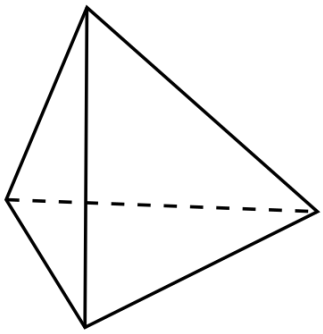
Topology. Euler characteristics `https://en.wikipedia.org/wiki/Euler_characteristic`.

- Number of vertices V, edges E, and faces F.
- Compute $\chi = V - E + F$ for several polyhedral: Cube, Tetrahedral, etc. (Cube: $V = 8$, $E = 12$, $F = 6$.)
- $V - E + F$ as invariant. Proof by adding a new edge to a face.
- A face must have no holes.
- Continuum limit.

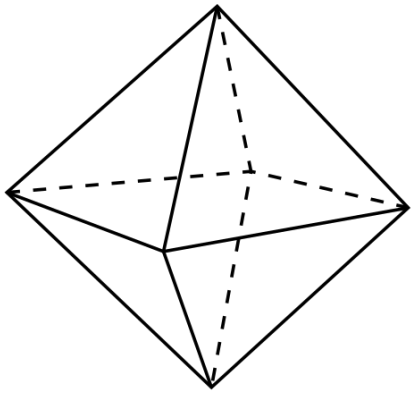
There is no use for Euler characteristics if it does not depend on anything.



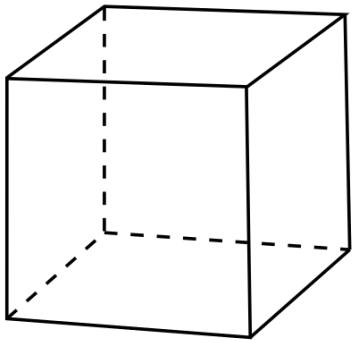
- $V - E + F$ for torus. (for the figure: $V = 12$, $E = 24$, $F = 12$.)
- $\chi = V - E + F = 2 - 2g$. Proof by cutting a handle.
- A story of an ant.
- What does it have to do with physics?



Tetrahedron
$V = 4,\ E = 6,\ F = 4.$



Octahedron
$V = 6,\ E = 12,\ F = 8.$



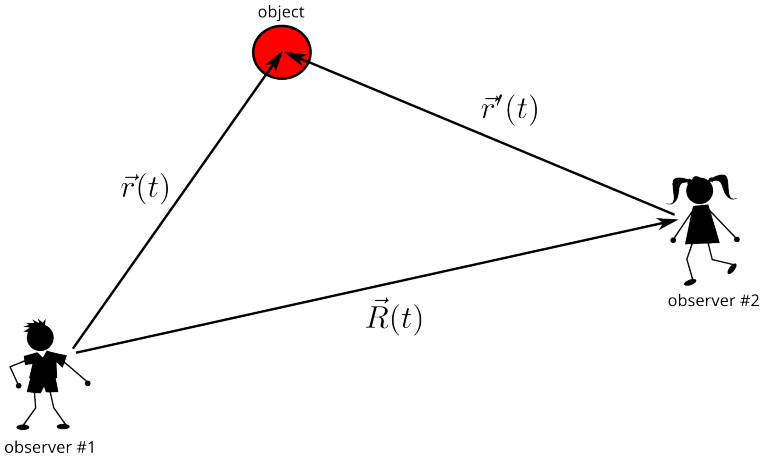
Cube
$V = 8,\ E = 12,\ F = 6.$

LECTURE 4
Mechanical world. Galilean invariance. Newton laws.

4.1. Mechanics.

4.1.1. How do we start?

- One body, one observer.
  - A body (simplification – point like object of a certain mass) in an empty space.
  - Process is independent of observer.
  - No universal frame of reference.
  - Position $\vec{r}$ depends on time: $\vec{r}(t)$. Velocity — the rate of change of the position: $\vec{v} = \frac{d\vec{r}}{dt} \equiv \dot{\vec{r}}$.
- One body, two observers.
  - Look at the figure: we have two observers and one object. At time t the observer #1 sees the object at position $\vec{r}(t)$, the observer #2 sees the object at position $\vec{r}'(t)$. The observer #1 sees the observer #2 at the position $\vec{R}(t)$. From the figure we see that
  $$\vec{r}(t) = \vec{R}(t) + \vec{r}'(t).$$
  This relation is valid for all times, so differentiating it with respect to time we get at every moment of time
  $$\vec{v} = \vec{V} + \vec{v}',$$
  where $\vec{v}$ is the velocity of the object at time t as measured by the observer #1, $\vec{v}'$ is the velocity of the object at time t as measured by the observer #2, and $\vec{V}$ is the velocity of the observer #2 at time t as measured by the observer #2.
  - Galilean invariance.
  - Galilean invariance in increments.
  $$dx' = dx - V_x dt, \quad dy' = dy - V_y dt, \quad \ldots$$
  $$dt' = dt.$$
- Time reversal. No universal clock.
- Two bodies, one observer.
  - Interactions. What is force?
  - Force is a (vector) measure of interaction.



- It requires two bodies: INTERaction.
- There interaction cannot depend on how we label the bodies, so the force form the first on the second should be the same (but opposite direction) as the force from the second on the first.
- A force AND ONLY A FORCE can lead to a change in velocity. Velocity is a vector. Change is velocity means change in magnitude and/or direction!
- Idea of locality in time (instantaneous) — change in velocity is described by derivatives of velocity over time. In particular $\vec{a} = \frac{d\vec{v}}{dt}$.

4.1.2. Newton's laws.

- If a body does not interact with anything, it moves with constant velocity.
- $\vec{F} = m\vec{a}$.
- $\vec{F}_{1\leftarrow 2} = -\vec{F}_{2\leftarrow 1}$.

$$\vec{F} = m\vec{a} \textbf{ is a differential equation!}$$

In fact, as it is a vector equation it is THREE SECOND ORDER DIFFERENTIAL EQUATIONS. In Cartesian coordinates:

$$m\frac{d^2x}{dt^2} = F_x(x,y,z,t)$$
$$m\frac{d^2y}{dt^2} = F_y(x,y,z,t)$$
$$m\frac{d^2z}{dt^2} = F_z(x,y,z,t)$$

These are typically non-linear differential equations. A complete solution requires two initial conditions for each equation.

4.1.3. Examples.

- Motion with constant acceleration in $1D$.

$$v = v_0 + at$$
$$x = x_0 + v_0 t + \frac{at^2}{2}$$

These are correct ONLY(!!!) for the case of constant acceleration.

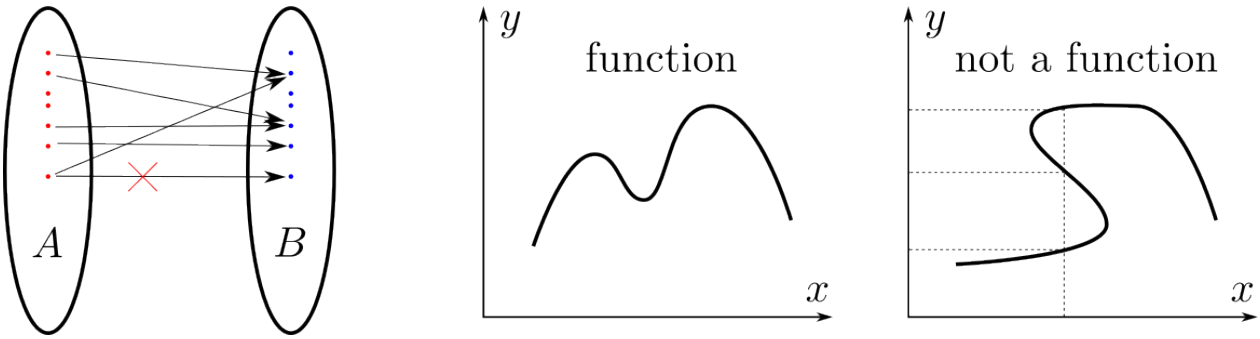
- Oscillator.
- Gravity.

LECTURE 5
Homework. Calculus of many variables.

5.1. Homework.

5.2. Calculus of many variables.

5.2.1. What is a function?

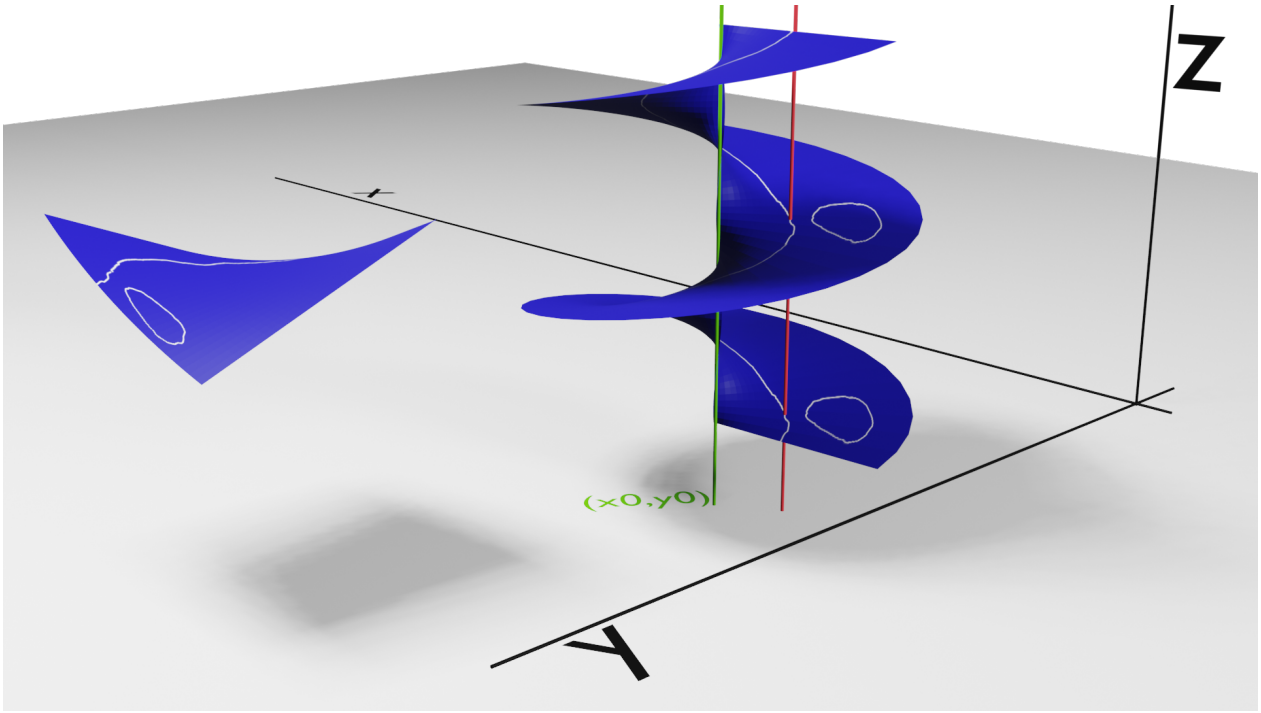


A map from one set A of elements/objects/points, to another set of elements/objects/points B, such, that any element of the sat A has no more than one image in the set B. When both sets A and B are the sets of real numbers we can call the elements of the set A to be our x and the elements of the set B to be our y. Then we can plot the relation between elements of A and B (or x and y). Sometimes we can write the relation between y and x as $y = f(x)$ with some known set of rules f. This set of rules tells us what to do with the number x in order to get the corresponding number y.

In this case x is called the argument of the function, and y the value of the function. If the function is smooth enough, then shifting the argument just a little bit the value will also change just a little bit. So if the shift dx of the argument x is infinitesimally small, then the change dy of the value $y = f(x)$ is also infinitesimally small, and we can write

$$dy = \frac{df}{dx}dx$$

Again, this is the shift of the value while we shift the argument. The shift of the value dy is called a differential.



5.2.2. Differential

- Let's now consider a function of two arguments $U(x, y)$. It gives a real number for each pair (x, y) of real numbers.
- Now when we say that we shift the argument we need to specify separately how we shift the argument x and the argument y. Correspondingly, we denote these shifts by dx and dy.
- Differential (the change of value of the function under the infinitesimal change of the arguments)

$$dU = \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy \equiv \vec{\nabla}U \cdot d\vec{r} \equiv \frac{\partial U}{\partial \vec{r}} \cdot d\vec{r} \equiv \mathrm{grad}U \cdot d\vec{r}.$$

  Notations:

$$\vec{\nabla}U \equiv \mathrm{grad}U \equiv \frac{\partial U}{\partial \vec{r}} \equiv \begin{pmatrix} \partial_x U \\ \partial_y U \end{pmatrix}, \qquad \partial_x U \equiv \frac{\partial U}{\partial x}, \quad \partial_y U \equiv \frac{\partial U}{\partial y}.$$

- Consider two function $f_x(x, y)$ and $f_y(x, y)$. When is a 1-form

$$f_x(x, y)dx + f_y(x, y)dy$$

  a differential of some function? In other words: What conditions do the functions $f_x(x, y)$ and $f_y(x, y)$ need to satisfy in order for the above 1-form to be a full differential? In other words: What conditions the functions $f_x(x, y)$ and $f_y(x, y)$ must satisfy in order for a function $U(x, y)$ such that $f_x(x, y) = \frac{\partial U}{\partial x}$ and $f_y(x, y) = \frac{\partial U}{\partial y}$ to exist?
- This condition is:

$$\frac{\partial f_x}{\partial y} = \frac{\partial f_y}{\partial x}$$

- Notice, this condition does not give us the function $U(x,y)$. However, if this conditions is satisfied, then such function exists! If it is not satisfied, then such function $U(x,y)$ simply does not exist!
- Examples:
  - Full differential: $ydx + xdy$

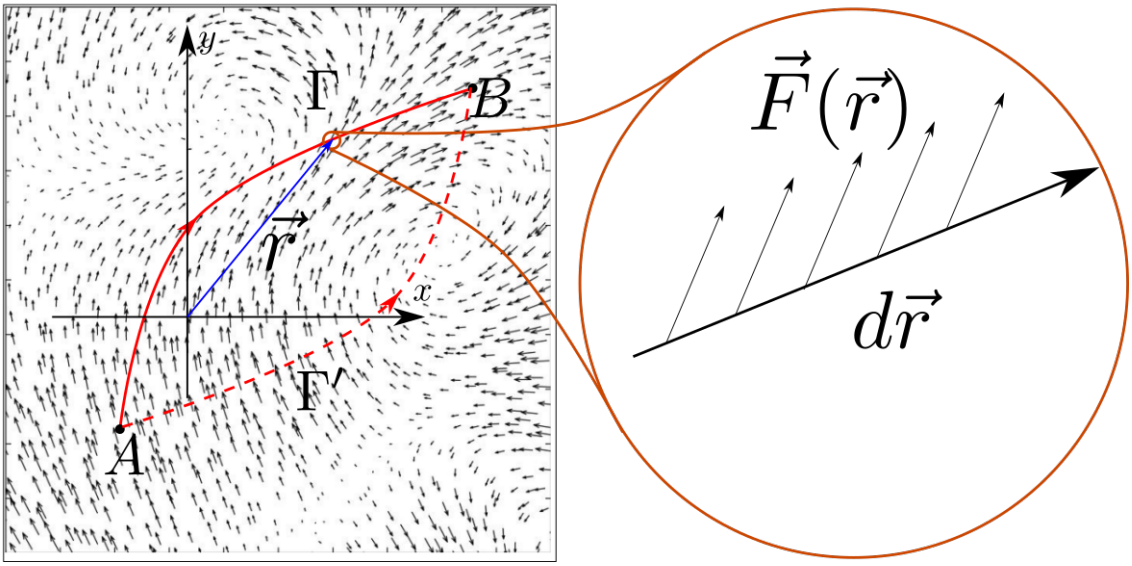
    $$f_x(x,y) = y, \quad f_y(x,y) = x, \qquad \frac{\partial f_x}{\partial y} = \frac{\partial f_y}{\partial x} = 1, \qquad U(x,y) = xy.$$

  - Not a differential: $ydx - xdy$

    $$f_x(x,y) = y, \quad f_y(x,y) = -x, \qquad \frac{\partial f_x}{\partial y} = 1 \neq \frac{\partial f_y}{\partial x} = -1$$

    There is no function U!

5.2.3. Integration of a vector field over a path.



We will work in $2D$, although all what we do here can be done in arbitrary dimensions.

Let's consider an arbitrary vector field $\vec{F}(x,y)$. What it means is that in each point of space x, y or $\vec{r}$ we have a vector $\vec{F}$ and this vector may be different in different points

Let's assume that we have two points A and B and a path Γ between them. The path is oriented, it goes from A to B, see figure.

We want to consider the following construction:

- We split the path Γ into infinitesimal pieces.
- Each piece is a strait infinitesimal vector $d\vec{r}$.
- For a piece at position $\vec{r}$ there is the vector $\vec{F}(\vec{r})$ IN THE SAME POSITION.
- This vector is almost constant for the whole infinitesimal piece.
- We then can define a number $\vec{F}(\vec{r}) \cdot d\vec{r}$ for each piece.
- We then sum up these number along the whole path.

This procedure is written as

$$\int_\Gamma \vec{F}(\vec{r}) \cdot d\vec{r}.$$

(A side note: If the path Γ is a closed loop, then this integral is called circulation of the vector field around the loop Γ. In this case the typical notation is $\oint_\Gamma \vec{F}(\vec{r}) \cdot d\vec{r}$)

Notice, that for given vector field $\vec{F}(\vec{r})$ this integral, in general,

- depends not only on A and B, but also on the path Γ.
- In particular, for two different paths Γ and Γ', both from A to B (see figure)

$$\int_\Gamma \vec{F}(\vec{r}) \cdot d\vec{r} \neq \int_{\Gamma'} \vec{F}(\vec{r}) \cdot d\vec{r}$$

However, there is a special situation when this integral does not depend on path. It means that $W = \int_\Gamma \vec{F}(\vec{r}) \cdot d\vec{r}$ will be the same for ANY path which goes from A to B.

This is the case, when there exists such a function $U(x, y)$ that

$$F_x(x, y) = \frac{\partial U}{\partial x}, \qquad F_y(x, y) = \frac{\partial U}{\partial y}.$$

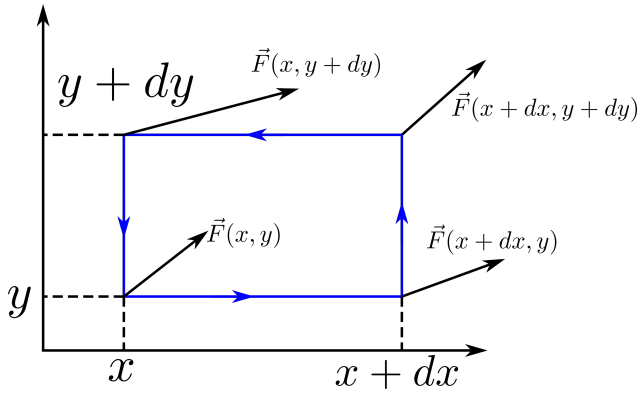
In this case we have

$$W = \int_\Gamma \vec{F}(\vec{r}) \cdot d\vec{r} = \int_\Gamma \left(\frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy \right) = \int_\Gamma dU = U(B) - U(A).$$

In particular for such vector field the integral W around ANY closed loop is zero! As in this case the initial and final points are the same, so subtracting U at initial point from U at final point is subtracting one number from the same number.

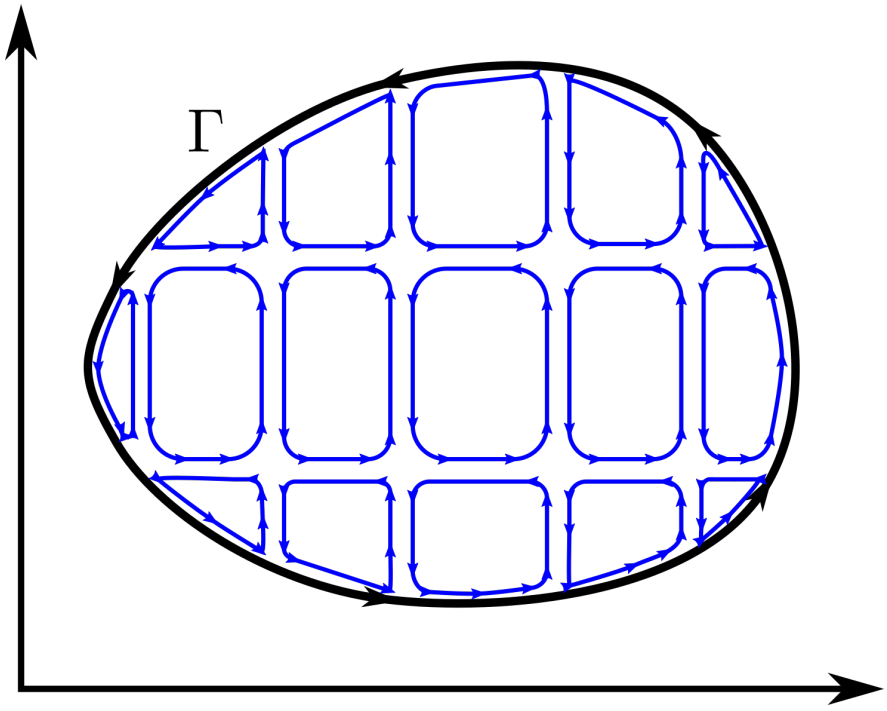
One can turn the last statement around: If for a particular vector field $\vec{F}(x, y)$ the integral W is zero for any closed loop, then there exist a function $U(x, y)$, such that $F_x(x, y) = \frac{\partial U}{\partial x}$ and $F_y(x, y) = \frac{\partial U}{\partial y}$.

Proof.



Consider an infinitesimally small loop (see figure). Then

$$W = \frac{F_x(x, y) + F_x(x + dx, y)}{2} dx + \frac{F_y(x + dx, y) + F_y(x + dx, y + dy)}{2} dy$$
$$- \frac{F_x(x + dx, y + dy) + F_x(x, y + dy)}{2} dx - \frac{F_y(x, y + dy) + F_y(x, y)}{2} dy$$



Now using

$$F_x(x+dx,y) = F_x(x,y) + \frac{\partial F_x(x,y)}{\partial x}dx$$
$$F_x(x+dx,y+dy) = F_x(x,y) + \frac{\partial F_x(x,y)}{\partial x}dx + \frac{\partial F_x(x,y)}{\partial y}dy$$
$$F_x(x,y+dy) = F_x(x,y) + \frac{\partial F_x(x,y)}{\partial y}dy$$

And the same for the y components, we find, that

$$W = \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) dxdy.$$

So if W for any loop is zero, then we have

$$\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = 0.$$

But this is exactly the requirement for the existence of the function U!

For a NON-conservative vector field we just proved that for any closed loop

$$W = \oint_\Gamma \vec{F}(\vec{r}) \cdot d\vec{r} = \int_{\mathcal{A}} \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) dxdy,$$

where $\mathcal{A}$ is the area inside the loop.

In order to see that we need to split a large loop into many infinitesimal ones.

LECTURE 6
Newton's law. Energy conservation. Motion in $1D$.

So far:

- Newton's law
$$\vec{F} = m\vec{a}, \qquad \vec{a} = \frac{d\vec{v}}{dt}, \qquad \vec{v} = \frac{d\vec{r}}{dt}$$
- Vector tangential to trajectory has the same direction as $\vec{v}$. The vectors $\vec{v}$ and $\vec{F}$ in general have different directions. A trajectory does not, in general, coincide with the force lines.
- Newton's law as a differential equation
$$m\frac{d^2\vec{r}}{dt^2} = \vec{F}(\vec{r}, t), \qquad \vec{r}(t = t_0) = \vec{r}_0, \qquad \vec{v}(t = t_0) = \vec{v}_0$$
- Example:
  - $\vec{F}(\vec{r}, t)$ does not depend on $\vec{r}$ and t. Say $\vec{F} = -mg\hat{y}$, where g is an arbitrary constant and $\hat{y}$ is a unit vector in the y direction. Then the equation of motion is
$$\frac{d^2\vec{r}}{dt^2} = -g\hat{y}, \quad \text{or} \quad \frac{d^2x}{dt^2} = 0, \quad \frac{d^2y}{dt^2} = -g$$
This equation(s) must be supplied with initial conditions, say at $t = 0$
$$x(t = 0) = x_0, \quad v_x(t = 0) = v_{x0}, \quad y(t = 0) = y_0, \quad v_y(t = 0) = v_{y0}.$$
  - The solution of the equation of motion with the given initial conditions is
$$x(t) = x_0 + v_{x0}t, \qquad y(t) = y_0 + v_{y0}t - \frac{gt^2}{2}.$$

6.1. What do we know and what do we need?

What we know:

- Newton's laws tell us that if we know the motion $\vec{r}(t)$, then we know all the forces which acted on the body.
- Also if we know all the forces, and the initial conditions, then treating $\vec{F} = m\vec{a}$ as a set of second order differential equations we can find $\vec{r}(t)$.

It is the second point that is most useful. But in order to use the Newton's laws this way we need to know all the forces. So the rest of physics is about how to compute the forces.

Due to locality the force which act of a point particle may depend only on instantaneous "state" (position, velocity, etc) of the particle. The simplest case is when the forces depend on the position only. In order to desctibe such forces it is natural and very convenient to use the idea of a force field.

- A force is the result of interaction between our particle and some other object.
- We then say, that that object "produces" a force field $\vec{F}(\vec{r}$ EVERYWHERE in the space (independent from where our particle is).
- After that we say, that if our particle at some moment of time t is at the position $\vec{r}$, then the force which acts on our particle at this very instant is $\vec{F}(\vec{r})$.

It turns out that ALL the fundamental forces can be described by such fields. (Magnetic field is a bit different, we will discuss it later.)

6.2. Energy conservation.

Now we assume that we know the force field $\vec{F}(\vec{r})$. So if a particle of mass m is at position $\vec{r}$, that we know the force which acts on it. No matter where the particle is, we know the force which acts on it.

- No we assume that the particle is moving from point A to point B along some trajectory Γ. We are not concerned why it is moving along this trajectory, there may be other forces in play. In this case we can define work W which is done by our force field during this motion. Work. Work of the force field $\vec{F}(\vec{r})$ is the integral of the force field $\vec{F}(\vec{r})$ along the path Γ.

$$W = \int_{A,\Gamma}^{B} \vec{F} \cdot d\vec{r}$$

  This work in general depends on the path Γ!!!!!
- Now we consider the situation when the particle's motion is due to the force field $\vec{F}$, then if Γ is the body TRAJECTORY resulted from the very same force, then using $\vec{F} = m\vec{a} = m\frac{d\vec{v}}{dt}$ we find

$$W = \int_{A,\Gamma}^{B} \vec{F} \cdot d\vec{r} = \int_{A,\Gamma}^{B} m\frac{d\vec{v}}{dt} \cdot \frac{d\vec{r}}{dt}dt = \int_{A,\Gamma}^{B} m\frac{d\vec{v}}{dt} \cdot \vec{v}dt$$
$$= \int_{A,\Gamma}^{B} \frac{d\frac{m\vec{v}^2}{2}}{dt}dt = \frac{m\vec{v}_B^2}{2} - \frac{m\vec{v}_A^2}{2} = K_B - K_A \equiv \Delta K,$$

  where K_A and K_B are the kinetic energy of the particle in the initial and final points of the path Γ.

  I remind that here Γ is the TRAJECTORY, which we do not know unless we have solved the problem already. So the above equation is not very useful for finding the trajectory. However, it tells us that the change of the kinetic energy equals the to the work done by the forces that cause the motion.
- Special case of CONSERVATIVE forces. We assume that the force field $\vec{F}(\vec{r})$ is such that there exists a function $U(\vec{r})$, such that (I consider here an example in $2D$, so that the position vector $\vec{r}$ has two components (x, y) and the force field $\vec{F}(\vec{r})$ is given

by two components $F_x(x,y)$ and $F_y(x,y)$)

$$F_x(x,y) = -\frac{\partial U}{\partial x}, \qquad F_y(x,y) = -\frac{\partial U}{\partial y}$$

A force that satisfies this condition is called CONSERVATIVE force.

- In this case

$$W = \int_{A,\Gamma}^{B} \vec{F}\cdot d\vec{r} = \int_{A,\Gamma}^{B} (F_x dx + F_y dy) = -\int_{A,\Gamma}^{B} \left(\frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy\right) = -\int_{A,\Gamma}^{B} dU = U_B - U_A$$

where U_A and U_B are the values of the function U in the initial and final points correspondingly.

For the CONSERVATIVE forces, work does not depend on path! It depends only on initial and final points!

- If initial and final points are the same, the work must be zero. So for any closed loop

$$\oint \vec{F}\cdot d\vec{r} = 0.$$

- A force filed $\vec{F}(\vec{r})$ is conservative if, and only if there is a function $U(\vec{r})$ such that

$$\vec{F} = -\frac{\partial U}{\partial \vec{r}} \equiv -\vec{\nabla}U \equiv -\mathrm{grad}U$$

- This function $U(\vec{r})$ is called the potential energy.

Consequences:

- Non-uniqueness of U.
- Voltage. Kirchhoff's law.
- Potential difference. Why do you need ground.
- In order to specify the full force field $\vec{F}(\vec{r})$ we only need to specify one function $U(\vec{r})$!!!
- Energy. For a motion of a particle from the initial point i to the final point f caused by a conservative force with the potential energy $U(\vec{r})$:

$$\Delta K = K_f - K_i = \int_{\Gamma} \vec{F}\cdot d\vec{r} = -\int_{\Gamma} \frac{\partial U}{\partial \vec{r}}\cdot d\vec{r} = -\int_{\Gamma} dU = -U_f + U_i$$

so

$$K_i + U_i = K_f + U_f, \qquad E = K + U, \qquad E_i = E_f$$

Full energy is conserved! $E = \frac{m\vec{v}^2}{2} + U(\vec{r})$.

- Time translation invariance. Energy conservation.
- Translation invariance. Momentum conservation.

6.3. Motion in 1D.

- Conservative forces in 1D. In 1D every force which depends only on coordinate $F(x)$ is a conservative force. We can always construct the potential energy

$$U(x) = -\int_{x_0}^{x} F(x')dx', \qquad F(x) = -\frac{\partial U(x)}{\partial x}.$$

(choosing any other point x_0' instead of x_0 only adds a constant $\int_{x_0}^{x_0'} F(x')dx'$ to the function $U(x)$.)

- Energy conservation: $E = \frac{mv^2}{2} + U(x)$. As $\frac{mv^2}{2} > 0$ the motion is only possible if
$$U(x) < E.$$
- Let's initial conditions be $x(t_0) = x_0$ and $v(t_0) = v_0$, so we can compute the energy at the initial moment of time t_0
$$E = \frac{mv_0^2}{2} + U(x_0).$$
- However, energy E is constant. So it will be the same at ANY moment of time. So assuming that at some other moment of time t the particle is at position x and has velocity v, we can write for any moment of time t
$$\frac{m}{2}\left(\frac{dx}{dt}\right)^2 + U(x) = E$$
$$\frac{dx}{dt} = \pm\sqrt{\frac{2}{m}}\sqrt{E - U(x)}$$
$$dt = \pm\sqrt{\frac{m}{2}}\frac{dx}{\sqrt{E - U(x)}}$$
$$t - t_0 = \pm\sqrt{\frac{m}{2}}\int_{x_0}^{x(t)}\frac{dx}{\sqrt{E - U(x)}}$$
- Oscillator.
  - Let's assume that our initial conditions are
  $$x(t=0) = 0, \qquad v(t=0) = v_0.$$
  - The force is $F = -kx$, the corresponding potential energy
  $$U(x) = \frac{kx^2}{2}, \qquad F(x) = -\frac{\partial U}{\partial x} = -kx$$
  - The energy is conserved, so we compute it at $t = 0$
  $$E = \frac{mv^2(t=0)}{2} + \frac{kx^2(t=0)}{2} = \frac{mv_0^2}{2}.$$
  - Now we have
  $$t = \pm\sqrt{\frac{m}{2}}\int_0^{x(t)}\frac{dx}{\sqrt{E - \frac{kx^2}{2}}}.$$
  - Taking this integral we find
  $$t = \sqrt{\frac{m}{k}}\arcsin\left(\sqrt{\frac{k}{2E}}x(t)\right)$$
  - Inverting this equation
  $$x(t) = \sqrt{\frac{2E}{m}}\sin\left(\sqrt{\frac{k}{m}}t\right)$$
  - Using the value of E and usual frequency $\omega = \sqrt{k/m}$ we get
  $$x(t) = \frac{v_0}{\omega}\sin(\omega t).$$

- Motion in 1D in arbitrary potential $U(x)$.
  - Let's draw the function $U(x)$ – see figure 1.
  - We also draw a line E – it is conserved, it is a constant.
  - As kinetic energy K is always positive, we must have
$$U(x) < E.$$
  - So the shaded regions on the figure are inaccessible/prohibited for a particle of energy E.
  - At the points of intersection of the lines $U(x)$ and E, the kinetic energy is zero, so the velocity is zero. These points are called turning points.
  - If a particle is in between two of such points it must go back and force in between them. If the particle has only one of such points, as of the far left and far right on the figure, then the particle will go to infinity. These are the only two possibilities.
  - If the particle is in between two turning points, say x_1 and x_2 on the figure. The period of its motion can be computed by
$$T = 2\sqrt{\frac{m}{2}}\int_{x_1}^{x_2}\frac{dx}{\sqrt{E-U(x)}}$$
The extra factor of 2 is because the particle should go back and forth.

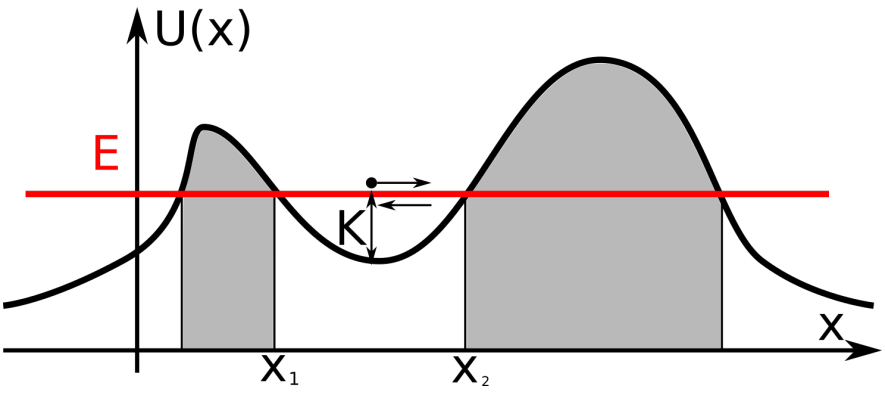


Figure 1. 1D potential $U(x)$ – black line.

LECTURE 7
Hamiltonian.

7.1. Why do we need a new formulation?

- Newtonian formulation deals with forces.
- A force field $\vec{F}(\vec{r})$ is given by three functions

$$F_x(\vec{r}), \qquad F_y(\vec{r}), \qquad F_z(\vec{r}).$$

- All fundamental forces are conservative.
- A conservative force field is given by a single function $U(\vec{r})$:

$$\vec{F} = -\vec{\nabla}U$$

- So all information we need is contained in one function, instead of three. Using the vector force field $\vec{F}(\vec{r})$ is an overkill.
- There must be a simpler way to formulate the Newtonian mechanics for the conservative forces.
- We want to reformulate the Newtonian mechanics in such a way as to avoid using forces and use only the potential energy $U(\vec{r})$.

There is more

- Symmetries are hard to notice in vector formulation.
- The idea of force cannot be used in quantum mechanics.
- The idea of force cannot be used in relativistic physics.

7.2. Hamiltonian formulation.

7.2.1. Energy conservation.

- Consider a particle moving in a potential field

$$m\frac{d^2\vec{r}}{dt^2} = -\frac{\partial U}{\partial \vec{r}}, \qquad +\text{initial conditions.}$$

- The trajectory $\vec{r}(t)$ is then defined by the potential energy function $U(\vec{r})$ and initial conditions.
- Consider energy as a function of velocity and coordinates: $E(\vec{v},\vec{r}) = \frac{m\vec{v}^2}{2} + U(\vec{r})$. At this point we want to think of $\vec{v}$ and $\vec{r}$ as independent variables.
- Full vs. partial derivatives.

  - There is no explicit time dependence in the function $E(\vec{v},\vec{r})$, so $\frac{\partial E}{\partial t} = 0$ — PARTIAL derivative of E over time is zero.
  - However, if the particle moves along its trajectory, then both $\vec{r}$ and $\vec{v}$ change with time: $\vec{r}(t)$, $\vec{v}(t)$.
  - So $E(\vec{v},\vec{r})$ may depend on time through the time dependence of $\vec{r}(t)$ and $\vec{v}(t)$.
  - This time dependence is computed by computing the FULL time derivative: $\frac{dE}{dt}$.
- The particle moves according to the equation of motion. If we solve it, we will know $\vec{r}(t)$ and $\vec{v}(t)$. We then can stick these functions into our function $E(\vec{v},\vec{r})$ and get $E(\vec{v}(t),\vec{r}(t))$ a function of t.
- Let's compute, how this energy $E(\vec{v}(t),\vec{r}(t))$ changes with time

$$\frac{dE}{dt} = \frac{\partial E}{\partial \vec{v}}\cdot\frac{d\vec{v}}{dt} + \frac{\partial E}{\partial \vec{r}}\cdot\frac{d\vec{r}}{dt} = m\vec{v}\cdot\frac{d^2\vec{r}}{dt^2} + \frac{\partial U}{\partial \vec{r}}\cdot\vec{v} = \vec{v}\cdot\left(m\frac{d^2\vec{r}}{dt^2} + \frac{\partial U}{\partial \vec{r}}\right) = 0.$$

  In other words energy is conserved on a given trajectory (the trajectory in a given potential field $U(\vec{r})$ is given by initial conditions). It is, however, different in different trajectories. So the Energy depends on the initial conditions, as expected.

7.2.2. Momentum. Dispersion relation

- Momentum for our system is simply

$$\vec{p} = m\vec{v}, \qquad \text{or} \qquad \vec{v} = \frac{\vec{p}}{m}$$

- The Newton equation of motion in terms of momentum is

$$\dot{\vec{p}} = -\frac{\partial U}{\partial \vec{r}}.$$

  (Notations: $\dot{\vec{p}} \equiv \frac{d\vec{p}}{dt}$.)
- We can also express the kinetic energy in terms of momentum.

$$K = \frac{\vec{p}^2}{2m}$$

  The kinetic energy in terms of momentum is called *dispersion relation*. It is not always a simple quadratic function. See figure for more complicated examples of dispersion relations of an electron in a crystal lattice.
- Consider a function $H(\vec{p},\vec{r}) = \frac{\vec{p}^2}{2m} + U(\vec{r})$. We take $\vec{p}$ and $\vec{r}$ as independent variables in this function.
- We then see, that our equations of motion are

$$\dot{\vec{p}} = -\frac{\partial H}{\partial \vec{r}}$$
$$\dot{\vec{r}} = \frac{\partial H}{\partial \vec{p}}$$

  Notice the minus sign in the first equation.
- These equations are called Hamiltonian equations. The equations must be supplied with the initial conditions: initial position and initial momentum.
- The function H is called Hamiltonian.
- Instead of one second order differential equation we have two first order differential equations.

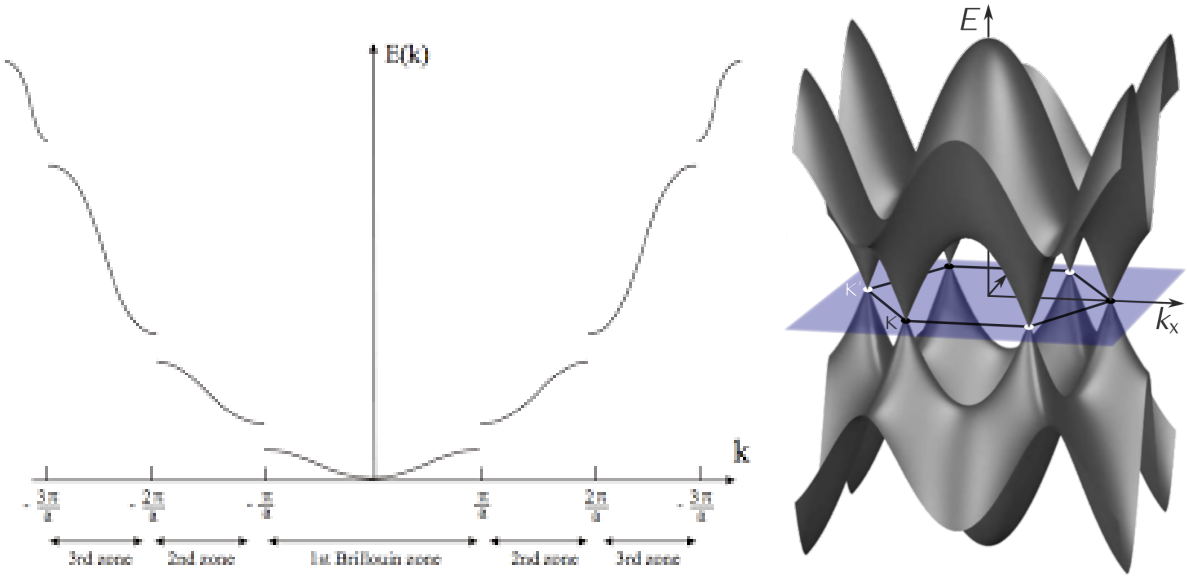


Figure 1
Examples of different dispersion relations: left, $1D$ electron on a simple $1D$ lattice; right, the dispersion relation of electrons in graphene.

Hamiltonian is a function of coordinates and momenta.

- Consider the value of the Hamiltonian on a trajectory: $\vec{p}(t)$, $\vec{r}(t)$: $H(\vec{p}(t), \vec{r}(t))$,

$$\frac{dH}{dt} = \frac{\partial H}{\partial \vec{p}} \cdot \dot{\vec{p}} + \frac{\partial H}{\partial \vec{r}} \cdot \dot{\vec{r}} = -\frac{\partial H}{\partial \vec{p}} \cdot \frac{\partial H}{\partial \vec{r}} + \frac{\partial H}{\partial \vec{r}} \cdot \frac{\partial H}{\partial \vec{p}} = 0$$

- Energy IS a value of a Hamiltonian on a trajectory.
- Notice that in derivation of the energy conservation the form of the Hamiltonian was not important. So taking an arbitrary function of coordinates and momentum as Hamiltonian and considering dynamics as given by the Hamiltonian equations, we will always have energy as conserved on trajectories.
- All mechanical problems will have exactly the same Hamiltonian equations of motion. Different problems differ only by one scalar function – the Hamiltonian.
- Second Hamiltonian equation gives the velocity for general dispersion relation.

7.2.3. Phase space.

- Hamiltonian is an arbitrary (specific for a given problem) function on a phase space.
- Phase space trajectories.

7.3. Functionals.

- Definition of functionals.
  - Correspondence/map "number to number" is called a function.
  - Correspondence/map "function to number" is called a functional.
- Examples.

LECTURE 8
Lagrangian.

- Homework.

8.1. Euler-Lagrange equation.

8.1.1. Functionals.

- Definition of functionals.
  - Correspondence/map "number to number" is called a function.
  - Correspondence/map "function to number" is called a functional.
- Examples.
- For functions which satisfy the boundary conditions $f(x_A) = f_A$ and $f(x_B) = f_B$ in many cases the functional can be represented by

$$\mathcal{S}[f(x)] = \int_{x_A}^{x_B} L(f'(x), f(x))dx.$$

  A function of two variables $L(z_1, z_2)$ and two pairs of numbers x_A, f_A and x_B, f_B define the functional.
  What it means:
  - It is a rule which gives a number for any function $f(x)$ which satisfies the boundary conditions $f(x_A) = f_A$ and $f(x_B) = f_B$.
  - If you have the function $L(z_1, z_2)$ of two variables and two pairs of numbers x_A, f_A and x_B, f_B, then you have the functional given above.
  - Then if you are given a function $f(x)$ (which satisfies the boundary conditions $f(x_A) = f_A$ and $f(x_B) = f_B$), you take its derivative $f'(x)$ and substitute it into $L(z_1, z_2)$ instead of argument z_1, you also substitute the function $f(x)$ instead of the argument z_2. You then have a function of x: $L(f'(x), f(x))$.
  - You then take the integral $\int_{x_A}^{x_B} L(f'(x), f(x))dx$, which gives you some number.
  - You return this number as a value of the functional on the function $f(x)$.
- Example: a length of an arbitrary path which goes from the point (x_A, y_A) to the point (x_B, y_B).
  - Let's assume, that the path is given by a function $y(x)$. This function must satisfy

$$y(x_A) = y_A, \qquad y(x_B) = y_B.$$

– The length of this path is

$$l[y(x)] = \int_{x_A}^{x_B} \sqrt{1+y'^2}dx.$$

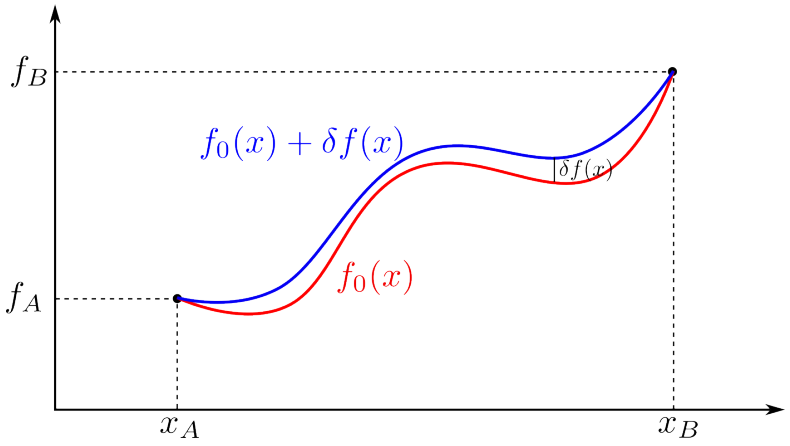
– $l[y(x)]$ is a functional. For any path $y(x)$ it gives back a number — its length!

– Looking for the function $L(y'(x), y(x))$ we must treat y and y' as independent variables. So our function $L(y'(x), y(x))$ is given by

$$L(y'(x), y(x)) = \sqrt{1+y'^2}.$$

– So in this particular example this function does not depend on its second argument.

8.1.2. Euler-Lagrange equation.



- We ask the following question: given a functional $\mathcal{S}[f(x)]$, what function $f(x)$ (which satisfies the boundary condition) gives us the minimal value of the functional?
- We consider only the functions which satisfy the boundary conditions.
- Let's assume that we found the function $f_0(x)$ that solves our problem.
- Let's change this function a little and see how the value of the functional will change. So we consider a function $f(x)$

$$f(x) = f_0(x) + \delta f(x), \qquad \delta f(x_A) = 0, \quad \delta f(x_B) = 0$$

The last two equalities are due to the fact, that the function $f(x)$ must satisfy the same boundary conditions.

- The new value of the functional is

$$\mathcal{S}[f(x)] = \int_{x_A}^{x_B} L(f_0'(x) + \delta f'(x), f_0(x) + \delta f(x))dx$$

Remember L is just a function of two variables, so we can write

$$L(f_0'(x) + \delta f'(x), f(x) + \delta f(x)) = L(f_0'(x), f_0(x)) + \frac{\partial L}{\partial f_0'}\delta f' + \frac{\partial L}{\partial f_0}\delta f,$$

So we have

$$\mathcal{S}[f(x)] = \mathcal{S}[f_0(x)] + \int_{x_A}^{x_B} \left(\frac{\partial L}{\partial f_0'}\frac{d\delta f}{dx} + \frac{\partial L}{\partial f_0}\delta f\right) dx,$$

Taking the integral by parts in the first term we get

$$\mathcal{S}[f(x)] = \mathcal{S}[f_0(x)] + \int_{x_A}^{x_B} \left(-\frac{d}{dx}\frac{\partial L}{\partial f_0'} + \frac{\partial L}{\partial f_0} \right) \delta f(x) dx$$

Now we see, that depending on $\delta f(x)$ the integral can be either positive, or negative. But it must never be negative, because $\mathcal{S}[f_0(x)]$ is the minimum! It means that the expression in the brackets must be zero! We then found, that the function $f_0(x)$ must be such as to satisfy the following equation

$$\frac{d}{dx}\frac{\partial L}{\partial f_0'} - \frac{\partial L}{\partial f_0} = 0$$

This is called Euler-Lagrange equation.

- Example: in the example above, for the length of a curve, we can ask which curve gives the minimal length.
  - Our function L does not depend on its second argument and is $L(y'(x), y(x)) = \sqrt{1+y'^2}$.
  - So the Euler-Lagrange equation for this case is

$$\frac{d}{dx}\frac{\partial L}{\partial y'} = \frac{\partial L}{\partial y}, \qquad y(x_A) = y_A, \qquad y(x_B) = y_B$$

    or

$$\frac{d}{dx}\frac{y'}{\sqrt{1+y'^2}} = 0, \qquad y(x_A) = y_A, \qquad y(x_B) = y_B.$$

  - This is a second order ordinary differential equation with boundary conditions. The general solution is

$$y(x) = ax + b,$$

    where the numbers a and b must be found from the boundary conditions.
  - The equation above $y = ax + b$ describes a straight line!!

8.2. Hamilton principle.

- Hamilton principle. Action. Minimal action.
- Lagrangian.
  - Lagrangian.

$$L(\dot{\vec{r}}, \vec{r}) = K - U.$$

  - Action

$$\mathcal{A}[\vec{r}(t)] = \int_{t_i}^{t_f} L(\dot{\vec{r}}, \vec{r}) dt$$

Lagrangian is a function of coordinates and velocities.

- Euler-Lagrange equation.

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\vec{r}}} = \frac{\partial L}{\partial \vec{r}}$$

- Examples:

– 1D motion in a potential $U(x)$

$$L(\dot{x}, x) = \frac{m\dot{x}^2}{2} - U(x).$$

Euler-Lagrange equation – left hand side:

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x}, \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{d}{dt}m\dot{x} = m\ddot{x}$$

Euler-Lagrange equation – right hand side:

$$\frac{\partial L}{\partial x} = -\frac{\partial U}{\partial x}$$

Euler-Lagrange equation:

$$m\ddot{x} = -\frac{\partial U}{\partial x}$$

- One can use any set of numbers as coordinates.
- Examples.
  – Oscillator.
  – Pendulum. We start by defining the coordinates. Our coordinate is the angle ϕ. We need to express the kinetic and the potential energy through ϕ and $\dot{\phi}$. First we find the kinetic energy $K = \frac{m\vec{v}^2}{2}$.

$$x = l\sin\phi, \qquad y = l\cos\phi.$$

So the velocities

$$v_x = \frac{dx}{dt} \equiv \dot{x} = l\dot{\phi}\cos\phi, \qquad v_y = \frac{dy}{dt} \equiv \dot{y} = -l\dot{\phi}\sin\phi$$

so the kinetic energy

$$K = \frac{m}{2}\vec{v}^2 = \frac{m}{2}\left(v_x^2 + v_y^2\right) = \frac{ml^2}{2}\dot{\phi}^2.$$

The potential energy

$$U = mgl(1 - \cos\phi)$$

and so on.
  – Pendulum with a spring.
  – Pendulum with a spring on a wedge.
  – Double pendulum. etc.

LECTURE 9
Oscillations. Dissipation. Resonance. Response.

- How crazy the Hamilton principle is!!

9.1. Euler formula

$$e^{i\phi} = \cos(\phi) + i\sin(\phi).$$

which also mean

$$\cos(\phi) = \frac{e^{i\phi} + e^{-i\phi}}{2}, \qquad \sin(\phi) = \frac{e^{i\phi} - e^{-i\phi}}{2i}$$

and

$$e^{i\pi} = -1.$$

It also means that

$$\Re e^{i\phi} = \cos(\phi), \qquad \Im e^{i\phi} = \sin(\phi), \qquad \left|e^{i\phi}\right|^2 = e^{i\phi}e^{-i\phi} = e^{i(\phi-\phi)} = 1$$

Any complex number $Z = A + iB$

$$\Re Z = A, \qquad \Im Z = B$$

can be written as

$$Z = |Z|e^{i\phi}, \qquad |Z| = \sqrt{(\Re Z)^2 + (\Im Z)^2}, \quad \sin\phi = \frac{\Im Z}{|Z|}$$

9.2. Why harmonic oscillations are so important.

Consider any system. It will have dissipation! and noise!

- Dissipation $\longrightarrow$ decreases mechanical energy of the oscillator.
- Noise:
  - Any system with dissipation will have noise.
  - Kicks a system out of unstable equilibriums. Or it does not allow the system to freeze in a wrong extremum.
- No matter how small the dissipation and noise are they together ensure that the system finds the minimum of the potential energy.
- We observe systems mainly when they are at or very close to equilibrium. It means that the systems that we observe are close to the minimum of their potential (or more precisely free) energy.

- Close to a minimum every function can be described as $\frac{kx^2}{2}$ – a harmonic oscillator.
- So in the majority of observations we observe responses of (coupled) harmonic oscillators.

9.3. Oscillators without dissipation.

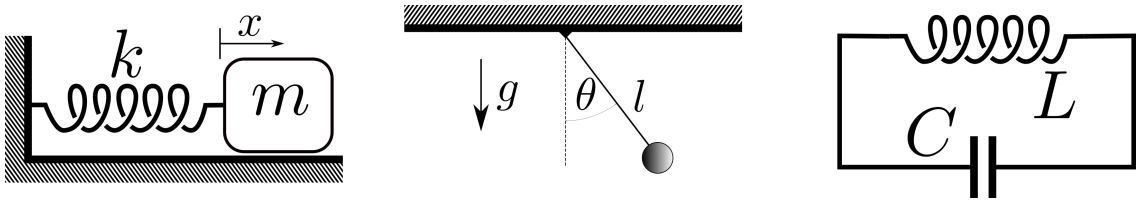


Figure 1. Different oscillators.

- Lagrangian

$$L = \frac{m}{2}\dot{x}^2 - \frac{k}{2}x^2$$

- Oscillator, pendulum, electric resonator (LC-circuit)

$$m\ddot{x} = -kx, \qquad ml\ddot{\theta} = -mg\sin\theta \approx -mg\theta, \qquad -L\ddot{Q} = \frac{Q}{C},$$

All of these equation have the same form

$$\ddot{x} = -\omega_0^2 x, \qquad \omega_0^2 = \begin{cases} k/m \\ g/l \\ 1/LC \end{cases}, \qquad x(t=0) = x_0, \quad v(t=0) = v_0.$$

- Notice the minus sign! This is a very important minus sign!!! It quarantines that the oscillator returns back – oscillates, instead of running away.
- Notice the simple structure of the equation — the second derivative of a function is linearly proportional to the function itself.
- Notice, that the frequency of oscillations does not depend on energy! It means that the frequency does not depend on initial conditions.
- Notice, that you can read the frequency of the oscillations directly from the equation — the frequency of oscillation is simply a square root of the proportionality coefficient (without the minus sign).
- The solution

$$x(t) = A\sin(\omega_0 t) + B\cos(\omega_0 t) = |C|\cos(\omega_0 t - \phi), \qquad B = x_0, \qquad \omega_0 A = v_0.$$

- $A = |C|\sin(\phi)$, $B = |C|\cos(\phi)$, or $|C| = \sqrt{A^2 + B^2}$ — amplitude; $\phi = \tan^{-1}(A/B)$ — phase.
- Second order differential equation — the general solution depends of two arbitrary constants: A and B, or $|C|$ and ϕ.
- Oscillates forever. Frequency is ω_0. The frequency can be read straight from the equation. The frequency does not depend on initial conditions. The frequency is the property of the h

- The solution can be written as

$$x(t) = \Re C e^{i\omega_0 t}, \qquad C = |C|e^{-i\phi},$$

where C is an arbitrary complex number. In this case the arbitrary constants are $\Re C$ and $\Im C$.

- One can see this by simply plugging the solution $x(t) = \Re C e^{i\omega_0 t}$ into the equation. $\ddot{x}(t) = -\Re\omega_0^2 C e^{i\omega_0 t}$,

$$\ddot{x} + \omega_0^2 x = -\Re\omega_0^2 C e^{i\omega_0 t} + \omega_0^2 \Re C e^{i\omega_0 t} = \Re\left(-\omega_0^2 C e^{i\omega_0 t} + \omega_0^2 C e^{i\omega_0 t}\right) = 0.$$

Notice, that because of the linearity of the equation and because all coefficients in equation are real the operation of taking the real part is trivial.

One can use Euler formula to see the connection between different forms of the solution

$$x(t) = \Re|C|e^{i(\omega_0 t - \phi)} = |C|\Re e^{i(\omega_0 t - \phi)} = |C|\cos(\omega_0 t - \phi).$$

Or we can do it differently

$$\begin{aligned} x(t) &= \Re|C|e^{-i\phi}e^{i\omega_0 t} = |C|\Re e^{-i\phi}e^{i\omega_0 t} = |C|\Re\left(\cos(\phi) - i\sin(\phi)\right)\left(\cos(\omega_0 t) + i\sin(\omega_0 t)\right) \\ &= |C|\Re\left(\cos(\phi)\cos(\omega_0 t) + \sin(\phi)\sin(\omega_0 t) + i(\cos(\phi)\sin(\omega_0 t) - \sin(\phi)\cos(\omega_0 t))\right) \\ &= \underbrace{[|C|\cos(\phi)]}_{B}\cos(\omega_0 t) + \underbrace{[|C|\sin(\phi)]}_{A}\sin(\omega_0 t) \end{aligned}$$

9.4. Oscillations with dissipation (friction).

Any real physical system will have dissipation. In this lecture we discuss how the dissipation effects the oscillations.

- Dissipation cannot be easily included into Hamiltonian or Lagrangian formalism. We need to revert back to the Newtonian formulation and simply add a new force, which provides the dissipation.
- Oscillations with friction:

$$m\ddot{x} = -kx - 2\alpha\dot{x}, \qquad -L\ddot{Q} = \frac{1}{C}Q + R\dot{Q},$$

  - Dissipative term is simply phenomenological at this stage.
  - The physics origin of such terms is typically a complicated problem.
  - Dissipative terms are time reversal NON-invariant.
- The sign of α.
  - The mechanical energy of an oscillator is $E = \frac{mv^2}{2} + \frac{kx^2}{2}$.
  - Let's compute, how it changes with time

$$\frac{dE}{dt} = mv\dot{v} + kxv = -vkx - 2\alpha v^2 + kxv = -2\alpha v^2.$$

  - Under the dissipation the mechanical energy must decrease at all times. Notice, that this requires, that

$$\alpha > 0.$$

  - The case $\alpha < 0$ would correspond to pumping of energy into the system.

- Consider

$$\ddot{x} = -\omega_0^2 x - 2\gamma \dot{x}, \qquad x(t=0) = x_0, \quad v(t=0) = v_0.$$

This is a linear equation with constant real coefficients. We look for the solution in the form $x = \Re C e^{i\omega t}$, where ω and C are complex constants.

$$\omega^2 - 2i\gamma\omega - \omega_0^2 = 0, \qquad \omega = i\gamma \pm \sqrt{\omega_0^2 - \gamma^2}$$

- Two solutions, two independent constants.
- Two cases: $\gamma < \omega_0$ and $\gamma > \omega_0$.
- In the first case (underdamping):

$$x = e^{-\gamma t}\Re\left[C_1 e^{i\Omega t} + C_2 e^{-i\Omega t}\right] = |C| e^{-\gamma t}\sin\left(\Omega t + \phi\right), \qquad \Omega = \sqrt{\omega_0^2 - \gamma^2}$$

Decaying oscillations ($\gamma > 0$). Shifted frequency.

– The constants $|C|$ and ϕ must be found from the initial conditions.

- In the second case (overdamping): ω is imaginary for both cases.

$$x = A_- e^{-\Gamma_- t} + A_+ e^{-\Gamma_+ t}, \qquad \Gamma_\pm = \gamma \pm \sqrt{\gamma^2 - \omega_0^2}, \qquad \Gamma_+ > \Gamma_- > 0$$

- For the initial conditions

$$x(t=0) = x_0, \qquad v(t=0) = 0$$

we find

$$A_- = x_0 \frac{\Gamma_+}{\Gamma_+ - \Gamma_-}, \qquad A_+ = -x_0 \frac{\Gamma_-}{\Gamma_+ - \Gamma_-}.$$

- For $t \to \infty$ the A_+ term can be dropped as $\Gamma_+ > \Gamma_-$, then

$$x(t \to \infty) \approx x_0 \frac{\Gamma_+}{\Gamma_+ - \Gamma_-} e^{-\Gamma_- t}$$

.

- At $\gamma \to \infty$,

$$\Gamma_+ \approx 2\gamma \to \infty, \qquad \Gamma_- \approx \frac{\omega_0^2}{2\gamma} \to 0$$

then

$$x(t) \approx x_0 e^{-\Gamma_- t}$$

As Γ_- is very small, the dynamics is very slow. The motion is arrested. The example is an oscillator in honey.

9.5. Comments on dissipation.

- Time reversibility. A need for a large subsystem.
- Locality in time.

9.6. Resonance

- Let's add an external force:

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = f(t), \qquad x(t=0) = x_0, \quad v(t=0) = v_0.$$

- The full solution is the sum of the solution of the homogeneous equation with any solution of the inhomogeneous one. This full solution will depend on two arbitrary constants. These constants are determined by the initial conditions.
- Let's assume, that $f(t)$ is not decaying with time. The solution of the inhomogeneous equation also will not decay in time, while any solution of the homogeneous equation will decay. So in a long time $t \gg 1/\gamma$ The solution of the homogeneous equation can be neglected. In particular this means that the asymptotic of the solution does not depend on the initial conditions.
- Let's now assume that the force $f(t)$ is periodic with some period. It then can be represented by a Fourier series $f(t) = \sum_i f_i \sin(\Omega_f^i + \phi_i)$. As the equation is linear the solution will also be a series, where each term corresponds to a force with a single frequency. So we need to solve

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = f\sin(\Omega_f t),$$

  where f is the force's amplitude.
- From the equation (linear) it is obvious, the amplitude of $x(t)$ will be proportional to the force amplitude f.
- Let's look at the solution in the form $x = f\Im C e^{i\Omega_f t}$, and use $\sin(\Omega_f t) = \Im e^{i\Omega_f t}$. We then get

$$C = \frac{1}{\omega_0^2 - \Omega_f^2 + 2i\gamma\Omega_f} = |C|e^{-i\phi},$$

$$|C| = \frac{1}{\left[(\Omega_f^2 - \omega_0^2)^2 + 4\gamma^2\Omega_f^2\right]^{1/2}}, \quad \tan\phi = \frac{2\gamma\Omega_f}{\omega_0^2 - \Omega_f^2}$$

$$x(t) = f\Im|C|e^{i\Omega_f t - i\phi} = f|C|\sin\left(\Omega_f t - \phi\right),$$

- Resonance frequency – the frequency at which the amplitude $|C|$ is the largest:

$$\Omega_f^r = \sqrt{\omega_0^2 - 2\gamma^2} = \sqrt{\Omega^2 - \gamma^2},$$

  where $\Omega = \sqrt{\omega_0^2 - \gamma^2}$ is the frequency of the damped oscillator.
- Phase changes sign at $\Omega_f^\phi = \omega_0 > \Omega_f^r$. Importance of the phase – phase shift.
- To analyze resonant response we analyze $|C|^2$.
- The most interesting case $\gamma \ll \omega_0$, then the response $|C|^2$ has a very sharp peak at $\Omega_f \approx \omega_0$:

$$|C|^2 = \frac{1}{(\Omega_f^2 - \omega_0^2)^2 + 4\gamma^2\Omega_f^2} \approx \frac{1}{4\omega_0^2}\frac{1}{(\Omega_f - \omega_0)^2 + \gamma^2},$$

  so that the peak is very symmetric.
- $|C|^2_{\max} \approx \frac{1}{4\gamma^2\omega_0^2}$.
- to find HWHM we need to solve $(\Omega_f - \omega_0)^2 + \gamma^2 = 2\gamma^2$, so HWHM $= \gamma$, and FWHM $= 2\gamma$.

- Q factor (quality factor). The good measure of the quality of an oscillator is $Q = \omega_0/\text{FWHM} = \omega_0/2\gamma$. (decay time) $= 1/\gamma$, period $= 2\pi/\omega_0$, so $Q = \pi\dfrac{\text{decay time}}{\text{period}}$.
- For a grandfather's wall clock $Q \approx 100$, for the quartz watch $Q \sim 10^4$, for atomic clocks $Q \sim 10^{11} - 10^{16}$.

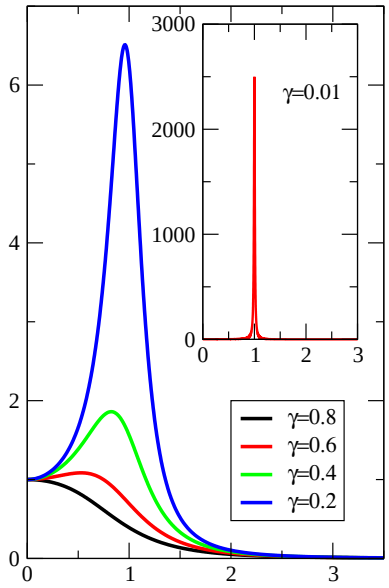


Figure 2. Resonant response. For insert $Q = 50$.

- In a system with the dissipation the resonance in velocity $\dot{x}$ happens at different frequency ω_0, than the resonance in coordinates $\sqrt{\omega_0^2 - 2\gamma^2}$.

9.7. Response.

- Response. The main quantity of interest. What is "property"?
  - We have an object in equilibrium.
  - We do something to it — shine light, apply a force, etc.
  - We observe how the object responses to our action.
  - This response IS what we call a property of the object: color, mass, stiffness, etc.
- The equation
$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = f(t).$$
The LHS is time translation invariant!
- Multiply by $e^{i\omega t}$ and integrate over time using integration by parts
$$\left(-\omega^2 - 2i\gamma\omega + \omega_0^2\right) x_\omega = \int_{-\infty}^{\infty} f(t) e^{i\omega t} dt, \quad \text{where} \quad x_\omega = \int_{-\infty}^{\infty} x(t) e^{i\omega t} dt$$
which gives
$$x_\omega = -\frac{\int_{-\infty}^{\infty} f(t') e^{i\omega t'} dt'}{\omega^2 + 2i\gamma\omega - \omega_0^2}$$

- The inverse Fourier transform gives

$$x(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} x_\omega = -\int_{-\infty}^{\infty} f(t')dt' \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{-i\omega(t-t')}}{\omega^2 + 2i\gamma\omega - \omega_0^2} = \int_{-\infty}^{\infty} \chi(t-t')f(t')dt'.$$

- Where the response function is ($\gamma < \omega_0$)

$$\chi(t) = -\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{-i\omega t}}{\omega^2 + 2i\gamma\omega - \omega_0^2} = \begin{cases} e^{-\gamma t} \frac{\sin(t\sqrt{\omega_0^2 - \gamma^2})}{\sqrt{\omega_0^2 - \gamma^2}} & , \quad t > 0 \\ 0 & , \quad t < 0 \end{cases} , \qquad \omega_\pm = -i\gamma \pm \sqrt{\omega_0^2 - \gamma^2}$$

- Causality principle. Poles in the lower half of the complex ω plane. True for any (linear) response function. The importance of $\gamma > 0$ condition.

9.8. General procedure.

General procedure: If we know the potential energy function $U(\vec{r})$

- First, find the position of the minimums.
- Find which minimum has the lowest energy.
- Use Taylor expansion of the Potential energy function around the minimum to the second order.
- Use it as k to find the oscillation/resonance frequency.

LECTURE 10
Spontaneous symmetry braking.

- Homework.

10.1. Spontaneous symmetry braking.

The mystery of a broken symmetry.

- The fundamental laws are translationally invariant, but the world around us is not.
- A squeezed rod buckles in some direction. Which direction will that be?
- A magnet below transition picks a particular direction of the magnetization.
- And so on.

The symmetry of a solution does not have to have the full symmetry of the equation.

This phenomenon is called spontaneous symmetry breaking. This idea is used very widely in physics. It is central for the Landau theory of the second order phase transitions. Such diverse phenomena as Higgs boson, magnetization in magnets, superfluidity, superconductivity, etc are all in the realm of this theory.

The phenomena mentioned above are quantum and as such requires a different machinery, but, remarkably there is a simple mechanical problem that illustrates one of the most important aspects of all of them.

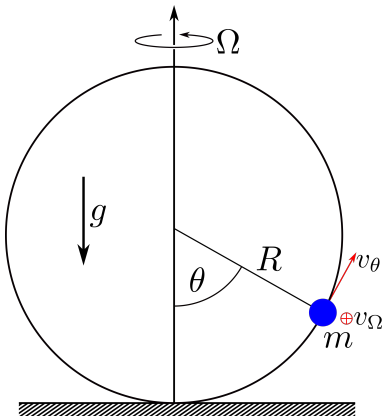


Figure 1. Bead on a vertical rotating hoop.

10.2. Example.

A bead on a vertical rotating hoop.

We have a hoop of radius R rotating with a constant and fixed(!) angular velocity Ω around a diameter in the vertical direction, see figure. There is a bead of mass m which can freely — without friction — move along the hoop. There is gravity acting on the bead. We want to write the equations of motion for the bead, analyze them, and see if we can learn something interesting.

"Something interesting" means that we want to learn some universal aspects. The aspects which do not depend on the details of the problem and can be used in developing intuition about more general and more complicated physical effects.

- The hoop is rotating with the constant/fixed angular velocity Ω, so its motion is known and no equation required for it (Notice, that this would be different should the hoop rotate freely, then its motion would be influenced by the motion of the bead and we would have to write the equations of motion for both the hoop and the bead.)
- Ω is a parameter of the problem. We have full control over it.
- The position of the bead at any moment of time is then fully described by just <u>one</u> generalized coordinate — the angle θ, see Fig. 1.

We start by writing the Lagrangian. The Lagrangian is a function of <u>two</u> variables: the generalized coordinate θ and the generalized velocity $\dot{\theta}$.

10.2.1. Effective potential energy.

- Potential energy:
$$U(\theta) = mgR(1 - \cos\theta).$$
- Kinetic energy:
  - The kinetic energy is $\frac{m\vec{v}^2}{2}$. The velocity of the bead has two components: $v_\theta = r\dot{\theta}$, and $v_\Omega = \Omega R \sin\theta$.
  - These two components are perpendicular to each other, see Fig. 1.
  - So $\vec{v}^2 = v_\theta^2 + v_\Omega^2$.
$$K = \frac{m}{2}R^2\dot{\theta}^2 + \frac{m}{2}\Omega^2 R^2 \sin^2\theta.$$
- The Lagrangian.
$$L = \frac{m}{2}R^2\dot{\theta}^2 + \frac{m}{2}\Omega^2 R^2 \sin^2\theta - mgR(1 - \cos\theta).$$
- The Lagrangian can be written as
$$L = \frac{m}{2}R^2\dot{\theta}^2 - U_{eff}(\theta),$$
where the "effective" potential energy is
$$U_{eff}(\theta) = -\frac{m}{2}\Omega^2 R^2 \sin^2\theta + mgR(1 - \cos\theta).$$
- Notice, that the effective potential energy has a symmetry: $\theta \to -\theta$. If one changes the sign of θ the potential energy does not change!

- The equation of motion.

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \frac{\partial L}{\partial \theta}.$$

or

$$R\ddot{\theta} = -\frac{1}{mR}\frac{\partial U_{eff}(\theta)}{\partial \theta} = (\Omega^2 R\cos\theta - g)\sin\theta.$$

10.2.2. Equilibrium points.

- There are four equilibrium points: $\frac{\partial U_{eff}}{\partial \theta} = 0$

$$\sin\theta = 0, \qquad \text{or} \qquad \cos\theta = \frac{g}{\Omega^2 R}$$

- Critical Ω_c. The second two equilibriums are possible only if

$$\frac{g}{\Omega^2 R} < 1, \qquad \Omega > \Omega_c = \sqrt{g/R}.$$

- Notice, that the equilibriums $\theta = 0$ and $\theta = \pi$ are symmetric under $\theta \to -\theta$ — under this transformation each these points turn into themselves. However, the other two equilibriums are NOT symmetric, under this transformation they turn into each other.
- Effective potential energy for $\Omega \sim \Omega_c$. Assuming $\Omega \sim \Omega_c$ we are interested only in small θ. So

$$U_{eff}(\theta) \approx \frac{1}{2}mR^2(\Omega_c^2 - \Omega^2)\theta^2 + \frac{3}{4!}mR^2\Omega_c^2\theta^4$$

$$U_{eff}(\theta) \approx mR^2\Omega_c(\Omega_c - \Omega)\theta^2 + \frac{3}{4!}mR^2\Omega_c^2\theta^4$$

- Spontaneous symmetry breaking. Plot the function $U_{eff}(\theta)$ for $\Omega < \Omega_c$, $\Omega = \Omega_c$, and $\Omega > \Omega_c$, see Fig. 2.
- Discuss universality.

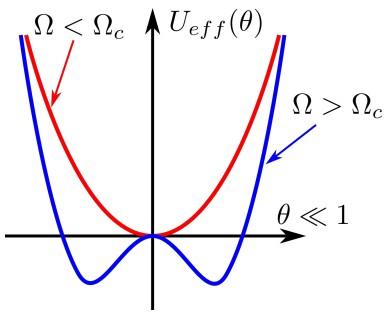


Figure 2. Effective potential for $|\theta| \ll 1$ and $\Omega \approx \Omega_c$.

10.2.3. Small oscillations about stable equilibria.

- Small oscillations around $\theta = 0$, $\Omega < \Omega_c$

$$mR^2\ddot{\theta} = -mR^2(\Omega_c^2 - \Omega^2)\theta, \qquad \omega = \sqrt{\Omega_c^2 - \Omega^2} \approx \sqrt{2\Omega_c(\Omega_c - \Omega)}.$$

- Small oscillations around θ_0, $\Omega > \Omega_c$.

$$U_{eff}(\theta) = -\frac{m}{2}\Omega^2 R^2 \sin^2\theta + mgR(1-\cos\theta),$$

$$\frac{\partial U_{eff}}{\partial\theta} = -mR(\Omega^2 R\cos\theta - g)\sin\theta, \qquad \frac{\partial^2 U_{eff}}{\partial\theta^2} = mR^2\Omega^2\sin^2\theta - mR\cos\theta(\Omega^2 R\cos\theta - g)$$

$$\left.\frac{\partial U_{eff}}{\partial\theta}\right|_{\theta=\theta_0} = 0, \qquad \left.\frac{\partial^2 U_{eff}}{\partial\theta^2}\right|_{\theta=\theta_0} = mR^2(\Omega^2 - \Omega_c^4/\Omega^2) \approx 2mR^2(\Omega^2-\Omega_c^2) \approx 4mR^2\Omega_c(\Omega-\Omega_c)$$

So the Tylor expansion gives

$$U_{eff}(\theta\sim\theta_0) \approx \text{const} + \frac{1}{2}4\Omega_c mR^2(\Omega-\Omega_c)(\theta-\theta_0)^2$$

The frequency of small oscillations then is

$$\omega = 2\sqrt{\Omega_c(\Omega-\Omega_c)}.$$

So What we have is that at $\Omega < \Omega_c$ the frequency of oscillations is $\omega = \sqrt{2\Omega_c(\Omega-\Omega_c)}$, while at $\Omega < \Omega_c$ the frequency of oscillations is $\omega = 2\sqrt{\Omega_c(\Omega-\Omega_c)}$. We can measure the frequency of oscillation by, say, resonance effects (exactly how to do it depends on the system). We then will see, that if we slowly change Ω through Ω_c, then the resonance frequency will change through zero in a predictable way.

If we also measure the width of the resonance in the two regimes we will be able to say something about the large subsystem (remember: width of the resonance comes from dissipation, dissipation comes from coupling to a large subsystem) which our resonator is coupled to.

10.2.4. Response.

- The effective potential energy for small θ and $|\Omega-\Omega_c|$

$$U_{eff}(\theta) = \frac{1}{2}a(\Omega_c-\Omega)\theta^2 + \frac{1}{4}b\theta^4.$$

- θ_0 for the stable equilibrium is given by $\partial U_{eff}/\partial\theta = 0$

$$\theta_0 = \begin{cases} 0 & \text{for} \quad \Omega < \Omega_c \\ \sqrt{\frac{a}{b}(\Omega-\Omega_c)} & \text{for} \quad \Omega > \Omega_c \end{cases}$$

  Plot $\theta_0(\Omega)$. Non-analytic behavior at Ω_c.
- Response: how θ_0 responses to a small change in Ω.

$$\frac{\partial\theta_0}{\partial\Omega} = \begin{cases} 0 & \text{for} \quad \Omega < \Omega_c \\ \frac{1}{2}\sqrt{\frac{a}{b}}\frac{1}{\sqrt{\Omega-\Omega_c}} & \text{for} \quad \Omega > \Omega_c \end{cases}$$

  Plot $\frac{\partial\theta_0}{\partial\Omega}$ vs Ω. The response *diverges* at Ω_c.
- Around the transition the role of noise (the response diverges, so no noise is small) is very important and very nontrivial.

LECTURE 11
Oscillations with time dependent parameters.

11.1. Oscillations with time dependent parameters.

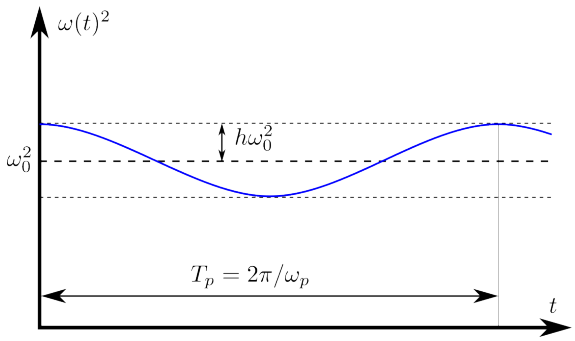
Let's consider the following problem

- The parameters of the oscillator (either k, or l for a pendulum, or C and L in circuit, etc) depend on time.
- There is no external force acting on the oscillator.
- In this case the equation of motion can generally be converted to

$$\ddot{x} = -\omega^2(t)x.$$

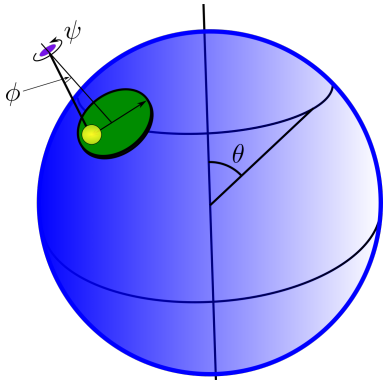
- The mechanical energy of the oscillator is not conserved in this case, as someone is doing work by changing the parameters.

It is most interesting when the dependence of parameters on time — the function $\omega^2(t)$ — is periodic, say with a period $T_p = 2\pi/\omega_p$. It is also most interesting, when the parameters do not change by much, so that we have almost intact oscillator with its own frequency close to ω_0.



- After the time T_p the whole system returns back where it was, but the state of the system does not have to be the same.

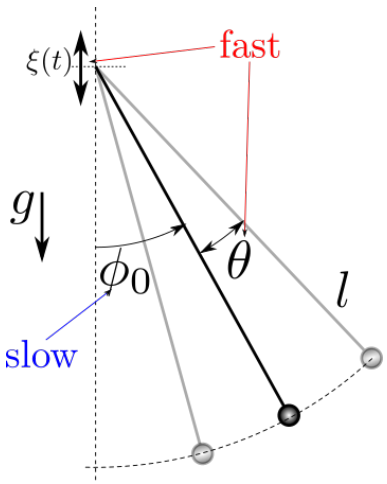
We will distinguish between three different cases: $\omega_p \ll \omega_0$, $\omega_p \gg \omega_0$, $\omega_p \sim \omega_0$. Below we consider an example of each case.



11.2. Slow change of parameters. Foucault pendulum.

- A pendulum is on Earth. Earth spins around its axis with frequency $\omega_p = 2\pi/24$hours. We, the observers, are on Earth, so we are NOT in the inertial frame of reference.
- The pendulum oscillates with it's own frequency ω_0.
- It is an example of slow change of the parameter — the direction of $\vec{g}$ as seen from the outside (from an inertial frame of reference).
- The pendulum is describe by TWO variables: ϕ — the usual pendulum angle and ψ — the orientation of the plane of oscillations (the plane the pendulum is in) as observed by us.
- Due to the Earth's rotation the angle ψ is changing with time.
- One can ask what will be the change of ψ after one rotation of the Earth.
- The result is $\Delta\psi = -2\pi\cos\theta$ — solid angle of the path.
- Earth does not have to rotate. It is enough to carry the pendulum along a closed path on the surface of Earth..
- The purely geometrical nature of such phase was first discovered in quantum mechanics: Berry phase 1984; classical: Hannay angle 1985.
  - Geometrical in nature.
  - Universal.

11.3. Fast change of parameters. Kapitza pendulum.



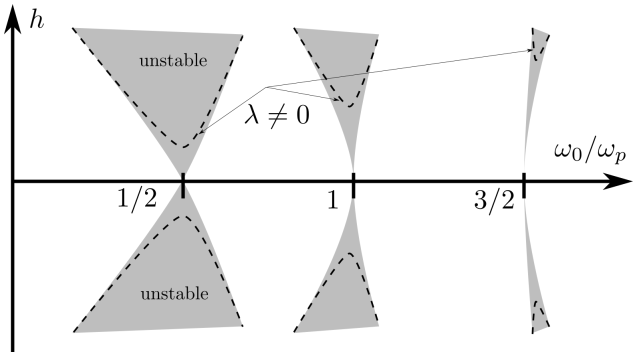
- $\omega_p \gg \omega_0$ — Kapitza pendulum. (demo) Criteria: $\overline{\left(\dot{\xi}\right)^2} \gg gl$.
- Importance of the time scales separation.
- Averaging out fast processes – a natural thing to do.
- Importance of non-linearity.
- Universal mechanism – averaging over fast degrees of freedom leads to the change of the dynamics of the slow degree of freedom through non-linearity.
- This is in essence the mechanism of all phase transitions in Statistical Mechanics and in the Quantum field theory.

11.4. Parametric resonance

- $\omega_p \sim \omega_0$ — parametric resonance ($\omega_p = \frac{2}{n}\omega_0$)

$$\ddot{x} = -\omega^2(t)x, \qquad \omega^2(t) = \omega_0^2(1 + h\cos(\omega_p t)), \quad h \ll 1$$

- Different from the usual resonance:
  - If the initial conditions $x(t=0) = 0$, $\dot{x}(t=0) = 0$, then $x(t) = 0$.
  - Frequency ω_p is a fraction of ω_0.
  - At finite dissipation one must have a finite amplitude h in order to get to the resonance regime.



- There is a direct analog of this (without dissipation) in Quantum Mechanics. It is called Bloch Theorem. This mechanism is the reason why we have metals, insulators and semiconductors. It is also the starting point for all our understanding of the electronic properties of solids.

LECTURE 12
Waves.

12.1. Waves in $1+1$ — one dimensional space and time.

- Waves. Ripples. Sound waves. Light waves. What are we describing? — water height, pressure, electric field, etc.
- More is different. Waves as collective excitations.
- Amplitude, phase, frequency, wave number, period, wavelength.
  - A typical sin wave:

$$A\sin\left(\omega t - kx - \phi_0\right)$$

    One can eliminate ϕ_0 simply by shifting zero of time or coordinate:

$$A\sin\left(\omega(t-\phi_0/\omega) - kx\right), \quad \text{or} \quad A\sin\left(\omega t - k(x+\phi_0/k)\right).$$

  - A is called amplitude.
  - Whatever is under sin is called phase.
  - ω is frequency, another frequency $f = \frac{\omega}{2\pi}$. Both frequencies have units s^{-1} — inverse seconds. However, typically this units for f are called Hz. Hz are never used for ω.
  - k is called a wave number. The units are m^{-1} — inverse meters.
- Consider one particular point of space, say x_0. An observer at this point sees a periodically changing whatever the wave is describing (electric field, magnetic field, pressure, water height, etc.) As sin is a periodic function, if time changes from t to $t + 2\pi/\omega$, then the phase changes by 2π. So the values of the sin at t and $t + 2\pi/\omega$ are the same. So

$$T = \frac{2\pi}{\omega} = \frac{1}{f}$$

  is a period (time period) of the wave.
- Consider an observation at a particular moment of time, say t_0 but in entire space. Measuring the wave at an arbitrary point x and at the point $x + 2\pi/k$ (for any x) we will get the same value. So $2\pi/k$ is the "space period" of the wave. It is called wave-length.

$$\lambda = 2\pi/k.$$

- Consider an observer who now sees the whole wave in space and time. The observer sees a wave going towards him/her. At time t the observer sees one crest of the wave at point x_0 and the next crest at the point $x_0 + \lambda$. At time $t + T$ he/she sees that the first crest have moved to the position of the second. So the observer sees that during time T the wave have moved the distance λ. So the speed of the wave is:

$$c = \frac{\lambda}{T} = \lambda f = \frac{2\pi f}{k} = \frac{\omega}{k},$$

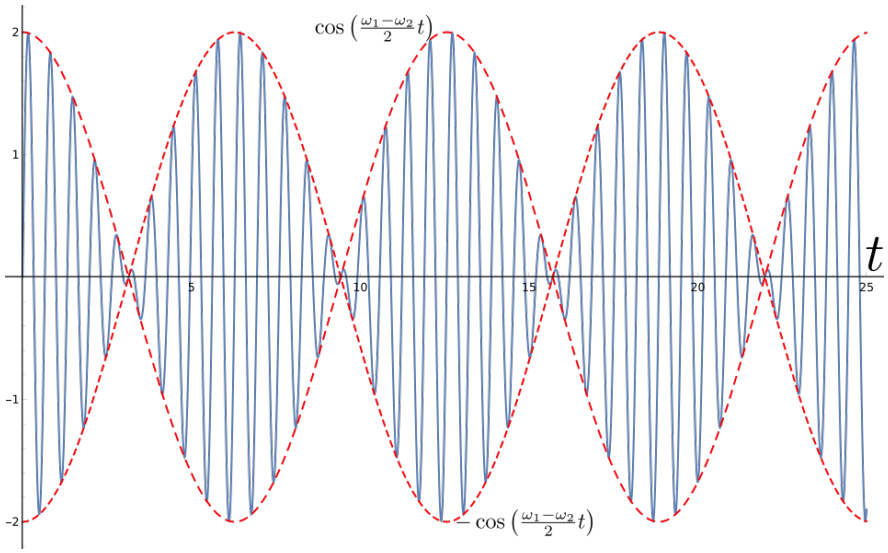
or

$$\omega = ck.$$

- Linearity. Superposition.
  - Acoustic beat `https://en.wikipedia.org/wiki/Beat_(acoustics)`

$$\sin(\omega_1 t) + \sin(\omega_2 t) = 2\cos\left(\frac{\omega_1 - \omega_2}{2}t\right)\sin\left(\frac{\omega_1 + \omega_2}{2}t\right)$$

    If ω_1 is close to ω_2 ($|\omega_1 - \omega_2| \ll \omega_1$), then this wave looks like a high frequency ($(\omega_1 + \omega_2)/2 \approx \omega_1$) sin with a small frequency ($(\omega_1 - \omega_2)/2$) envelope. The figure shows the case of $\omega_1 = 10$, $\omega_2 = 9$.



  - Interference. Let's take two waves of the same frequency and amplitude at the same point of space, but there is a phase shift between them.

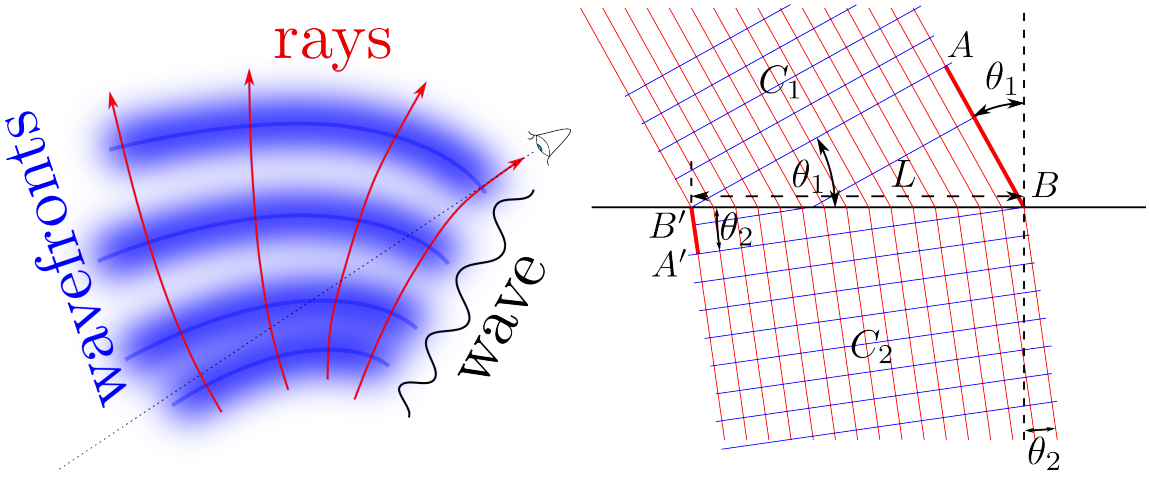
$$\sin(\omega t + \phi_1) + \sin(\omega t + \phi_2) = 2\cos\left(\frac{\phi_1 - \phi_2}{2}\right)\sin\left(\omega t + \frac{\phi_1 + \phi_2}{2}\right)$$

    Notice, in particular, that if the phase shift $\Delta\phi = \phi_1 - \phi_2 = \pi$, then the result is zero. But if $\Delta\phi = 0$, then the result is a wave with twice the amplitude.
    Phase itself is not measurable, it depends on when you started your watch. However, the phase shift between two waves is measurable. (Compare this with the non-uniqueness of potential energy).
    Another important point is this. We often measure not the wave itself, but the intensity. The intensity of light, for example. The intensity is proportional to the square of the wave. So when amplitude increases by a factor of 2 the intensity increases by a factor of 4!

You can watch the video here for a demo and some more explanations `https://www.stem.org.uk/elibrary/resource/26770`.

12.2. Waves in more than one spacial dimension.

- Wave fronts and rays.
- Snell's Law. It takes the same time for the wave front to travel from A to B in

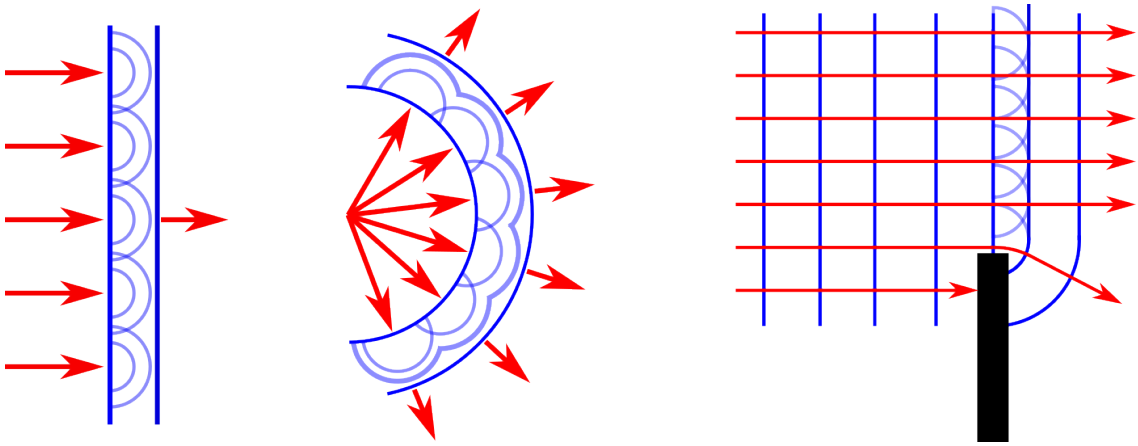


the media 1 with speed c_1 as for it to travel from B' to A' in the media 2 with the speed c_2, so $\frac{|AB|}{c_1} = \frac{|B'A'|}{c_2}$. If the distance $|B'B|$ is L, then $|AB| = L\sin\theta_1$ and $|B'A'| = L\sin\theta$. So we get

$$\frac{\sin\theta_1}{c_1} = \frac{\sin\theta_2}{c_2}.$$

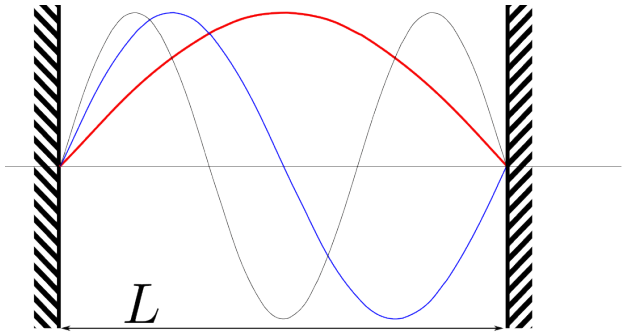
Notice, that the wave front picture allows us to understand the mechanism for the light to "chose" the path of quickest travel!

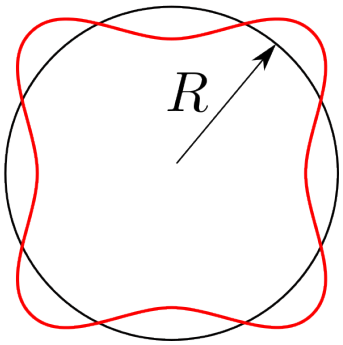
- Green's picture. Huygens's principle. `https://en.wikipedia.org/wiki/Huygens%E2%80%93Fresnel_principle`.
- Diffraction. `https://en.wikipedia.org/wiki/Diffraction`



12.3. Resonators. Doppler effect. Anderson localization.

- Resonator.
- Wave in a loop.

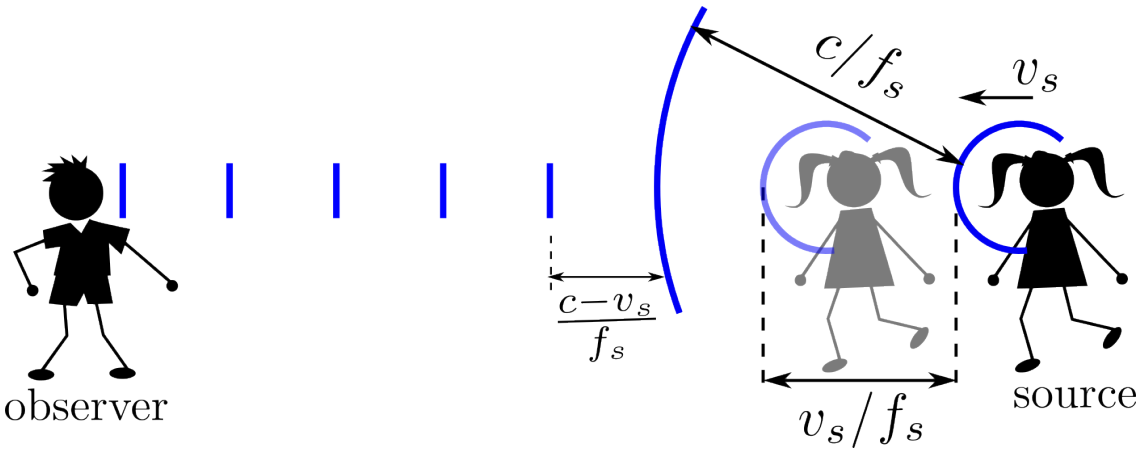




- Acoustic Doppler effect `https://en.wikipedia.org/wiki/Doppler_effect`. The source of frequency f_s is moving towards the stationary observer with velocity v_s. The observer hears the frequency f_o:

$$f_o = f_s \frac{c}{c - v_s}.$$

Discuss the role of the medium.



- Anderson localization.
- Difference between waves and classical particles (particle's stream).

LECTURE 13
Currents

- Homework.

13.1. Waves vs. particles.

Consider a wave and a stream of classical particles (like a water stream from a hose).

- Both carry energy.
- Both carry momentum.

However,

- Only waves have interference.
- Only waves have diffraction.

and

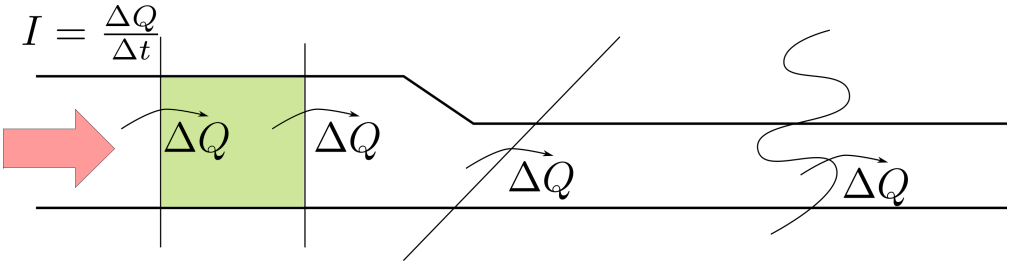
- Only particles can be described as having coordinates.
- Only for particles can we use the number (or density) of particles.

Waves have phase! Particles have particle number!

But also

- Waves can be a description of the collective excitation of particles.
- Can particles be collective excitation of waves? What does it even mean?

13.2. Electric current



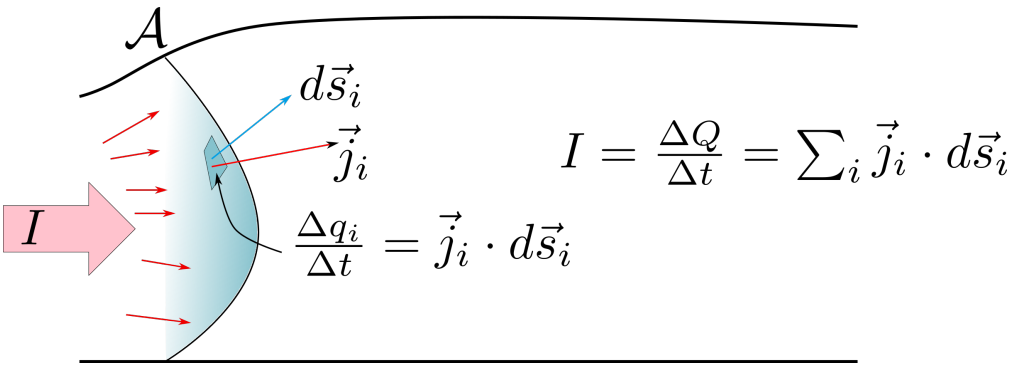
- Consider a wire carrying an electric current I.
- How do we define this current?
- We consider an arbitrary cross-section of the wire.
- We start the measurement at time t and run it for small time interval Δt until time $t + \Delta t$. We measure how much charge crosses this cross-section during this time

interval. We call this amount of charge ΔQ. Then we define the current at time t as

$$I(t) = \lim_{\Delta t \to 0} \frac{\Delta Q}{\Delta t}.$$

- In order for this definition to make sense the result must not depend on which cross-section we have used, as this cross-section is arbitrary.
- It is indeed so, as can be seen considering the volume of the wire between two cross-section (the greenish volume in the figure above). As the total charge in this volume cannot change (due to strong Coulomb interaction) the amount of charge ΔQ entering the greenish volume must be equal to amount of charge leaving this volume.
- Notice the importance of the Coulomb interaction in establishing $I(t)$. Notice also, that this argument (independence I of the cross-section) implies that the wire capacitance is zero. In particular it is incorrect if there is a capacitor in the wire. This nonzero wire capacitance also cannot be ignored for ultra-fast electronics.

13.3. Electric current density, general current density.



- The definition of the current above assumes a wire. This is not fully satisfactory, as it does not allow as to discuss currents in the bulk — We can only talk about the total current through a cross-section.
- Moreover, The definition of I requires a finite cross-section area, so this definition cannot be made local.
- Consider a motion of the charge in the large bulk. At a given moment of time, lets consider a point $\vec{r}$ of the space.
- There are some charges in this point — let's say they are electrons.
- We say that there is a density of electrons $\rho(\vec{r})$ at this point.
- Each electron has a charge e, so there is a density of charge $e\rho(\vec{r})$.
- These electrons are moving at the point $\vec{r}$ with average velocities $\vec{v}(\vec{r})$.
- Consider an infinitesimally small piece of flat surface of area ds at the same point $\vec{r}$. The orientation of this piece can be describe as a vector perpendicular to the surface. So we say that the area and the orientation are given by the vector $d\vec{s}$.
- Now we ask: How much charge crosses the area $d\vec{s}$ during the small time interval Δt?
- The charges are moving with the velocity $\vec{v}$, but only the component of the velocity along the vector $d\vec{s}$ (perpendicular to the piece of surface) matters. So all the electrons in the volume $\Delta t \vec{v} \cdot d\vec{s}$ cross the piece of surface.

- So the charge $\Delta q = e\rho\Delta t\vec{v}\cdot d\vec{s}$ crosses the piece of surface during the time interval Δt.
- Notice that $\frac{\Delta q}{\Delta t} = e\rho\vec{v}\cdot d\vec{s}$. This expression can be written as

$$\frac{\Delta q}{\Delta t} = \vec{j}\cdot d\vec{s}, \qquad \vec{j} = e\rho\vec{v}.$$

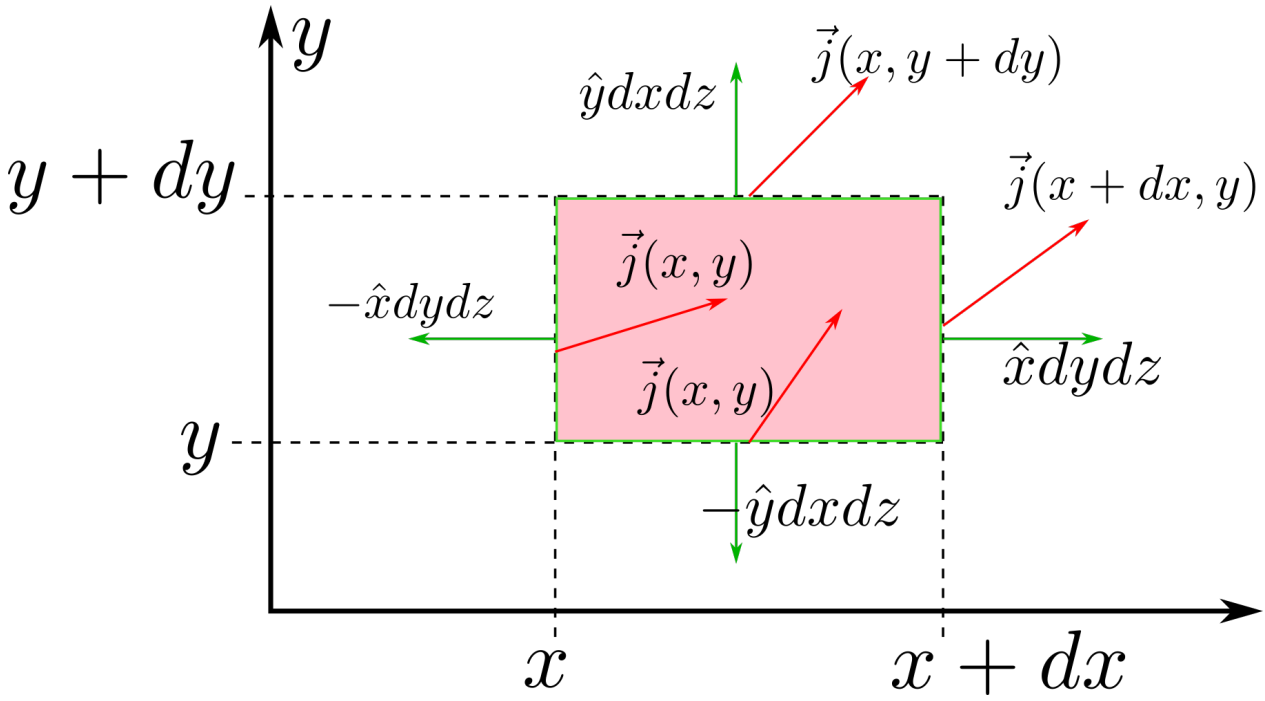
- The vector(!) $\vec{j}$ does not depend on which piece $d\vec{s}$ we considered (but the piece MUST BE at the same point) $\vec{j}(\vec{r},t)$ is called the "current density" at point $\vec{r}$ and time t.
- The current density in general depends on the coordinate and time $\vec{j}(\vec{r},t)$.
- This definition does not depend on any properties of the electric charge. So it can be used to define "mass current density", "energy current density", "cars current density", "people current density", etc. whatever moves in bulk, even "stars current density" in astronomy.
- Let's go back and see how the "charge current density" $\vec{j}$ is related to the current I. Consider a cross-section $\mathcal{A}$ of a wire. We split the cross-section surface $\mathcal{A}$ into infinitesimal "tiles" of area ds_i (i labels the tiles). These "tiles" have their orientations, so we say that each "tile" is defined by the vector $d\vec{s}_i$, see the figure above. The total charge ΔQ crossing the cross-section $\mathcal{A}$ during time Δt is sum of the charges $\Delta q_i = \vec{j}_i\cdot d\vec{s}_i\Delta t$, so

$$\Delta Q = \sum_i \Delta q_i = \Delta t\sum_i \vec{j}_i\cdot d\vec{s}_i$$

or

$$I = \frac{\Delta Q}{\Delta t} = \sum_i \vec{j}_i\cdot d\vec{s}_i = \int_{\mathcal{A}} \vec{j}\cdot d\vec{s}.$$

13.4. Charge, mass, etc. conservation laws.



What does it mean that the charge is conserved? Or mass is conserved?

- Looking at the picture: The change of the charge dQ inside the pink volume over time dt is given by

$$-\frac{dQ}{dt} = -\hat{x}dydz\cdot\vec{j}(x,y,z) + \hat{x}dydz\cdot\vec{j}(x+dx,y,z) - \hat{y}dxdz\cdot\vec{j}(x,y,z) + \hat{y}dxdz\cdot\vec{j}(x,y+dy,z)$$
$$-\hat{z}dxdy\cdot\vec{j}(x,y,z) + \hat{z}dxdy\cdot\vec{j}(x,y,z+dz)$$

  (the last line is for the third direction. Remember we are in 3D.)
- As for any vector $\vec{A}$ we have $\vec{A}\cdot\hat{x} = A_x$, $\vec{A}\cdot\hat{y} = A_y$, and $\vec{A}\cdot\hat{z} = A_z$, the above equation is

$$-\frac{dQ}{dt} = dydz\,(j_x(x+dx,y,z) - j_x(x,y,z)) + dxdz\,(j_y(x,y+dy,z) - j_y(x,y,z))$$
$$+dxdy\,(j_z(x,y,z+dz) - j_z(x,y,z))$$

- We rewrite it further as

$$-\frac{dQ}{dt} = dydzdx\Big(\frac{j_x(x+dx,y,z) - j_x(x,y,z)}{dx} + \frac{j_y(x,y+dy,z) - j_y(x,y,z)}{dy}$$
$$+\frac{j_z(x,y,z+dz) - j_z(x,y,z)}{dz}\Big)$$

- Which is in turn

$$-\frac{dQ}{dt} = dydzdx\Big(\frac{\partial j_x}{\partial x} + \frac{\partial j_y}{\partial y} + \frac{\partial j_z}{\partial z}\Big)$$

- As the charge Q inside is given by $\rho dxdydz$, where ρ is the charge density, we see, that $\frac{dQ}{dt} = \frac{d\rho}{dt}dxdydz$. So we have

$$-\dot{\rho} = \frac{\partial j_x}{\partial x} + \frac{\partial j_y}{\partial y} + \frac{\partial j_z}{\partial z}$$

- We then derived the mathematical expression for the local conservation of charge!

$$\dot{\rho} + \nabla\vec{j} = 0.$$

- There is another way to derive the same relation.
  - Consider an arbitrary and finite volume Ω with the boundary which we will call $\partial\Omega$.
  - The total charge inside this volume is $Q = \int_\Omega \rho dV$.
  - So $\dot{Q} = \int_\Omega \dot{\rho} dV$.
  - By definition of the current density $\dot{Q} = -\int_{\partial\Omega}\vec{j}\cdot d\vec{s}$.
  - So we have an equation

$$\int_\Omega \dot{\rho}dV + \int_{\partial\Omega}\vec{j}\cdot d\vec{s} = 0.$$

  - We use the Gauss theorem on the second term

$$\int_\Omega \left(\dot{\rho} + \nabla\vec{j}\right)dV = 0.$$

  - As this relation holds for ANY volume Ω the integrand must be zero!

$$\dot{\rho} + \nabla\vec{j} = 0.$$

- It is the statement of the conservation of charge, as the only physical statement we used in this derivation is the fact, that the charge in the pink volume can only change by exactly the amount that crossed the boundary. This means that the charge cannot appear or disappear, it can only continuously travel from one point to another. But this is exactly the meaning of charge conservation!
- We have not used any specific properties of the electric charge. So the conservation law for any other conserved quantity (energy, mass, even momentum) can be written in exactly the same way.
- This equation $\dot{\rho} + \nabla\vec{j} = 0$ is also called "continuity equation".

13.5. Electric circuits.

- Current. As we saw current is defined in a cross-section! In a circuit it means that it is defined using only ONE POINT of a circuit! This is why one must break a wire to insert the ammeter.
- Voltage. Potential DIFFERENCE. Voltage is defined using TWO points of a circuit — it is the DIFFERENCE. That is why a voltmeter is connected to TWO points in a circuit.
- A device is a relation between the current THROUGH the device and the voltage ACROSS the device.
- Inductance:
$$V = -L\frac{dI}{dt}.$$
- Capacitor. $V = Q/C$, differentiating over time
$$\frac{dV}{dt} = \frac{1}{C}I.$$
- Resistor. Ohm's law.
$$V = IR, \qquad \vec{j} = \sigma\vec{E}.$$
- Kirchhoff's laws/rules.
- Phasor diagrams.
$$V_L = i\omega L I_L, \qquad V_C = -i\frac{1}{\omega C}I_C, \qquad V_R = RI_R.$$

LECTURE 14
Gauss theorem. Lorenz force.

- Exam announcement.

There is an unfortunate confusion in standard physics textbooks. Many of them use "Gauss theorem" and "Gauss law" interchangeably. These are, however, two very different things. The confusion comes from the fact that in many cases they both are used together. I want to distinguish between them:

- Gauss theorem — purely mathematical theorem. It can be and it is used in many different settings. It has no physical content whatsoever.
- Gauss law — physical statement. It tells us how electric filed can be computed from a known distribution of electric charges.

We will start with Gauss theorem.

14.1. On boundaries.

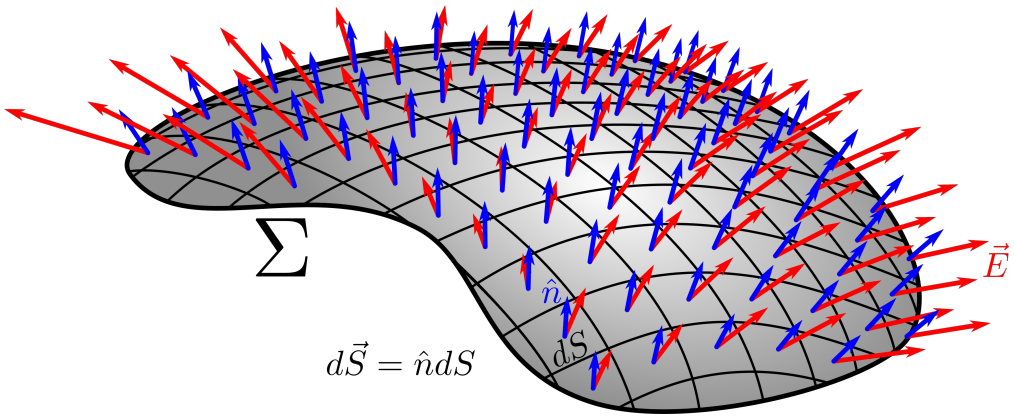
- A boundary of an interval is a set of two points – the interval's end points.
- A boundary of a surface is a line.
- A boundary of a piece of volume is a surface.
- Boundary of a surface or volume Ω is DENOTED as $\partial\Omega$.
- A boundary has no boundary $\partial\partial\Omega = 0$.

14.2. Gauss theorem.

14.2.1. Definition of flux of vector field through a surface.

Consider an arbitrary vector field $\vec{E}(\vec{r})$. Consider an arbitrary surface Σ. The flux of the vector field $\vec{E}$ through the surface Σ is **defined** by the following procedure

- We take the surface Σ and split it into infinitesimal "tiles".
- We enumerate these tiles by $i = 1, 2, \ldots$.
- The area of each tile is dS_i.
- For each tile we define the unit vector $\hat{n}_i$ perpendicular to the tile i. (Any two neighboring tiles have almost the same $\hat{n}$s)
- We then DEFINE the vector of area for each tile $d\vec{S}_i = \hat{n}_i dS_i$.
- At the position of the tile i the vector field $\vec{E}(\vec{r})$ has a (vector) value $\vec{E}_i$.
- The infinitesimal flux of the field $\vec{E}$ trough the tile i is DEFINED by $\vec{E}_i \cdot d\vec{S}_i$.



- The total flux of the vector field $\vec{E}$ through the surface Σ is **defined** as the sum of the infinitesimal fluxes through all the tiles $\Phi_{\vec{E},\Sigma} = \sum_i \vec{E}_i \cdot d\vec{S}_i$.

The above procedure is formally denoted as

$$\Phi_{\vec{E},\Sigma} = \int_\Sigma \vec{E} \cdot d\vec{S}.$$

- In order to use this definition one MUST know the vector field $\vec{E}$ AND the surface Σ.
- The flux then depends on both $\vec{E}$ AND Σ. If we change $\vec{E}$ the flux will change, if we change Σ the flux will change.
- This is exactly the same construction as the one we used to find the current I through the current density $\vec{j}$.
- There is no "correct" way to define which of the two directions perpendicular to the surface the vector $d\vec{S}$ is pointing to, but if we chose one of these direction the direction of the vector for the neighboring tiles is set. (It does not mean that the direction can be defined globally. Surfaces for which the direction can be globally defined are called orientable. These will be the only surfaces we consider.)
- Remember, that $\vec{E}$ and $d\vec{S}$ are at the same point.

14.2.2. Gauss theorem.

Consider a piece of volume Ω with the boundary $\partial\Omega$. The boundary $\partial\Omega$ is a surface. We can compute the flux of a vector field $\vec{E}$ through the surface $\partial\Omega$

$$\oint_{\partial\Omega} \vec{E} \cdot d\vec{S}.$$

- The surface $\partial\Omega$ is closed. We can now define that $d\vec{S}$ is pointing to outside.

The Gauss theorem states that for any (smooth) vector field $\vec{E}$ and for an ARBITRARY volume Ω:

$$\oint_{\partial\Omega} \vec{E} \cdot d\vec{S} = \int_\Omega \nabla \cdot \vec{E} dV.$$

- This is purely mathematical theorem. It has no physical content. We have not even discussed what the field $\vec{E}$ is. It can be any vector field.
- Let me also emphasize that the theorem works for an ARBITRARY volume Ω.

There is a formulation of the Gauss theorem in any dimensions.

14.2.3. An example of the Gauss theorem in $1D$.

An example of the Gauss theorem in $1D$ is the familiar formula

$$\int_a^b \frac{df(x)}{dx} dx = -f(a) + f(b)$$

On the left hand side we have $1D$ bulk integral of $1D$ "divergence" $\frac{df}{dx}$ of $1D$ "vector field" $f(x)$ over $1D$ "volume" — the interval $[a, b]$. On the right hand side we have the "boundary integral" the sum (with the correct signs) of the function value in the points a and b.

14.2.4. An example of the Gauss theorem in $2D$.

Consider a flat $2D$ space — a plane. There is a $2D$ vector field $\vec{B}(x, y)$. We draw an arbitrary shape Σ in it. The boundary of this shape is $\partial\Sigma$. The Gauss theorem states

$$\int_{\partial\Sigma} \vec{B} \cdot d\vec{s} = \int_\Sigma \left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} \right) dxdy.$$

The vector $d\vec{s}$ is perpendicular (pointing outside) to the boundary $\partial\Sigma$ at each boundary point.

One can bring this into a more familiar form: Let's define a vector field $\vec{A}(x, y)$ by the following rule

$$A_x(x, y) = -B_y(x, y), \qquad A_y(x, y) = B_x(x, y).$$

Also for any vector $d\vec{s}$ of the boundary we define a vector $d\vec{l}$ by

$$dl_x = -ds_y, \qquad dl_y = ds_x.$$

Geometrically the vector $d\vec{l}$ is ALONG (counterclockwise) the boundary $\partial\Sigma$ at each point. Then we see

$$\vec{B} \cdot d\vec{s} = B_x ds_x + B_y ds_y = \vec{A} \cdot d\vec{l}$$

Also

$$\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = \mathrm{curl}\vec{A}.$$

So we have

$$\oint_{\partial\Sigma} \vec{A} \cdot d\vec{l} = \int_\Sigma \mathrm{curl}\vec{A}\, dxdy.$$

This is a simple (planar) version of the Stokes theorem. The boundary of a $2D$ object Σ is a line/path $\partial\Sigma$, so in the left hand side of the above equation is circulation of the vector field $\vec{A}$ around the path $\partial\Sigma$.

14.2.5. Example of the use of the Gauss theorem

- Current as a flux of current density field.
  - By the definition of the current density and by **the charge conservation law** the total charge inside a volume Ω changes according to

    $$\dot{Q} = -\oint_{\partial\Omega} \vec{j} \cdot d\vec{S} = -\int_\Omega \nabla \cdot \vec{j} dV.$$

    The last equality is the Gauss theorem.
  - By the definition of the charge density $\rho(\vec{r})$ we have

    $$Q = \int_\Omega \rho dV, \qquad \text{and} \qquad \dot{Q} = \int_\Omega \dot{\rho} dV$$

- So we have
$$\int_\Omega \dot{\rho} dV = -\int_\Omega \nabla \cdot \vec{j} dV, \qquad \text{or} \qquad \int_\Omega \left(\dot{\rho} + \nabla \cdot \vec{j}\right) dV = 0.$$
- As this is correct for any Ω we must have
$$\dot{\rho} + \nabla \cdot \vec{j} = 0.$$
The continuity equation!

14.3. Current density.

Computing current density through local quantities.

- Current density. Charge density ρ, collective velocity $\vec{v}$:
$$\vec{j} = \rho \vec{v}.$$
If charges are electrons and the density of electrons is n, then $\rho = en$, where e is the charge of an electron.
- Continuity equation
$$\dot{\rho} + \nabla \cdot (\rho \vec{v}) = 0.$$

14.4. Lorenz force.

If we know position and velocities of all charges, we can find $\vec{j}$ etc. We just need to solve the Newton equations: $\vec{F} = m\vec{a}$. But what is $\vec{F}$? Force on a charge q.

- Lorenz force.
$$\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}.$$
- Problem with Lorenz force.
  - The Lorenz force depends on the velocity of the particle.
  - The velocity of the particle is different in different frames of reference.
  - But the force (and hence acceleration) must be the same in all frames of reference.
  - It means that when we go into another frame of reference the electric and magnetic fields must transform through each other.
  - It means that electric and magnetic fields should obey very similar equations. As these equations must be the same in all frames of references.

Examples:

- Cyclotron radius, cyclotron frequency.
- Force on a piece of wire.

Next question: Where $\vec{E}$ and $\vec{B}$ come from?

LECTURE 15
Maxwell Equations: Gauss electric and magnetic laws.

- Homework.

15.1. Gauss Law for electric field.

15.1.1. Gauss theorem.

Do not confuse Gauss law with the Gauss theorem. The Gauss theorem states that for any (smooth) vector field $\vec{E}$ and for arbitrary volume Ω:

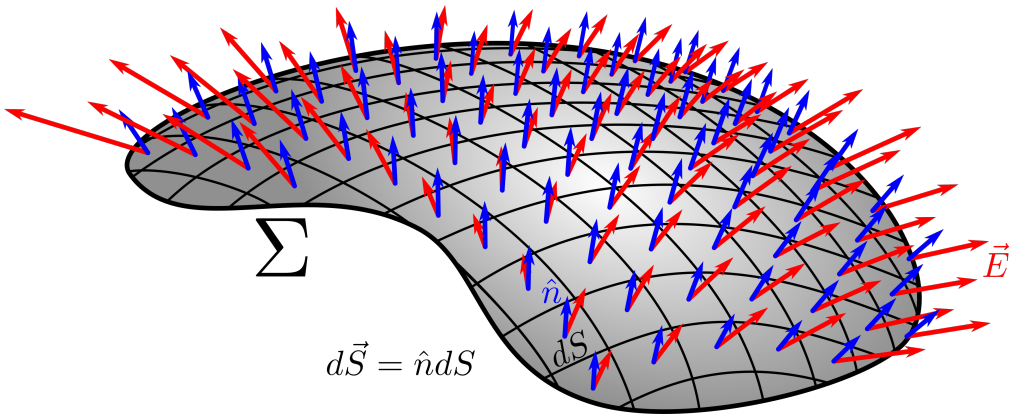
$$\oint_{\partial\Omega} \vec{E} \cdot d\vec{S} = \int_{\Omega} \nabla \cdot \vec{E} dV.$$

Gauss theorem is a very general theorem it has no physics content.

15.1.2. Gauss law.

Gauss law is physics law. It has very specific physics content.

We start by saying that we consider electric field $\vec{E}(\vec{r})$. Electric field is a vector field in space so we can use the general constructions discussed before for this particular case.

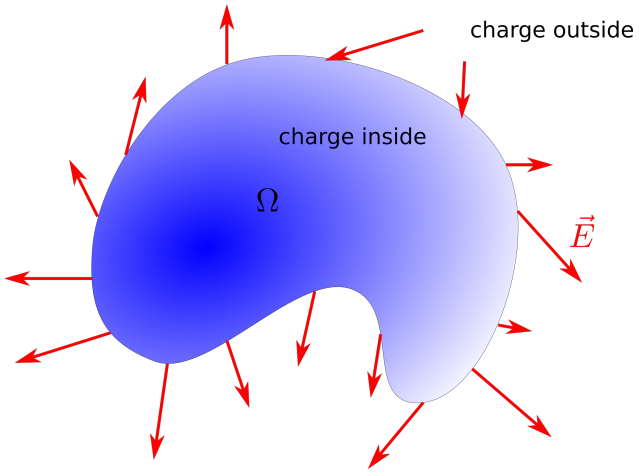


15.1.2.1. *Electric field flux.*

- We can define the flux of a Electric field $\vec{E}(\vec{r})$ through an ARBITRARY surface Σ.

$$\Phi_E = \int_{\Sigma} \vec{E} \cdot d\vec{S}.$$

- This is a general DEFINITION of the flux. In particular, if we have a flat piece of surface of area $\mathcal{A}$ and a uniform electric field E perpendicular to that piece of surface, then the flux is $E\mathcal{A}$.



15.1.2.2. *Gauss law.* Now we can state the Gauss law.

- We have some charge distribution with arbitrary charge density $\rho(\vec{r})$.
- This charge distribution produces some electric field $\vec{E}(\vec{r})$.
- For ANY/ARBITRARY volume of space Ω we have
- Gauss's Law:

$$\oint_{\partial\Omega} \vec{E}\cdot d\vec{S} = \frac{Q}{\epsilon_0} = \frac{1}{\epsilon_0}\int_{\Omega} \rho dV,$$

where Q is the total electric charge INSIDE Ω.

Notice, that this is very different from the statement of the Gauss theorem, as it gives the relation between two different physical quantities — the electric field and the electric charge.

- In words this law states:

For ANY piece of volume the flux of the electric field through the boundary of this volume (the left hand side) equals (with the factor $1/\epsilon_0$) to the total charge inside the volume (the right hand side).

- This law has a very clear physical content: the charges are the sources of the electric field flux.
- This law is just another form of the Coulomb law.
- In fact if we take the Coulomb law as the established physical law we can prove the Gauss law mathematically, and vice verse.

$$\text{Coulomb law} \Longleftrightarrow \text{Gauss law}.$$

- The Coulomb law is the law deduced from the experiment. So the Gauss law is also an experimental fact. It is a physics law.

15.1.2.3. *Examples of the use of the Gauss law.* The Gauss law relates the total charge inside a volume with the total electric field flux through the boundary of the volume. As the total flux is the sum of the (infinite number of) terms each containing the electric field at different points, it in general is useless in this form if one wants to find the electric field at each point

of space. Moreover, the Gauss law is scalar (it is the dot product), so it is definitely not enough to find a vector field.

However, if the charge distribution has high enough symmetry, then from this symmetry we can find out the direction of the field and restrict the possible coordinate dependence of the field strongly enough in order for the Gauss theorem to give the electric field. Again, it is the symmetry that provides the additional information needed to find the electric field from the Gauss law.

I want to emphasize the crucial role of the symmetry in all the examples below.

- Charged sphere.
- Charged infinite plane.
- Electric field of a charged infinite wire.

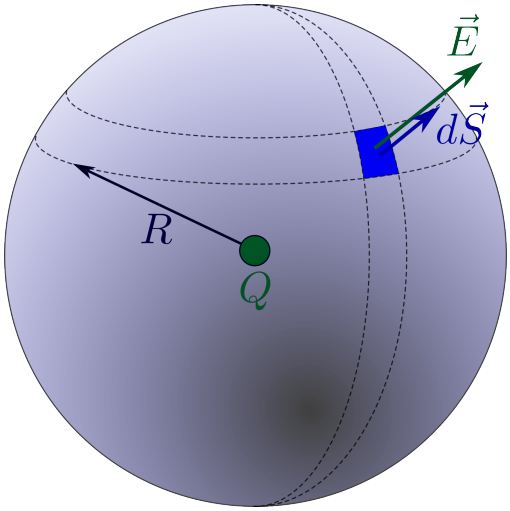
15.1.2.4. *Local form of the Gauss's Law.* The Gauss law is valid for ANY volume. We can use this fact in order to reformulate the Gauss law through only local quantities. For that we need to use the Gauss theorem.

- Using Gauss theorem we write
$$\oint_{\partial\Omega} \vec{E} \cdot d\vec{S} = \int_{\Omega} \nabla \cdot \vec{E} dV$$
- So the Gauss law can be written as
$$\int_{\Omega} \nabla \cdot \vec{E} dV = \frac{1}{\epsilon_0} \int_{\Omega} \rho dV, \qquad \text{or} \qquad \int_{\Omega} \left(\nabla \cdot \vec{E} - \frac{1}{\epsilon_0} \rho \right) dV = 0$$
- As it must be correct for ANY Ω we have the local form of the Gauss's Law
$$\nabla \cdot \vec{E} = \frac{\rho(\vec{r})}{\epsilon_0}.$$
This can be viewed as yet another form of the Coulomb law.
- But unlike the Coulomb law which is only valid for the point-like charges, this formulation is valid for any charge distribution. And it is local! It gives the divergence of electric field at the point $\vec{r}$ through the charge density at the very same point $\vec{r}$.
- This is still only one equation for $3D$ vector field — vector $\vec{E}$ has three components, each depends on the position vector $\vec{r}$. So we have three unknown functions and only one equation. In general case, we are two equations short.
- However, in the STATIC case (nothing is moving and nothing depends on time) electric field is a potential field! So it can be written as
$$\vec{E} = -\nabla \phi,$$
where ϕ is the electric potential.
- The local Gauss law then has a form
$$\Delta \phi = -\frac{\rho(\vec{r})}{\epsilon_0},$$
where Δ is a Laplace operator $\Delta \equiv \operatorname{div} \operatorname{grad} \equiv \nabla \cdot \nabla \equiv \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$.

  Now one linear differential equation with one unknown field ϕ! With appropriate boundary conditions it will have a unique solution!
- The solution of this equation is a mathematical problem. The physics is in the Gauss law itself and in the boundary conditions for the above equation. Examples: conductor, insulators.

15.1.2.5. *Coulomb law, Gauss law.*

- We know that according to the Coulomb law the electric field from the point like charge Q at the coordinate origin is $\vec{E} = \frac{kQ\vec{r}}{r^3}$, where $k = \frac{1}{4\pi\epsilon_0}$.
- However, let's assume that we do not know that. We did some measurement and found out that there is electric field everywhere and it is described by $\vec{E} = \frac{kQ\vec{r}}{r^3}$. We can use the Gauss law in order to find the distribution of charges that produce this observed electric field.
- In homework we checked, that for such field $\nabla \cdot \vec{E} = 0$ everywhere except $r = 0$. So according to the Gauss law the charge density is also zero everywhere except $r = 0$. The point $r = 0$ is singular and must be treated carefully.
- Let's consider a sphere of radius R. The flux through the boundary of this sphere is $\int \vec{E} \cdot d\vec{S} = 4\pi kQ = \frac{1}{\epsilon_0}Q$.



- So according to the integral form of the Gauss law there is the total charge Q inside this sphere of radius R.
- Moreover, we see, that the electric field flux does not depend on R!
- If we take $R \to 0$ the flux will stay the same. So we must conclude, that the whole charge Q is at the point which belongs to ALL these spheres. There is only one such a point and it is at the center of all spheres.
- So the electric field $\vec{E} = \frac{kQ\vec{r}}{r^3}$ is produced by a point-like charge Q at the coordinate origin.

15.2. Gauss Law for magnetic field.

- There are no magnetic charges.
- For the flux of the magnetic field $\vec{B}(\vec{r})$ through any closed surface $\partial\Omega$ is zero

$$\oint_{\partial\Omega} \vec{B} \cdot d\vec{S} = 0.$$

- It's local version (using Gauss theorem)

$$\nabla \cdot \vec{B} = 0.$$

LECTURE 16
Maxwell Equations: Faraday's and Ampere's Laws.

- Exam 1 solutions.
- Notations
  - For any (smooth) vector field $\vec{A}(\vec{r})$ we can define the following fields

$$\begin{aligned} \mathrm{curl}\vec{A} &\equiv \nabla \times \vec{A}, && \text{— a vector field.} \\ \mathrm{div}\vec{A} &\equiv \nabla \cdot \vec{A}, && \text{— a scalar field.} \end{aligned}$$

  - For a scalar field $U(\vec{r})$ we can define the following vector fields

$$\mathrm{grad}U \equiv \nabla U, \qquad \text{— a vector field.}$$

- We will also use a Laplace operator.

$$\Delta f \equiv \nabla \cdot \nabla f \equiv \nabla^2 f \equiv \mathrm{div}\,\mathrm{grad} f \equiv \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

  where f can be either scalar or vector field. In the later case the operator acts on the components of the field independently.
- The following simple relations can be proven by simply considering the components of the fields:

$$\begin{aligned} \nabla \times \nabla U &= 0, && \text{for any smooth function } U. \\ \nabla \cdot \nabla \times \vec{A} &= 0, && \text{for any smooth vector field } \vec{A}. \end{aligned}$$

16.1. What we know so far.

So far we know

- The Lorenz force tells us that the equations for electric $\vec{E}$ and magnetic $\vec{B}$ fields must be very similar.
- From experiment we know that charge is conserved.
- From experiment we know that there are no magnetic charges.
- From experiment we know that charged particles interact with Coulomb force.

The charge conservation gives us the continuity equation

$$\nabla \cdot \vec{j} + \dot{\rho} = 0$$

We rewrite the Coulomb force as Gauss law:

$$\text{Gauss's law:} \qquad \oint_{\partial\Omega} \vec{E}\cdot d\vec{S} = \tfrac{1}{\epsilon_0}\int_\Omega \rho dV, \qquad \nabla\cdot\vec{E} = \frac{\rho}{\epsilon_0}$$

$$\text{Gauss's law magnetic:} \qquad \oint_{\partial\Omega} \vec{B}\cdot d\vec{S} = 0, \qquad \nabla\cdot\vec{B} = 0$$

The last equation is the same equation for the magnetic field $\vec{B}$ as the first for the electric field $\vec{E}$ with the addition that there is no magnetic charges.

This is not enough:

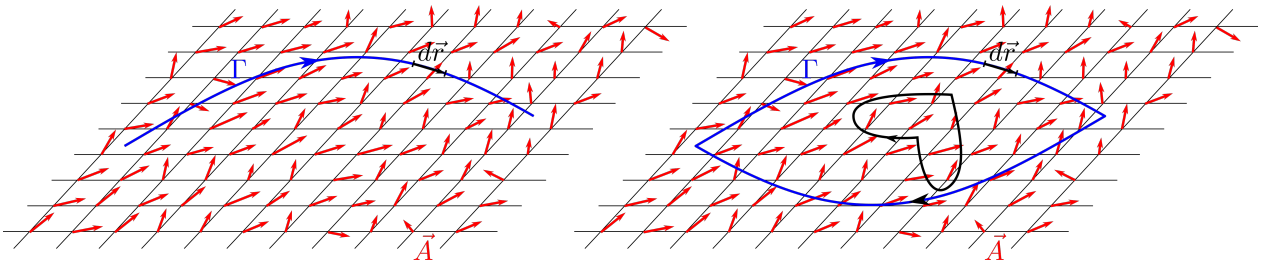
- First, we have only two scalar equation for 6 components of the fields 3 for electric field and 3 for magnetic.
- Second, there is no time derivatives — no dynamics of the fields.

We need more experimental observations to complete the set of equations for electric $\vec{E}$ and magnetic $\vec{B}$ fields. These EXPERIMENTAL observations have the form of two laws:

- Faraday's law;
- Ampere's law.

16.2. Some more math.

16.2.1. Circulation of a vector field.



Another construction for the vector fields

- Circulation of a vector field.
- For any vector field $\vec{A}(\vec{r})$ and any **oriented** path Γ we can compute

$$\int_\Gamma \vec{A}\cdot d\vec{r}.$$

  Notice, that the value of this integral depends on both: the vector field $\vec{A}(\vec{r})$ and on path Γ. We must know both in order to be able to compute this integral.
- If the path Γ is closed and **oriented**, then such an integral

$$\oint_\Gamma \vec{A}\cdot d\vec{r}.$$

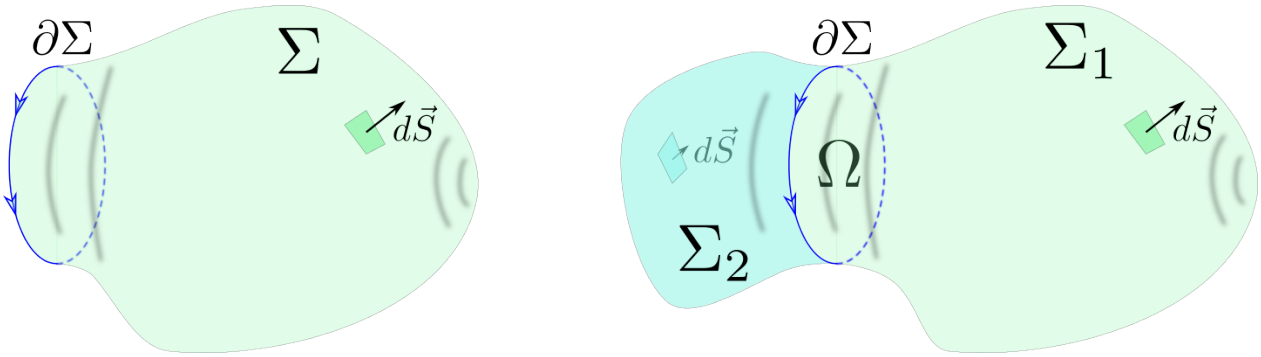
  is called circulation. The value of this integral sill depends on both: the vector field $\vec{A}(\vec{r})$ and on the closed path Γ.
- Notice, that if we change the orientation of the path Γ the integral changes sign.
- Example: work of a force vector field over a path Γ.

$$\mathcal{W} = \int_\Gamma \vec{F}\cdot d\vec{r}.$$

- Example of a circulation: work of a force vector field over a closed path Γ.

$$\mathcal{W} = \oint_\Gamma \vec{F} \cdot d\vec{r}.$$

16.2.2. Stokes' theorem



"Gauss" theorem for circulation. Stokes' theorem.

Let's take an arbitrary vector field $\vec{A}(\vec{r})$ and an arbitrary piece of surface Σ with the boundary $\partial\Sigma$.

- The boundary $\partial\Sigma$ of the surface Σ is a closed path.
- We **chose** the orientation of $\partial\Sigma$.
- We can define the circulation of $\vec{A}$ over $\partial\Sigma$

$$\oint_{\partial\Sigma} \vec{A} \cdot d\vec{r}.$$

- The orientation of $\partial\Sigma$ induces the orientation of Σ by right hand rule.
- The Stokes theorem states.

$$\oint_{\partial\Sigma} \vec{A} \cdot d\vec{r} = \int_\Sigma \nabla \times \vec{A} \cdot d\vec{S}.$$

  The circulation of a smooth vector field $\vec{A}(\vec{r})$ on the boundary $\partial\Sigma$ of the surface Σ equals to the flux of the vector field $\vec{\nabla} \times \vec{A}$ through the surface Σ.
- Notice, that if I change the orientation of $\partial\Sigma$, then the orientation of Σ also changes, both integrals change sign and the theorem still holds.

I will not prove this theorem. However, there is one point I want to emphasize: **The left hand side (LHS) depends on the boundary $\partial\Sigma$. The right hand side (RHS) depends on the surface Σ.** But one can draw many surfaces with the same boundary!

In order for this theorem to make sense the RHS must not depend on the surface Σ as long as this surface has the boundary $\partial\Sigma$.

- Independence of $\int_\Sigma \nabla \times \vec{A} \cdot d\vec{s}$ of Σ.
  - Consider Σ_1 and Σ_2 with the common boundary $\partial\Sigma$, see figure.
  - The orientation of both Σ_1 and Σ_2 is induced by the orientation of $\partial\Sigma$.
  - We can compute the fluxes of the vector field $\nabla \times A$ through both surfaces:

$$\Phi_{\Sigma_1} = \int_{\Sigma_1} \nabla \times \vec{A} \cdot d\vec{S}, \qquad \Phi_{\Sigma_2} = \int_{\Sigma_2} \nabla \times \vec{A} \cdot d\vec{S}$$

    The theorem states, that both fluxes MUST be equal to $\oint_{\partial\Sigma} \vec{A} \cdot d\vec{r}$, so the fluxes MUST be equal to each other!

– The flux of the vector field $\nabla \times \vec{A}$ through the closed surface $\Sigma_1 \cup \Sigma_2$ (the surface which is just Σ_1 and Σ_2 glued together along their common boundary with area vector pointing outside) is $\Phi_{\Sigma_1 \cup \Sigma_2} = \Phi_{\Sigma_1} - \Phi_{\Sigma_2}$

$$\Phi_{\Sigma_1} - \Phi_{\Sigma_2} = \Phi_{\Sigma_1 \cup \Sigma_2} = \int_{\Sigma_1 \cup \Sigma_2} \nabla \times \vec{A} \cdot d\vec{S}.$$

– In the RHS we see the flux of the vector field $\nabla \times \vec{A}$ through a closed surface. Denoting Ω the volume inside $\Sigma_1 \cup \Sigma_2$ (or $\partial\Omega = \Sigma_1 \cup \Sigma_2$) and using Gauss theorem for the vector field $\nabla \times \vec{A}$ we have

$$\Phi_{\Sigma_1} - \Phi_{\Sigma_2} = \Phi_{\Sigma_1 \cup \Sigma_2} = \int_{\Sigma_1 \cup \Sigma_2} \nabla \times \vec{A} \cdot d\vec{S} = \int_{\Omega} \nabla \cdot \nabla \times \vec{A} dV = 0,$$

where Ω is the volume inside $\Sigma_1 \cup \Sigma_2$ (or $\partial\Omega = \Sigma_1 \cup \Sigma_2$). The last equality is valid because for ANY smooth vector field $\nabla \cdot \nabla \times \vec{A} = 0$.

– So we have

$$\Phi_{\Sigma_2} = \Phi_{\Sigma_1}.$$

So indeed the RHS of the Stokes theorem is independent of Σ.

- Example of a circulation: work of a force vector field over a closed path.

$$\mathcal{W} = \int_{\partial\Sigma} \vec{F} \cdot d\vec{r} = \int_{\Sigma} \nabla \times \vec{F} \cdot d\vec{s}.$$

if the force is a potential force, then $\vec{F} = -\nabla U$ and

$$\mathcal{W} = -\int_{\Sigma} \nabla \times \nabla U \cdot d\vec{s} = 0.$$

or if $\nabla \times \vec{F} = 0$ everywhere, then the work of this force on ANY closed path is zero — the force is a potential/conservative force.

16.3. Faraday's Law.

You are familiar with this law in the form

$$\mathcal{E} = -\frac{d\Phi_B}{dt}.$$

This law tells us that if we have a wire loop, then the EMF in the wire is given by the rate of change of the flux of the magnetic field. This is the law given by the experiment.

(NOTE: just to make sure, I want to point out, that the flux Φ_B can change because the magnetic field changes and because the area changes.)

This law in this formulation, however, is incomprehensible.

- What is a flux through a loop? We only know the flux through a surface.
- What is EMF?
- Why do we need a wire?

We start with the first question. In order to define the flux we draw an oriented surface Σ with our wire as a boundary. Then the flux is $\Phi_B = \int_{\Sigma} \vec{B} \cdot d\vec{S}$.

Now the second question. By the definition of the voltage we know the EMF is work done by the electric field on a unit charge which is moved along the contour. The contour here is simply the boundary of the surface Σ. So we have $\mathcal{E} = \oint_{\partial\Sigma} \vec{E} \cdot d\vec{r}$, where the boundary is oriented in accordance with the orientation of Σ.

- Faraday's Law, Circulation of Electric field. (zero in statics)

$$\oint_{\partial\Sigma} \vec{E}\cdot d\vec{r} = -\frac{\partial}{\partial t}\int_{\Sigma} \vec{B}\cdot d\vec{S}.$$

- Notice, we do not need any wire!
- However, there is arbitrariness in choosing Σ, only its boundary is fixed.
- Faraday's Law is independent of Σ — it only depends on $\partial\Sigma$.

$$\Phi_{\Sigma_1}-\Phi_{\Sigma_2} = \Phi_{\Sigma_1\cup\Sigma_2} = \int_{\Sigma_1\cup\Sigma_2} \vec{B}\cdot d\vec{S} = (\text{Gauss theorem}) = \int_{\Omega} \nabla\cdot\vec{B}dV = (\text{Gauss mag. law}) = 0.$$

- Local version of the Faraday's law. Using Stokes theorem

$$\oint_{\partial\Sigma} \vec{E}\cdot d\vec{r} = \int_{\Sigma} \nabla\times\vec{E}\cdot d\vec{S}$$

we get $\int_{\Sigma} \nabla\times\vec{E}\cdot d\vec{S} = -\frac{\partial}{\partial t}\int_{\Sigma}\vec{B}\cdot d\vec{S}$. As it must be true for ANY Σ we have

$$\nabla\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0$$

Now this equation has a time derivative of the magnetic field. We also need an equation which has a time derivative of the electric field.

16.4. Ampere's Law.

You might have seen the Ampere's law in the following form:

$$\oint_{\Gamma} \vec{B}\cdot d\vec{r} = \mu_0 I,$$

where Γ is some contour and the current I goes through this contour. This is the law given by experiment.

This formulation, however, has some problems.

- What is the orientation of the contour Γ?
- What does it mean: current I goes through the contour?

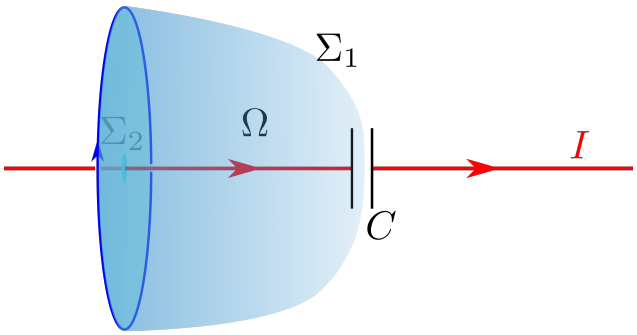
We can reformulate this law using the vector field circulation construction. We then say that there is a surface Σ and its boundary $\partial\Sigma$. We also say that the boundary and the surface are oriented in a consistent way, the same as was used in the Stokes theorem. Then the current I through the contour is simply $\int_{\Sigma}\vec{j}\cdot d\vec{S}$.

- Ampere's Law, Circulation of Magnetic field.

$$\oint_{\partial\Sigma} \vec{B}\cdot d\vec{r} = \mu_0\int_{\Sigma}\vec{j}\cdot d\vec{S}$$

However, there is a problem with this formulation also. As written it depends on Σ. Compare two surfaces on the figure Σ_1 and Σ_2. If the current I depends on time, say it is AC current, then the current that crosses Σ_2 is not zero, while the current which crosses Σ_1 is zero.

So the RHS of Ampere's law depends on our choice of the surface Σ, while the LHS does not depend on it. We need to fix this problem in order for the Ampere's law to be usable.



- Consider the RHS of the Ampere's law for two arbitrary surfaces Σ_1 and Σ_2 with common (oriented) boundary:

$$\int_{\Sigma_1}\vec{j}\cdot d\vec{S}-\int_{\Sigma_2}\vec{j}\cdot d\vec{S}=\int_{\Sigma_1\cup\Sigma_2}\vec{j}\cdot d\vec{S}=\text{Gauss Theorem}=\int_{\Omega}\nabla\cdot\vec{j}dV=\text{Continuity Eq.}$$
$$=-\frac{d}{dt}\int_{\Omega}\rho dV=\text{Gauss Law}=-\epsilon_0\frac{d}{dt}\int_{\Omega}\nabla\cdot\vec{E}dV=\text{Gauss Theorem}=-\epsilon_0\frac{d}{dt}\int_{\Sigma_1\cup\Sigma_2}\vec{E}\cdot d\vec{S}$$
$$=-\epsilon_0\frac{d}{dt}\int_{\Sigma_1}\vec{E}\cdot d\vec{S}+\epsilon_0\frac{d}{dt}\int_{\Sigma_2}\vec{E}\cdot d\vec{S}$$

- We see, that

$$\int_{\Sigma_1}\vec{j}\cdot d\vec{S}+\epsilon_0\frac{d}{dt}\int_{\Sigma_1}\vec{E}\cdot d\vec{S}=\int_{\Sigma_2}\vec{j}\cdot d\vec{S}+\epsilon_0\frac{d}{dt}\int_{\Sigma_2}\vec{E}\cdot d\vec{S}$$

  So that the combination $\int_{\Sigma}\vec{j}\cdot d\vec{S}+\epsilon_0\frac{d}{dt}\int_{\Sigma}\vec{E}\cdot d\vec{S}$ is independent of Σ (it is the same for two different arbitrary Σs with common boundary).
- If there is no electric field $\vec{E}=0$, then this combination is the same as just $\int_{\Sigma}\vec{j}\cdot d\vec{S}$.

Ampere's law, corrected.

$$\oint_{\partial\Sigma}\vec{B}\cdot d\vec{r}=\mu_0\int_{\Sigma}\vec{j}\cdot d\vec{S}+\mu_0\epsilon_0\frac{d}{dt}\int_{\Sigma}\vec{E}\cdot d\vec{S}.$$

Using Stokes theorem on the left hand side of this equation we find the local form of the Ampere's law:

$$\nabla\times\vec{B}-\mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}=\mu_0\vec{j}.$$

16.5. Full Set of Maxwell equations.

Gauss's law:	$\oint_{\partial\Omega}\vec{E}\cdot d\vec{S}=\frac{1}{\epsilon_0}\int_{\Omega}\rho dV,$	$\nabla\cdot\vec{E}=\frac{\rho}{\epsilon_0}$
Gauss's law magnetic:	$\oint_{\partial\Omega}\vec{B}\cdot d\vec{S}=0,$	$\nabla\cdot\vec{B}=0$
Faraday's law:	$\oint_{\partial\Sigma}\vec{E}\cdot d\vec{r}=-\frac{d}{dt}\int_{\Sigma}\vec{B}\cdot d\vec{S},$	$\nabla\times\vec{E}+\frac{\partial\vec{B}}{\partial t}=0$
Ampere's law:	$\oint_{\partial\Sigma}\vec{B}\cdot d\vec{r}=\mu_0\int_{\Sigma}\vec{j}\cdot d\vec{S}+\mu_0\epsilon_0\frac{d}{dt}\int_{\Sigma}\vec{E}\cdot d\vec{S},$	$\nabla\times\vec{B}-\mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}=\mu_0\vec{j}$

LECTURE 17
Maxwell equations: Full set. Gauge invariance.

17.1. Full set of Maxwell equations.

Full set of Maxwell equations:

Gauss's law:	$\oint_{\partial\Omega} \vec{E} \cdot d\vec{S} = \frac{1}{\epsilon_0} \int_\Omega \rho dV,$	$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$
Gauss's law magnetic:	$\oint_{\partial\Omega} \vec{B} \cdot d\vec{S} = 0,$	$\nabla \cdot \vec{B} = 0$
Faraday's law:	$\oint_{\partial\Sigma} \vec{E} \cdot d\vec{r} + \frac{d}{dt} \int_\Sigma \vec{B} \cdot d\vec{S} = 0,$	$\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$
Ampere's law:	$\oint_{\partial\Sigma} \vec{B} \cdot d\vec{r} - \mu_0\epsilon_0 \frac{d}{dt} \int_\Sigma \vec{E} \cdot d\vec{S} = \mu_0 \int_\Sigma \vec{j} \cdot d\vec{S},$	$\nabla \times \vec{B} - \mu_0\epsilon_0 \frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{j}$

In addition we should supply

- Initial conditions.
- Boundary conditions.
- "Material law"— how charges (and currents) respond to fields $\vec{E}$ and $\vec{B}$: fixed charges and wires; free particles — Lorenz force; metals — Ohm's law for low frequencies; plasmonic modes; insulators — polarization; magnets; superconductors; topological insulators; etc.

17.1.1. What we already know.

- Coulomb law. The non-trivial spherically symmetric solution of the static Gauss and Faraday's equations with the boundary condition $\vec{E}(\vec{r} \to \infty) \to 0$ gives the Coulomb law.
- Biot-Savart law. It is derived in Sec. 17.5.
- Charge conservation – Gauss's and Ampere's laws.
  - The time derivative of the Gauss law gives

$$\nabla \cdot \frac{\partial \vec{E}}{\partial t} = \frac{1}{\epsilon_0} \frac{\partial \rho}{\partial t}.$$

  - The div of the Ampere's law gives (we use $\nabla \cdot \nabla \times \vec{B} = 0$ for any $\vec{B}$)

$$-\mu_0\epsilon_0 \nabla \cdot \frac{\partial \vec{E}}{\partial t} = \mu_0 \nabla \cdot \vec{j}.$$

– Comparing these two equations we get

$$\nabla \cdot \vec{j} + \frac{\partial \rho}{\partial t} = 0.$$

The charge conservation law (the continuity equation).

17.1.2. Analysis of the equations.

- This is a set of linear, non-homogeneous partial differential equations.
  - As all partial differential equations the Maxwell equations requires one to set up boundary conditions.
  - Non-homogeneity means that we have non-zero right hand site with some functions $\rho(\vec{r}, t)$ and $\vec{j}(\vec{r}, t)$ — charge density and current density. So we need to know these functions to find the solution.
  - Linearity means that there is principle of superposition: if we have $\rho = \rho_1 + \rho_2$ and $\vec{j} = \vec{j}_1 + \vec{j}_2$, then the solution will be $\vec{E} = \vec{E}_1 + \vec{E}_2$ and $\vec{B} = \vec{B}_1 + \vec{B}_2$. Although one must be careful, as the boundary conditions may break linearity.
- Units.
  - From Faraday's law $\frac{[E]}{[l]} = \frac{[B]}{[t]}$, or $[E] = \frac{[l]}{[t]}[B]$.
  - From Ampere's law $\frac{[B]}{[l]} = [\mu_0 \epsilon_0] \frac{[E]}{[t]}$.
  - So $\frac{1}{[\mu_0 \epsilon_0]} = \frac{[l]^2}{[t]^2}$ – units of the square of the **velocity**.
- We have 8 equations for only 6 unknown functions $\vec{E}$ and $\vec{B}$. So the system of equations is over-complete.
  - An over-complete system will not in general have a solution. It means, that the system has no solution for arbitrary $\vec{j}(\vec{r}, t)$ and $\rho(\vec{r}, t)$.
  - It will have a solution only if certain constrains are satisfied by the RHS — functions $\vec{j}(\vec{r}, t)$ and $\rho(\vec{r}, t)$. (Think about an over-complete *system* of equations for x like this one $\begin{cases} x = a \\ x = b \end{cases}$. For arbitrary a and b it has no solution — there is no x which satisfies both equations simultaneously. It has a solution if and only if a certain constraint is satisfied by the RHS — the numbers a and b. It this particular case this constraint is $a = b$.)
  - As we have 8 equations for 6 functions we must have two constraints on the RHS. These are:
  - The first constraint is that the charge is conserved $\nabla \cdot \vec{j} + \partial\rho/\partial t = 0$. It comes from Gauss's and Ampere's laws.
  - The second one is trivial and comes from Gauss's magnetic and Faraday's laws. (If we had magnetic charges, this constraint would give us the conservation of magnetic charge.)

17.2. Gauge fields.

- Solve magnetic Gauss's and Faraday's laws (both equations have zeros on the right hand sides.)

$$\vec{B} = \nabla \times \vec{A}, \qquad \vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t}.$$

- The fields ϕ and $\vec{A}$ are called potential and vector potential respectively.

If we express $\vec{E}$ and $\vec{B}$ through the gauge fields $\vec{A}$ and ϕ the magnetic Gauss's law and the Faraday's law are automatically satisfied (notice, that these the laws that have zeros on RHS) The other two laws can be written as ($\Delta \equiv \nabla^2$.)

$$-\Delta\phi - \frac{\partial \nabla \cdot \vec{A}}{\partial t} = \frac{\rho}{\epsilon_0}$$
$$-\Delta\vec{A} + \vec{\nabla}(\nabla \cdot \vec{A}) + \mu_0\epsilon_0\vec{\nabla}\frac{\partial\phi}{\partial t} + \mu_0\epsilon_0\frac{\partial^2\vec{A}}{\partial t^2} = \mu_0\vec{j}$$

- Notice, that now we have four equations and four unknowns ϕ and $\vec{A}$. But we still have one constraint on the functions ρ and $\vec{j}$ in the right hand sides. So effectively we have only three equations. It means that we have a freedom to chose a "gauge" for our fields ϕ and $\vec{A}$.
- This freedom, however, must not change the physical fields $\vec{E}$ and $\vec{B}$.

17.3. Gauge invariance.

- Gauge transformation, for any $f(\vec{r}, t)$ the transformation

$$\vec{A} \to \vec{A} + \nabla f, \qquad \phi \to \phi - \frac{\partial f}{\partial t}$$

  does not change $\vec{E}$ and $\vec{B}$. But $\vec{E}$ and $\vec{B}$ are the only physically observable fields. So no matter what physical property we compute the result must be invariant under these gauge transformations.
- Remember, that this gauge freedom appears because of the continuity equation = charge conservation. The opposite is also true, if we demand, that our theories have gauge invariance, then the charge will be conserved.
- The el-magnetic gauge invariance is the simplest example of such invariance. It is called $U(1)$ symmetry.
- Gauge invariance imposes huge restriction on what we can accept as a theory. So it is a great tool to develop new theories/understanding.
- The gauge invariance is the major tool in all modern physics.
- All fundamental forces do have corresponding gauge fields.

Gauge symmetry (gauge freedom) allows us to chose any gauge we want. This choice is done by imposing an additional constraint on the fields ϕ and $\vec{A}$. This constraint is called gauge fixing.
There are many particularly useful gauges. I give below two examples.

17.4. Examples of gauge fixing.

Coulomb gauge. This gauge is given by the following gauge fixing condition

$$\nabla \cdot \vec{A} = 0.$$

The Maxwell equations then become

$$-\Delta\phi = \frac{\rho}{\epsilon_0}$$
$$-\Delta\vec{A} + \mu_0\epsilon_0\vec{\nabla}\frac{\partial\phi}{\partial t} + \mu_0\epsilon_0\frac{\partial^2\vec{A}}{\partial t^2} = \mu_0\vec{j}$$

Lorenz gauge. This gauge is given by the following gauge fixing condition

$$\nabla\cdot\vec{A} + \mu_0\epsilon_0\frac{\partial\phi}{\partial t} = 0.$$

The Maxwell equations then become

$$-\Delta\phi + \mu_0\epsilon_0\frac{\partial^2\phi}{\partial t^2} = \frac{\rho}{\epsilon_0}$$
$$-\Delta\vec{A} + \mu_0\epsilon_0\frac{\partial^2\vec{A}}{\partial t^2} = \mu_0\vec{j}$$

Notice, that both equations in this gauge can be written as

$$\left(-\Delta + \mu_0\epsilon_0\frac{\partial^2}{\partial t^2}\right)\begin{pmatrix}\phi\\ \vec{A}\end{pmatrix} = \begin{pmatrix}\rho/\epsilon_0\\ \mu_0\vec{j}\end{pmatrix}.$$

Also notice, that the combination $1/\sqrt{\epsilon_0\mu_0}$ has units of velocity.

17.5. Biot-Savart law.

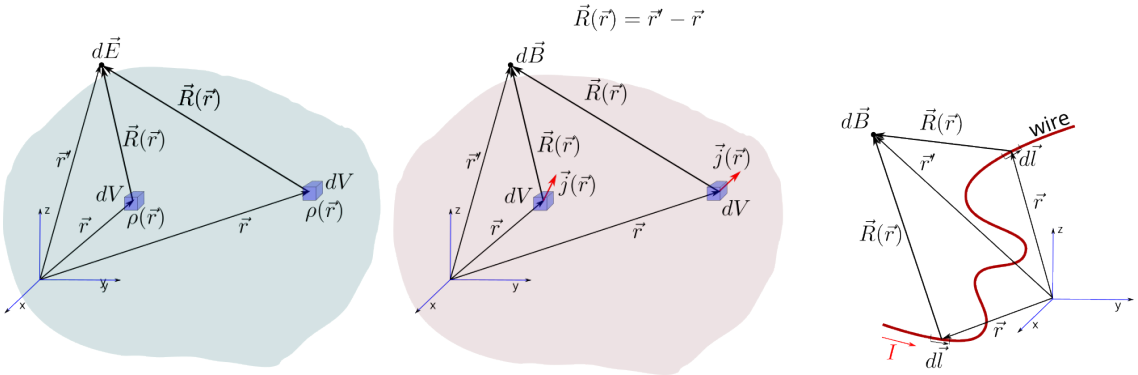


Figure 1. Illustration for the Coulomb and Biot-Savart laws.

In particular, if we are looking for the static solutions, meaning that neither ρ nor $\vec{j}$ depend on time and there is no EM waves around then neither ϕ nor $\vec{A}$ will depend on time (more precisely we can find a solution when neither ϕ nor $\vec{A}$ depend on time) Both Coulomb Lorenz gauges then give ($\partial_t\phi = 0$ and $\partial_t\vec{A} = 0$).

$$-\Delta\phi = \frac{\rho}{\epsilon_0}$$
$$-\Delta\vec{A} = \mu_0\vec{j}$$

Notice, that the equations look exactly the same. We know that the solution of the first equation for a point like charge is given by the Coulomb potential

$$d\phi = \frac{1}{4\pi\epsilon_0}\frac{\rho dV}{R}$$

So the solution of the second equation (for the "point like" current) must be

$$d\vec{A} = \frac{\mu_0}{4\pi}\frac{\vec{j}dV}{R}$$

So for any STATIC distribution of charges and currents we can find the electric and magnetic fields taking the gradient of $d\phi$ and the curl of $d\vec{A}$.

$$d\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{\rho dV\vec{R}}{R^3}$$

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{dV\vec{j}\times\vec{R}}{R^3}$$

So for any static distribution of charges and currents we can find the electric and magnetic fields using the Coulomb and Biot-Savart laws.

The familiar form of the Biot-Savart law

$$d\vec{B} = -\frac{\mu_0}{4\pi}\frac{I\vec{R}\times d\vec{l}}{R^3}.$$

is obtained by assuming the current density is inside the small piece of wire of length dl and cross-section dS, then $\vec{j}dV = \vec{j}dSdl = Id\vec{l}$.

17.6. Light.

- Maxwell equations in vacuum — no static solutions.
- Wave equation.
- General solution of the wave equation.
- Speed of light.

LECTURE 18
Let there be light! Electromagnetic waves. Speed of light.

- Homework.

18.1. Let there be light!

We saw that the Maxwell equation contain everything that we know about electric and magnetic fields. Here we will find out what else they have.

$$\begin{aligned} \text{Gauss's law:} &\qquad \nabla \cdot \vec{E} = 0 \\ \text{Gauss's law magnetic:} &\qquad \nabla \cdot \vec{B} = 0 \\ \text{Faraday's law:} &\qquad \nabla \times \vec{E} + \tfrac{\partial \vec{B}}{\partial t} = 0 \\ \text{Ampere's law:} &\qquad \nabla \times \vec{B} - \mu_0 \epsilon_0 \tfrac{\partial \vec{E}}{\partial t} = 0 \end{aligned}$$

- Maxwell equations show the dynamics of the fields $\vec{E}$ and $\vec{B}$ themselves, independent of the dynamics of the sources/charges/currents.
- Consider Maxwell equations in vacuum — there are no static solutions.
- However, there are dynamical solutions.

  Acting by $\nabla\times$ on Faraday's law and using $\nabla \times \nabla \times \vec{E} = \nabla(\nabla \cdot \vec{E}) - \Delta \vec{E}$ and the Gauss and Ampere's laws we get

$$\Delta \vec{E} - \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} = 0$$

  Acting by $\nabla\times$ on Ampere's's law and using $\nabla \times \nabla \times \vec{B} = \nabla(\nabla \cdot \vec{B}) - \Delta \vec{B}$ and the magnetic Gauss and Faraday's laws we get the same equation for the magnetic field.
- Wave equation, $1D$.

$$\frac{\partial^2 \vec{E}}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} = 0, \qquad c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}.$$

  – Boundary condition: $\vec{E}(t, x \to \pm\infty) \to 0$.
  – Initial condition: $\vec{E}(t = 0, x) = \vec{E}_0(x)$ – it can be thought as a boundary condition in time.

- General solution of the wave equation.

$$\vec{E}(x,t) = \vec{E}_0(x \pm ct).$$

(According to the Gauss magnetic and Ampere's laws magnetic field will also be generated.)

- Speed of light.

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}.$$

I want you to marvel at the following thing: The Maxwell equations we "derived" considering experiments (Coulomb force, Faraday's, Amperes) none of which suggested any idea of light. It is only after all these experiments were understood as a set of Maxwell equations we are able to see light.

It was first recognized by Maxwell, then demonstrated and carefully measured experimentally by Hertz.

18.1.1. Problem with the speed of light.

- In Galilean/Newtonian mechanics any velocity depends on the frame of reference the observer is in. However, the Maxwell equations are valid in any frame of reference, so in any frame of reference the e-m wave propagates with the velocity c. This is a clear contradiction.
- Both Maxwell equations/theory as well as Galilean/Newtonian mechanics were thoroughly tested in many many experiments.
- Idea of Aether — a special universal frame of reference. Michelson-Morley experiment. `https://en.wikipedia.org/wiki/Michelson%E2%80%93Morley_experiment`.

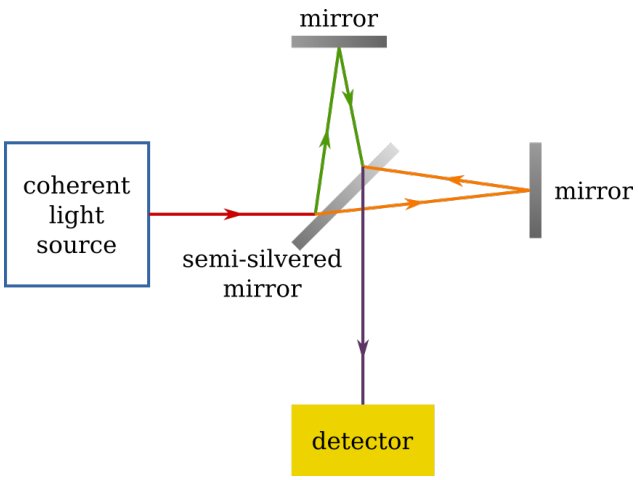


Figure 1. A Michelson interferometer uses the same principle as the original experiment. But it uses a laser for a light source.

18.2. Lorenz transformation.

What are the space-time transformations that leave the Maxwell equations invariant?

- Galilean transformation. Transformations that leave the Newton' equation invariant:

$$dx = dx' + Vdt', \qquad dt = dt'$$

(I use this transformation in dx, dt form, as it is much easier to deal with the small intervals of space and time.)

- We want to find the transformation which leaves the Maxwell equations invariant.
- This transformation will change both space and time and electric and magnetic fields.
- However, as Maxwell equations are linear and so is the wave equation we can simplify the problem by asking: what transformation of space and time will leave the wave equation invariant?
- We write the wave equation in the form

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{E} = 0.$$

Here we see, that the equation has the following form: an operator acts on the electric field $\vec{E}$ and gives zero.

- We want to find the transformation $t(x', t')$, and $x(x', t')$ which leaves the operator invariant

$$\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \to \frac{\partial^2}{\partial x'^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t'^2}.$$

- Lorenz transformation. Transformations that leave the wave operator invariant. Look for the transformation in the form

$$dx = Adt' + Bdx', \qquad dt = Cdt' + Ddx',$$

where A, B, C, and D are constants. This gives us the transformation of variables $x(x', t')$ and $t(x', t')$.

- In particular, from the definition of differential

$$dx = \frac{\partial x}{\partial x'}dx' + \frac{\partial x}{\partial t'}dt', \qquad dt = \frac{\partial t}{\partial x'}dx' + \frac{\partial t}{\partial t'}dt'$$

we find

$$\frac{\partial x}{\partial x'} = B, \quad \frac{\partial x}{\partial t'} = A, \quad \frac{\partial t}{\partial x'} = D, \quad \frac{\partial t}{\partial t'} = C.$$

- Now we use the chain rule and write

$$\frac{\partial}{\partial x'} = \frac{\partial x}{\partial x'}\frac{\partial}{\partial x} + \frac{\partial t}{\partial x'}\frac{\partial}{\partial t} = B\frac{\partial}{\partial x} + D\frac{\partial}{\partial t}$$
$$\frac{\partial}{\partial t'} = \frac{\partial x}{\partial t'}\frac{\partial}{\partial x} + \frac{\partial t}{\partial t'}\frac{\partial}{\partial t} = A\frac{\partial}{\partial x} + C\frac{\partial}{\partial t}$$

So that

$$\frac{\partial^2}{\partial x'^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t'^2} = \left(B^2 - \frac{1}{c^2}A^2\right)\frac{\partial^2}{\partial x^2} + \left(D^2 - \frac{1}{c^2}C^2\right)\frac{\partial^2}{\partial t^2} + 2\left(BD - \frac{1}{c^2}AC\right)\frac{\partial^2}{\partial x \partial t}$$

In order for the wave equation not to change its form we must have

$$B^2 - \frac{1}{c^2}A^2 = 1, \qquad D^2 - \frac{1}{c^2}C^2 = -\frac{1}{c^2}, \qquad BD - \frac{1}{c^2}AC = 0$$

We have three equation with four unknowns. The solution depends on one parameter γ and can be written as

$$A = \frac{c\gamma}{\sqrt{1-\gamma^2}}, \quad B = C = \frac{1}{\sqrt{1-\gamma^2}}, \quad D = \frac{\gamma/c}{\sqrt{1-\gamma^2}}$$

and hence the transformation which lives the wave equation invariant is:

$$dx = \frac{\gamma c dt'}{\sqrt{1-\gamma^2}} + \frac{dx'}{\sqrt{1-\gamma^2}}, \qquad cdt = \frac{cdt'}{\sqrt{1-\gamma^2}} + \frac{\gamma dx'}{\sqrt{1-\gamma^2}},$$

This is called Lorentz transformation.

- At this stage γ is an arbitrary parameter. These transformation rules do not have any physical content. It is so far just a mathematical statement that Lorenz transformation with arbitrary γ will leave the wave (in fact Maxwell) equations invariant.
- In order to understand the physical meaning of these transformations we need to figure out what γ is.
- In order to do that, let's assume, that γ is small, but γc is not small. Then the Lorenz transformation becomes

$$dx = \gamma c dt' + dx', \qquad cdt = cdt'.$$

- Comparing the Lorenz transformation to the Galileo transformation we find that $\gamma = V/c$ and then

$$dx = \frac{V dt'}{\sqrt{1-V^2/c^2}} + \frac{dx'}{\sqrt{1-V^2/c^2}}, \qquad cdt = \frac{cdt'}{\sqrt{1-V^2/c^2}} + \frac{V dx'/c}{\sqrt{1-V^2/c^2}},$$

- Now the Lorenz transformation has physical content! It tells us how to go from one frame of references to another!
- We can use these transformation in the full set of Maxwell equation and see how the electric and magnetic fields transform between different frames of references.
- We can also check, that the under this transformation the Lorenz force does not change (for small velocities) when one moves from one frame of references to another!
- This transformation also tells us that our space-time has a very different structure than what was thought before.

LECTURE 19
Special theory of relativity.

What we have done so far:

- Intuition: Translation, time translation invariance, and universality of time $\longrightarrow$ Galilean inviriance $\longrightarrow$ Newtonian mechanics. Experiments to check the validity.
- Experiments with magnetic and electric fields: Lorenz force, Gauss laws (both), Faraday's law, Ampere's law + writing it all in the form that makes sense $\longrightarrow$ Maxwell equations. Experiments to check the validity.
- Comparing the Newton's dynamics and Maxwell equations $\longrightarrow$ conundrum $\longrightarrow$ Lorenz transformation.
- Time is NOT universal $\longrightarrow$ Galilean inviriance is only approximate $\longrightarrow$ Newtonian mechanics is only approximate, it works only if speeds are much less then the speed of light (whether we can use the Newtonian mechanics or not depends on the problem and on the accuracy we need. The Newtonian mechanics will always have the corrections of the order of $(v/c)^2$. In many cases these corrections are beyond the resolution of our experimental devices.)

Lorenz transformation:

- Lorenz transformation. There are two frames of references. the second frame is moving with respect to the first with velocity V along the x direction. The time interval dt' and the space interval dx', dy', and dz' for some process are measured in the moving frame (frame #2), then in the stationary frame (frame #1) the time and space intervals for the SAME process are

$$cdt = \frac{cdt'}{\sqrt{1-V^2/c^2}} + \frac{Vdx'/c}{\sqrt{1-V^2/c^2}}, \qquad dx = \frac{Vdt'}{\sqrt{1-V^2/c^2}} + \frac{dx'}{\sqrt{1-V^2/c^2}}, \qquad dy = dy', \qquad dz = dz'.$$

- The inverse of the Lorenz transformation has the same form:

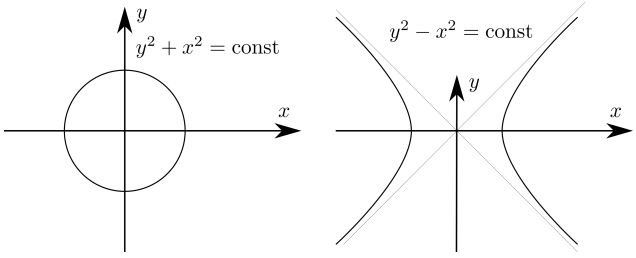
$$cdt' = \frac{cdt}{\sqrt{1-V^2/c^2}} - \frac{Vdx/c}{\sqrt{1-V^2/c^2}}, \qquad dx' = -\frac{Vdt}{\sqrt{1-V^2/c^2}} + \frac{dx}{\sqrt{1-V^2/c^2}}, \qquad dy' = dy, \qquad dz' = dz,$$

  with $V \to -V$, as expected, as from the point of view of the frame #2, the frame #1 is moving along x direction with the velocity $-V$.
- These transformations tell us that our space-time has a very different structure than what was thought before.

- Lorenz transformation is the transformation that leaves the interval ds DEFINED as
$$(ds)^2 = c^2(dt)^2 - (dx)^2$$
invariant.

$$(ds)^2 = \left(\frac{cdt'}{\sqrt{1-V^2/c^2}} + \frac{Vdx'/c}{\sqrt{1-V^2/c^2}}\right)^2 - \left(\frac{Vdt'}{\sqrt{1-V^2/c^2}} + \frac{dx'}{\sqrt{1-V^2/c^2}}\right)^2 = c^2(dt')^2 - (dx')^2.$$

- Consider that an observer who measures time in his/her OWN frame of reference. So one has a watch and simply looks at this watch. In this process the watch is not moving, so $dx = dy = dz = 0$, then the interval $ds = cdt$. So the interval is simply time in ones own frame of reference. The interval is also called PROPER TIME. Remember that the interval is the same in every frame of reference.
- $ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$ — metric of space-time!
- Event is a point of a space-time. Interval ds is the "distance" between the Events.
- This "distance"/interval between the events is computed the same way in all inertial frames of references and is independent in which inertial frame of reference it is computed in.
- This provides a true metric for the space-time. So the full space-time has geometry!
- A space (space-time) with such metric is called Minkowski space `https://en.wikipedia.org/wiki/Minkowski_space`. The metric is called Minkowski metric.
- Lorenz transformation is a "rotation" of the space-time. (remember, rotation is the transformation which leaves the distance between any two points unchanged.)
- GPS, LHC.



19.1. Consequences.

We want to introduce two frames of references: "primed" and "un-primed". The primed frame moves with velocity V as observed from un-primed frame. All quantities such as distances and times in primed frame will have prime: dx', dt', etc. All quantities in un-primed frame will not have primes: dx, dt, *etc.*

I want to *very strongly* emphasize, that in order to figure out all these consequences one MUST very carefully specify what experiment is being performed. Without doing that, our normal intuition will fool us at every step.

I want to compare Galilean and Lorenzian worlds. So here are both Galilean and Lorenz transformations.

Galilean,	Lorenz
$dx = dx' + Vdt'$,	$dx = \frac{Vdt'}{\sqrt{1-V^2/c^2}} + \frac{dx'}{\sqrt{1-V^2/c^2}}$
$dt = dt'$,	$cdt = \frac{cdt'}{\sqrt{1-V^2/c^2}} + \frac{Vdx'/c}{\sqrt{1-V^2/c^2}}$

19.1.1. Restriction on causality.

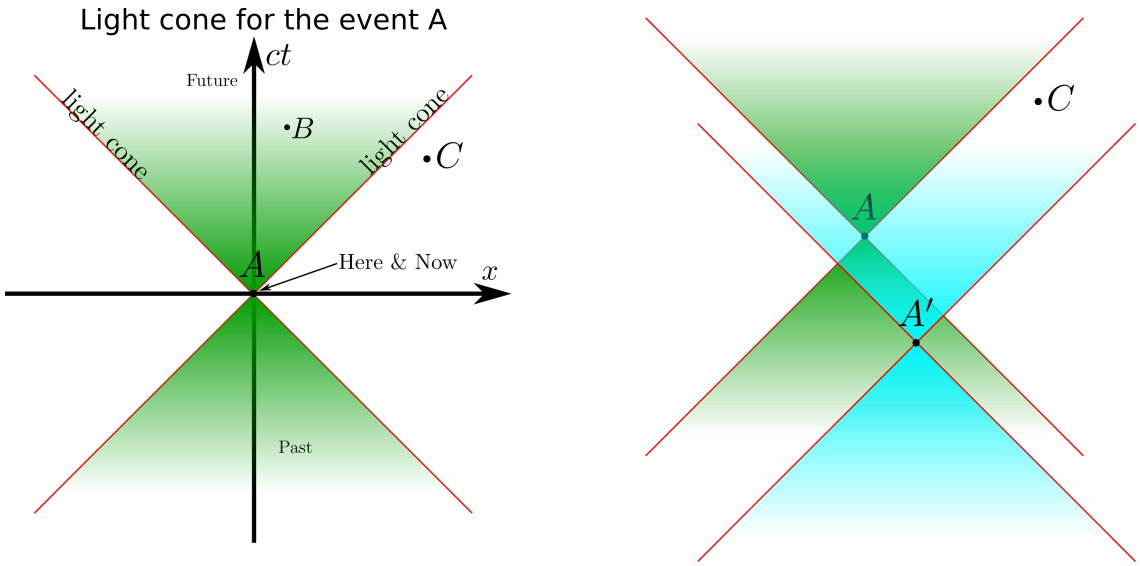


Figure 1

Look at the space-time x, ct depicted on the figure. The origin (event A in the picture) is "here and now" point. The red lines are the world lines of light – it travels with the speed of light either to the left or to the right $x = \pm ct$, so these are lines at 45 degrees in the x, ct plane. If we consider more than one space dimension, then it will be "light-cone".

- Any two events on a red line have interval $ds = 0$.
- For any event inside the cone, say point B, for the interval between the events A and B we have $(ds)^2 > 0$, so the interval between the events A and B is real.
- As interval is invariant under the change of frame of reference, this interval will be real in any other frame of reference.
- However, when the other event, say point C, is outside of the cone, then for the interval between the events A and C we have $(ds)^2 < 0$, so the interval between the events A and C is imaginary.
- As interval is invariant under the change of frame of reference, this interval will be imaginary in any other frame of reference.

Imaginary interval cannot happen for any physical process.

- Interval is the "proper time" — time measured in ones own frame of references.
- It cannot be imaginary for any physical process.

What it means is that events A and C cannot be parts of the same physical process. Or in other words the events A and C cannot have causal relation: one cannot be the cause of another, even though one (the event C on the picture) is later in time than the other (the event A on the picture).
This is what it means that nothing can travel faster than light.

However, both event A and event C can be consequences of an event A'. So the events A and C can be correlated!

19.1.2. Universality of the speed of light.

The speed of light is the same in all frames of references, in striking contrast to the Galilean world.

In the Galilean world if in the primed frame of references a body is moving with the speed of light then measuring its position after time dt' our primed friend finds that the position has shifted by $dx' = cdt'$. The un-primed observer in the Galilean world finds that $dx = dx' + Vdt' = (c+V)dt'$, and $dt = dt'$. So in the un-primed frame the velocity of the body is $\frac{dx}{dt} = c + V \neq c$.

This is not so in the Lorenz/Minkowski/Einstein/our world.

Again in the primed frame we have the body's velocity $c = dx'/dt'$, so $dx' = cdt'$. Then in un-primed frame we have

$$dx = \frac{Vdt'}{\sqrt{1-V^2/c^2}} + \frac{dx'}{\sqrt{1-V^2/c^2}} = \frac{V+c}{\sqrt{1-V^2/c^2}}dt'$$
$$cdt = \frac{cdt'}{\sqrt{1-V^2/c^2}} + \frac{Vdx'/c}{\sqrt{1-V^2/c^2}} = \frac{c+V}{\sqrt{1-V^2/c^2}}dt'.$$

The right hand sides of the two equations are identical, so we have in the un-primed frame $dx = cdt$, or the velocity v of the body which the un-primed observer observes $v = \frac{dx}{dt} = c$.

So if a body moves with the speed of light in one frame of reference, it moves with the speed of light in ALL frames!

Notice, that this statement is obvious from the fact that the interval ds is the same in all frames of references, so if it is 0 in one frame it is zero in any other.

$$dx' = cdt', \quad (ds)^2 = (cdt')^2 - (dx')^2 = 0 = (cdt)^2 - (dx)^2, \quad dx = cdt.$$

19.1.3. Simultaneity is not absolute.

Events that are simultaneous in one frame of reference are not necessarily simultaneous in another, in striking contrast to the Galilean world.

In the Galilean world $dx = dx' + Vdt'$, and $dt = dt'$. Consider two events that happen in the un-primed frame simultaneously, then $dt = 0$ — this is what simultaneous means, at distance l from one another, so $dx = l$. Then in the primed frame we have $dt' = dt = 0$, and $dx' = dx - Vdt' = dx'$

$$dt' = dt = 0, \qquad dx' = dx - Vdt' = dx = l.$$

So your primed friend sees the two events also happening simultaneously and also at distance l from one another.

This is not so in the Lorenz/Minkowski/Einstein/our world.

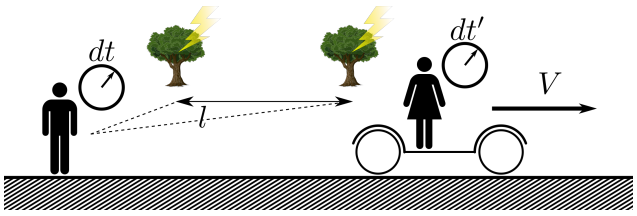


Figure 2

- The experiment is the following:
- Two events happen simultaneously a distance l from one another in our un-primed frame of reference.
- Simultaneous means the time between the events is zero $dt = 0$. For this case we have

$$dt = 0, \qquad dx = l$$

- We use the Lorenz transformation to find dt' and $l' = dx'$ — the time interval and the distance between the events in the primed frame.

$$dt' = -\frac{V}{c^2}\frac{l}{\sqrt{1 - V^2/c^2}}, \qquad l' = dx' = \frac{l}{\sqrt{1 - V^2/c^2}}.$$

- So our primed friend sees the two events as not simultaneous ($dt' \neq 0$) and happening at different distance to one another ($l' \neq l$).

(Notice also that events at finite distance form one another, that are simultaneous in at least one frame of reference cannot have causal relation, $(ds)^2 < 0$.)

19.1.4. Velocities in different frames.

In Galilean world if the velocity of an object in the primed frame is v' and the velocity of the primed frame with respect to un-primed frame is V. then the velocity of the object in the un-primed frame is $v = v' + V$.

This is not so in the Lorenz/Minkowski/Einstein/our world.

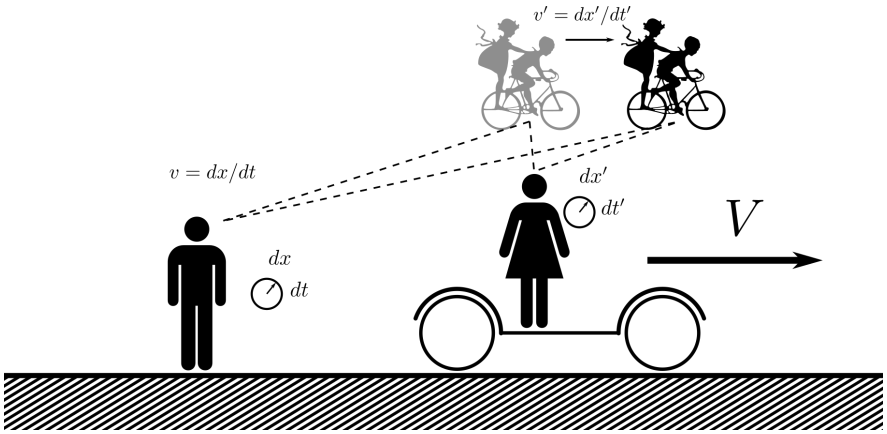


Figure 3

- The experiment is the following:
- A person in the primed frame is measuring the velocity of some object.
- For this the primed person measures the distance dx' the object travels during time dt'.
- Both dx' and dt' are valid for the primed frame of references. The velocity of the object in the primed frame is $v' = dx'/dt'$.

- We then use the Lorenz transformation and $dx' = v'dt'$ and find dx and dt in the un-primed frame.

$$dx = \frac{Vdt'}{\sqrt{1-V^2/c^2}} + \frac{v'dt'}{\sqrt{1-V^2/c^2}} = \frac{V+v'}{\sqrt{1-V^2/c^2}}dt'$$
$$cdt = \frac{cdt'}{\sqrt{1-V^2/c^2}} + \frac{Vv'dt'/c}{\sqrt{1-V^2/c^2}} = \frac{c+Vv'/c}{\sqrt{1-V^2/c^2}}dt'$$

- The velocity in the un-primed frame is $v = dx/dt$.
- So we find:

$$v = \frac{V+v'}{1+\frac{Vv'}{c^2}}.$$

- If both $v', V \ll c$, then $v = V + v'$ – our usual Galilean result!
- If $v' = c$, then $v = c$! The e.-m. wave indeed travels with the same speed in all frames of references!

19.1.5. Time change.

In the Galilean world, where $dt' = dt$, the time between two events is the same for all observers.

This is not so in the Lorenz/Minkowski/Einstein/our world.

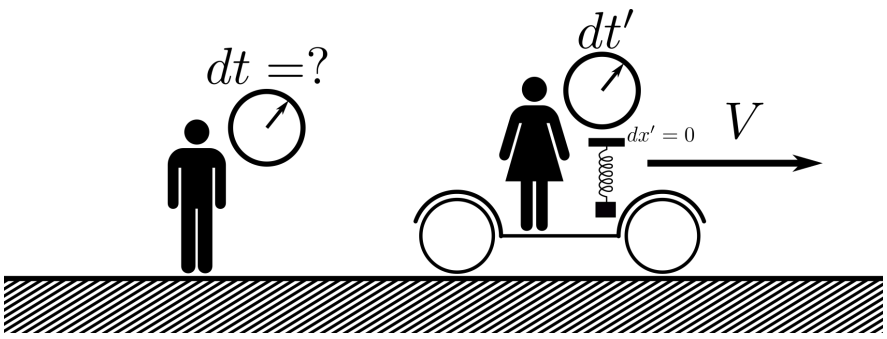


Figure 4

- The experiment is the following:
- In the moving (primed) frame the two events happen at the same place/coordinate, so $dx' = 0$.
- In the moving (primed) frame the two events have the time interval dt'.
- So the time interval dt in the frame of reference at rest is:

$$dt = \frac{dt'}{\sqrt{1-V^2/c^2}} > dt'.$$

- So the time interval dt for the un-primed observer is longer, than the time interval for the primed observer.
- This also can be seen from the invariance of the interval.
  - As $dx' = 0$, dt' is the proper time for the primed frame, so the interval between two events is $(ds)^2 = c^2(dt')^2$.

– The same two events in the un-primed frame happen at time dt and distance $dx = V dt$ (as the primed frame moves with velocity V) from one another. So the interval between the same events is $(ds)^2 = c^2(dt)^2 - (dx)^2 = c^2(dt)^2(1 - V^2/c^2)$.
– As the interval between the same events must be the same in all frames of reference we have $c^2(dt')^2 = c^2(dt)^2(1 - V^2/c^2)$, or

$$dt = \frac{dt'}{\sqrt{1 - V^2/c^2}} > dt'.$$

- From the point of view of the un-primed observer the time in the primed frame of reference slows down. As un-primed observer sees that all processes in the primed frame slow down, including chemical processes such as aging.
- Notice, that from the point of view of the primed observer it is the un-primed observer who is moving with velocity $-V$.
- Then from the point of view of the primed observer the time slows down in the un-primed frame of reference.
- It may seem counter-intuitive, but it must be so, otherwise we would be able to tell who is moving and who is standing.
- Twin's paradox.

19.1.6. Length change.

The length of the objects measured in different frames are also different. In Galilean world $dx = dx' + V dt'$, and $dt = dt'$. When one measures the length of a stick, one must check the positions of both ends of the stick simultaneously. So if one measures the length of the stick in the un-primed frame, then $dt = 0$, but in Galilean world it means $dt' = dt = 0$. So $dx = dx'$ and the length of the stick is the same.

This is not so in the Lorenz/Minkowski/Einstein/our world.

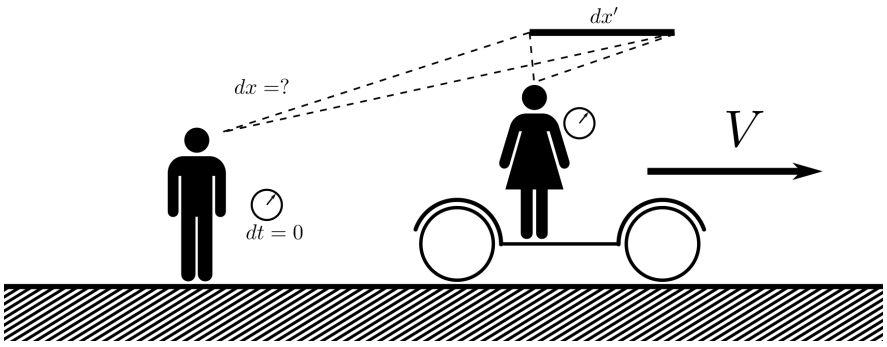


Figure 5

- The experiment is the following:
- A stick in the primed frame of reference is measured by a person in the same primed frame of reference (so the stick is not moving with respect to this person)
- The result is dx'.
- The length of this stick is now measured in the un-primed frame of references.
- In order to do that the researcher must note the positions of the ends of the stick at the same moment of time in his un-primed frame!

- So for this measurement $dt = 0$.

  (Notice, that from the primed frame the un-primed researcher is doing it wrong, as in the primed frame he/she is not taken the position of the both ends *simultaneously*.)
- From the Lorenz transformation we see that this means $cdt' = -\frac{V}{c}dx'$.
- Using this in the Lorenz transformation for dx we find:

$$dx = \frac{-V^2/c^2 + 1}{\sqrt{1 - V^2/c^2}}dx' = dx'\sqrt{1 - V^2/c^2}.$$

19.1.7. Doppler effect.

There is no Galilean version of Maxwell equations.

We already know, that the speed of light is the same for every observer. However, different observers see this light differently.

This effect exists only in the Lorenz/Minkowski/Einstein/our world.

I want to emphasize, that this is very different from the Doppler effect for the sound waves. The Doppler effect for the sound waves only exists because there is a media through which sound propagates. There is no such media for the light. It propagates through the vacuum.

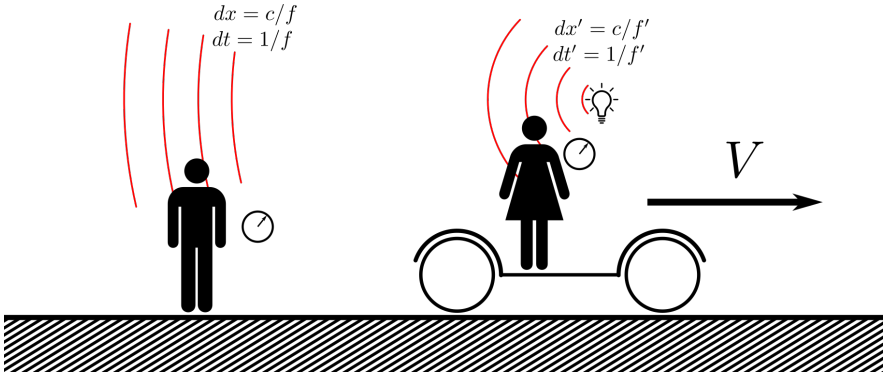


Figure 6

- The experiment is the following
- The light source is in the primed frame. It moves with respect to the un-primed observer with velocity V directly away.
- The light source has frequency f' and travels with velocity c in the primed frame.
- In the primed frame the distance between two wave fronts is $dx' = c/f'$, the time between them is just one period $dt' = T' = 1/f'$.
- Using the Lorenz transformation we find that in the un-primed frame

$$dx = \frac{V/f'}{\sqrt{1 - V^2/c^2}} + \frac{c/f'}{\sqrt{1 - V^2/c^2}}, \qquad cdt = \frac{c/f'}{\sqrt{1 - V^2/c^2}} + \frac{V/f'}{\sqrt{1 - V^2/c^2}}.$$

- First we notice, in the un-primed frame that $cdt = dx$ as it must be — the speed of light is the same for both observers.
- Second, we notice, that for the un-primed observer the frequency of the light is $1/dt$.

$$f = \frac{1}{dt} = \sqrt{\frac{c - V}{c + V}}f'.$$

- So the frequency (color for the visible part of the spectrum) of the light is different for different observers.
- This is Doppler effect.

There are special mathematical notations that make it much easier to work in Minkowski, (or any other) space.

LECTURE 20
Special theory of relativity. General theory of relativity.

- Evaluations will begin on June 21, and end on August 8.

20.1. Doppler effect and its uses.

When a source of light of frequency f' moves away from us with velocity V, we observe the light of a different frequency f

$$\text{light: } f = \sqrt{\frac{c-V}{c+V}}f', \qquad \text{acoustic: } f = \frac{c}{c+V}f'.$$

For light c is the speed of light. For acoustic c is the speed of sound in the media.

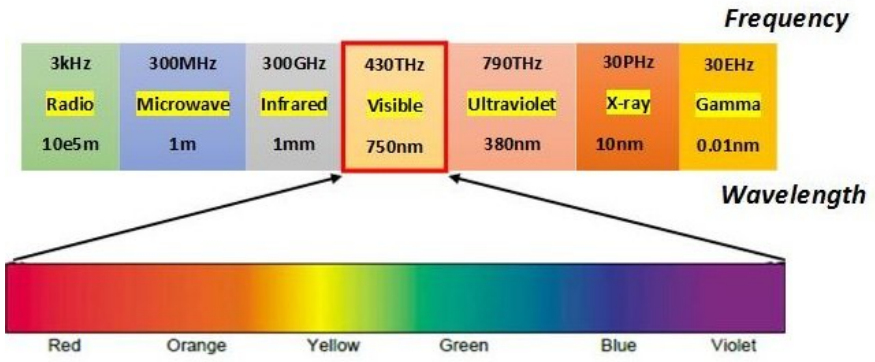


Figure 1

- Red shift.
- Blue shift.
- Velocity of the stars and galaxies. We know the spectrum of light of a star (there are a few types of stars that have different spectrum). This spectrum has features at some particular frequencies. When we observe the stars or galaxies, we notice that the frequencies of these features are shifted. From that we know how the velocity at which the stars are moving away (or towards) us.

- Hubble constant. (Edwin Hubble, PNAS, 15 (3) 168-173 (1929), https://doi.org/10.1073/pnas.15.3.168).

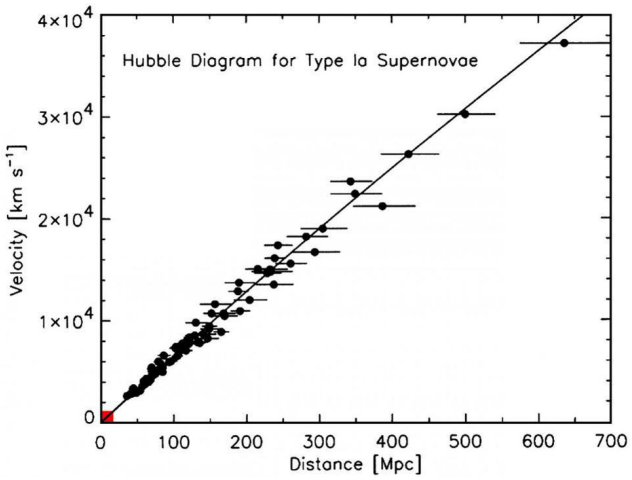


Figure 2. Taken from Robert Kirshner, PNAS, 101 (1) 8-13 (2004), https://doi.org/10.1073/pnas.2536799100.

- Universe expansion.
- Distance to the stars and galaxies.
- Light year, parsec (3.3 light years) https://en.wikipedia.org/wiki/Parsec
- Astronomical distances: https://en.wikipedia.org/wiki/List_of_nearest_galaxies
  - Distance to the Sun $\sim$ 8 light minutes.
  - Distance to the closest other star, Alpha Centauri: 4.367 light years.
  - The diameter of the Milky Way Galaxy: $\sim 100 - 180$ thousand light years. Milky way galaxy has a lot of small satellite galaxies.
  - The distance to the next large galaxy, Andromeda: 2.5 million light years.

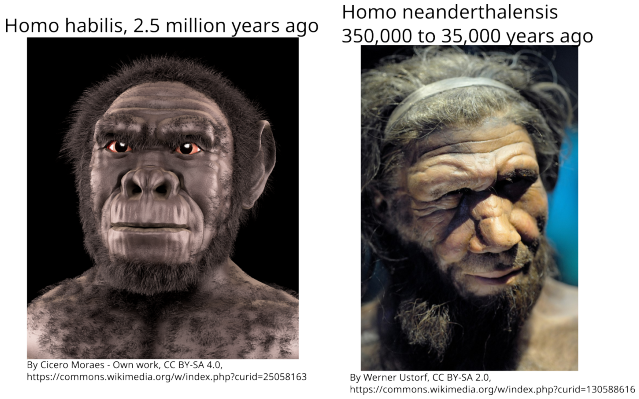


Figure 3. Time line of human evolution for comparison.

  - James Webb telescope resolves galaxies that are $\sim$ 13.6 billion light years away.
- Look into the past.
  - Hubble telescope, James Webb telescope.
  - Microwave background radiation https://en.wikipedia.org/wiki/Cosmic_microwave_background.

20.2. Dynamics.

In order to construct the relativistic dynamics we need to realize a key point:

- The key point The action must be invariant under Lorenz transformation.

The Action $\mathcal{S} = \int_{t_i}^{t_f} L(x, \dot{x})dt$ can be rewritten using the connection between the Lagrangian and Hamiltonian

$$L = p\dot{x} - H,$$

where p is momentum and H is the Hamiltonian. So we can write

$$\mathcal{S} = \int (pdx - Hdt)$$

and we demand, that H and p transform under the Lorenz transformation into another pair H' and p' in such a way, that the expression $pdx - Hdt$ remains invariant.

Using Lorenz transformation:

$$dt = \frac{dt'}{\sqrt{1 - V^2/c^2}} + \frac{Vdx'/c^2}{\sqrt{1 - V^2/c^2}}, \qquad dx = \frac{Vdt'}{\sqrt{1 - V^2/c^2}} + \frac{dx'}{\sqrt{1 - V^2/c^2}}$$

in $pdx - Hdt$ and collecting all terms with dx' and all terms with dt' together we get

$$pdx - Hdt = dx' \left(\frac{p}{\sqrt{1 - V^2/c^2}} - \frac{VH/c^2}{\sqrt{1 - V^2/c^2}} \right) - dt' \left(\frac{H}{\sqrt{1 - V^2/c^2}} - \frac{Vp}{\sqrt{1 - V^2/c^2}} \right)$$

Now we use our "key point", – the action must have the same form in any frame. So we must this as $p'dx' - H'dt'$. Comparing this with above we conclude

$$p' = \frac{p}{\sqrt{1 - V^2/c^2}} - \frac{HV/c^2}{\sqrt{1 - V^2/c^2}}, \qquad H' = \frac{H}{\sqrt{1 - V^2/c^2}} - \frac{Vp}{\sqrt{1 - V^2/c^2}}.$$

This is how one translates the Hamiltonian and the momentum between the frames of references.

The next step is the following: by simple substitution we can check, that the above transformations give

$$(H')^2 - (cp')^2 = (H)^2 - (cp)^2.$$

Let's see what it means.

- The value of the Hamiltonian on a trajectory is called Energy.
- Let's take a particle which is at rest in a primed frame of reference.
- In this frame its momentum is zero.
- We cannot say this about the energy as energy is defined only up to a constant, so lets call it E_0.
- In ANY un-primed frame this particle will have some energy E (or Hamiltonian) and a momentum p.
- According to our transformation rule we must have

  $$E^2 - c^2p^2 = E_0^2.$$

- It means, that a free moving particle must have a dispersion relation

  $$E^2 = E_0^2 + c^2p^2,$$

  where E_0 is the energy of the particle in it's own frame of references. It is called "rest energy".

We would like to know what this "rest energy" is. The key point is to recognize, that for small momentum we must recover the classical $\frac{p^2}{2m_0}$ value up to a constant.

So we write for small p:

$$E = \sqrt{E_0^2 + c^2p^2} \approx E_0 + \frac{c^2p^2}{2E_0}$$

Comparing the momentum dependent term with $\frac{p^2}{2m_0}$ we find

$$E_0 = m_0c^2, \qquad E^2 = c^2p^2 + m_0^2c^4.$$

m_0 is called "rest mass". It is exactly the mass of a particle at rest.

This is still not the celebrated $E = mc^2$, as in our equation it is simply the rest mass and the rest energy. In the non-relativistic mechanics mass is conserved, so the rest energy would also be conserved, hence this energy cannot be used to do work, as work is the CHANGE of energy.

- Momentum and velocity.

  Energy as a function of momentum is Hamiltonian, so we can write the Hamiltonian equations of motion:

  $$\dot{x} = \frac{\partial E(p)}{\partial p}, \qquad \dot{p} = -\frac{\partial E(p)}{\partial x}$$

  The first equation gives $v = \dot{x}$:

  $$v = \frac{pc^2}{\sqrt{p^2c^2 + m_0^2c^4}}, \quad \text{or} \quad p = \frac{m_0v}{\sqrt{1 - v^2/c^2}}.$$

  One can also say, that

  $$p = mv, \qquad m = \frac{m_0}{\sqrt{1 - v^2/c^2}}.$$

- Energy and velocity.

  Using $p = mv$ in $E(p)$ we find

  $$E = \sqrt{c^2p^2 + m_0^2c^4} = \frac{m_0c^2}{\sqrt{1 - v^2/c^2}} = mc^2.$$

- Example. Nuclear binding energy energy.
  - A nucleus consists of N neutrons and P protons.
  - We know the rest mass of each neutron m_N and each proton m_P.
  - We measure the mass of the nucleus M.
  - The binding energy is

  $$E = (Nm_n + Pm_P - M)c^2.$$

- From the Hamiltonian $H(p, x) = \sqrt{p^2c^2 + m_0^2c^4}$ one can compute the Lagrangian $L(x, \dot{x}) = p\dot{x} - H = -m_0\sqrt{c^2 - \dot{x}^2}$ and hence action

  $$\mathcal{S} = \int L(x, \dot{x})dt = -m_0 \int \sqrt{c^2 - \dot{x}^2}dt = -m_0 \int \sqrt{(cdt)^2 - (dx)^2} = -m_0 \int ds,$$

  where ds is interval! Notice

  - The action is purely geometrical. It is simply the length of a path in Minkowski metric.
  - As interval is invariant under the change of frame of references, the action is also invariant.
  - Another way of looking at the action is to remember, that the interval is the proper time. So minimizing the action means maximizing the proper time (there is minus in front of the integral).

20.3. A bit of general theory of relativity.

- Inertial and gravitational masses.
  - Compare the Newton's third law and Newton's gravity

$$\vec{F} = m\vec{a}, \qquad F = \frac{GmM}{R^2}.$$

    These two laws assume very different experiments: In the first, one applies a force and measures the acceleration; in the second one keeps two masses stationary at distance R to each other and measures the force.
  - In the first experiment we measure the response of a free object to an applied force. In the second we measure the gravity force between two objects.
  - How come the masses in the two laws are the same? More precisely, how come if we double the mass m in the first experiment we will also double the force in the second experiment?
  - It does not happen to any other force. The Coulomb force, for example, will not double.
- Non-inertial frame of references.
  - Imagine, that you are staying in a closed box (big enough, but you cannot see the outside) Consider two situations: in one the box is accelerating with a constant acceleration; in the second a planet is sitting close to the box.
  - Is there any way for you do distinguish between these?
  - Because the inertial (third law) and gravitational (Newton's gravity) masses are the same, there is no way to tell if your box is accelerating, or you are in the gravitational field.
  - In fact the only way to do that is to look at infinity. The gravity decays at infinity, but the "fake" force in the accelerated frame does not.
  - This suggests, that the gravity is equivalent to the local non-inertial frame. This translates to the local curvature of the space-time.
- Space-time metric depends on the distribution of mass. In free space the metric is Minkowski metric.
- The action of a free particle in the curved space-time is the total interval of the path computed in the metric of the curved space-time. When curvature of the space-time is small (mass densities are small), the action will look like the motion in Newtonian gravity.
- For large masses/densities the curvature is large and the gravity is very different from the Newtonian.
- Black holes `https://en.wikipedia.org/wiki/Black_hole`. Schwarzschild radius: $R_s = \frac{2GM}{c^2}$.

- Gravitational lensing `https://en.wikipedia.org/wiki/Gravitational_lens`.
- Gravitation waves: The gravitational field has its own dynamics $\longrightarrow$ the waves of gravity can propagate! `https://en.wikipedia.org/wiki/Gravitational_wave`

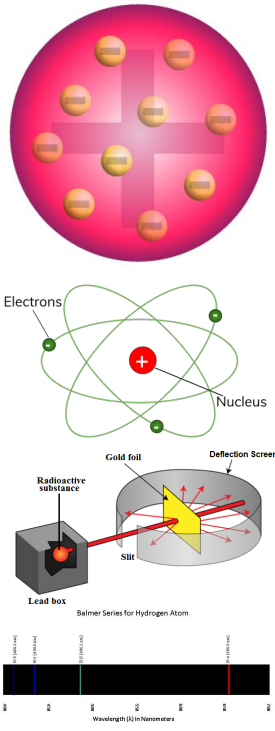
LECTURE 21
Problems with classical theory.

- Homework.
- Evaluations begin today, and end on August 8.

21.1. Waves vs stream of particles

- Common features:
  - Energy flux. Amount of energy which crosses a unit area per unit time.
  - Momentum flux. Amount of momentum which crosses a unit area per unit time
- Difference
  - Waves: diffraction and interference, or, in one word, phase.
  - Particles: number of particles.

21.2. Particles are waves.



- Atom stability. In the classical theory if an electron orbits the positively charged center the electron, as it moves with acceleration, will emit e.-m. waves with the frequency equal to the frequency of the electron's motion. By emitting the e.-m. wave the atom will lose the energy. As the electron loses the energy its orbit must shrink until it reaches the central nucleus. At this point the atom collapses. One can compute the time it takes for an atom to collapse: $t \approx 2 \times 10^{-11}$s, `http://www.physics.princeton.edu/~mcdonald/examples/orbitdecay.pdf`
  - Plum Pudding model. (Proposed by J. J. Thomson in 1904)
  - Rutherford experiment. (1911)
- Atomic spectra (see more in `https://en.wikipedia.org/wiki/Emission_spectrum`). In the classical theory an electron orbiting a nucleus should emit e.-m. waves of all frequencies. In other words the spectrum of emitted light should be continuous. However in the experiment the spectrum is discrete – consists of several sharp spectral lines.

- If an atom emits light the electron must change its state/orbit. The fact, that there is only discrete and very sharp lines means that there must be only discrete states/orbits allowed for an electron.
- It looks as if there is some sort of diffraction for electrons. So that only few states are available for an electron — the same as for a wave in a resonator. In particular it would mean that the electron cannot collapse on the nucleus. And when electron transition from one state to another it emits light of a very specific frequency.

21.3. Waves are particles.

21.3.1. Black body radiation.

More information can be found here: `https://en.wikipedia.org/wiki/Black_body`

A black body in physics means a body which absorbs all light that is coming at it. It does not mean that it cannot emit light. However, the light emitted by such a body has a very universal spectrum which depends only on the Temperature of the body (it was first recognized by Kirchhoff in 1860).

- Spectrum is intensity of light at each frequency. One measures it by filtering the emitted light through a filter which allows only the light of some particular frequency (in a small frequency interval around that frequency) through. Then one measures the intensity of the filtered light.
- Intensity is energy crossing a unit area per unit time.

Using a bit of Maxwell equations and statistical mechanics one can compute this spectrum. The results of this calculation is very strange

$$\tilde{u}_{\text{classical}}(\lambda, T) = \frac{8\pi k_B T}{\lambda^4}$$

(shown on the figure as a black line).

- It keeps going up forever as wavelength decreases, of frequency increases $f = c/\lambda$. So any black body should emit a lot of X-rays, γ-rays, etc.
- Even worse, the total integral under this curve diverges! The total integral is the total energy emitted per unit area, per UNIT TIME. So in one second a black body should emit infinite energy!

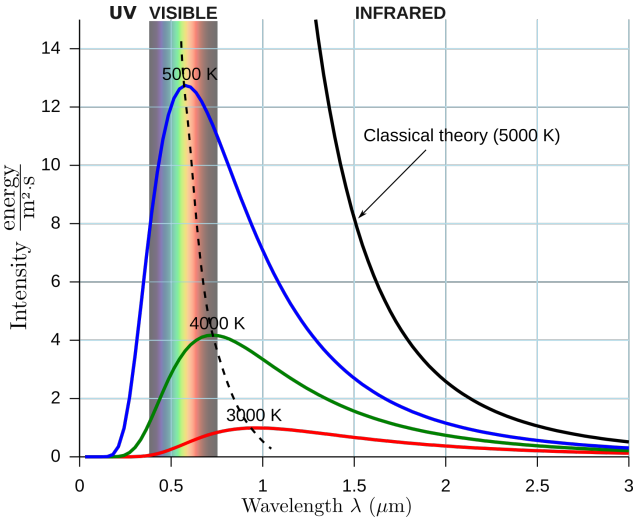
The problem was so severe, that it was called the Ultraviolet Catastrophe. It was recognized already by Kirchhoff in 1859.

But the experiment obviously does not show any of this. The picture shows the real intensity of the black-body radiation. The most important observations:

- it is indeed universal — the curves for different temperature can be rescaled into each other.
- The intensity goes to zero when $\lambda \to 0$ (or $f \to \infty$)!
- Has a maximum, which shifts with temperature.

$$\lambda_{\max} T = 2.898 \times 10^{-3} m \cdot K.$$

The fact that the maximum exists means that there is some internal energy scale in the problem. Nothing in the Maxwell equations suggests of such energy scale.



Planck's semi-empirical formula:

$$u(f,T)=\frac{8\pi hf^3}{c^3}\frac{1}{e^{hf/k_BT}-1},\qquad \tilde{u}(\lambda,T)=\frac{8\pi hc}{\lambda^5}\frac{1}{e^{hc/\lambda k_BT}-1}$$

where h is the Planck constant

$$h=6.6\times10^{-34}J\cdot s.$$

Often used $\hbar=\frac{h}{2\pi}$. See `https://en.wikipedia.org/wiki/Planck_constant`.

Notice, that if $hc\ll\lambda k_BT$, or $\lambda\gg hc/k_BT$ then

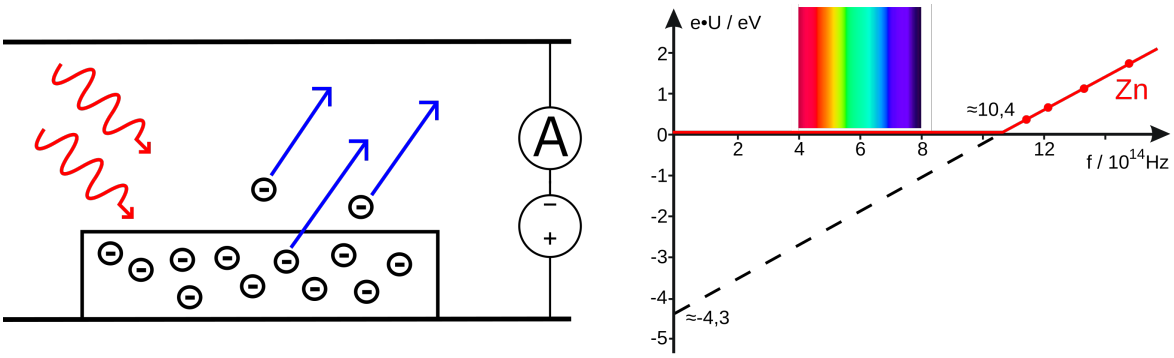
$$\tilde{u}(\lambda,T)\approx\tilde{u}_{\text{classical}}(\lambda,T).$$

- Planck's formula suggests that light consists of particles with energy

$$\epsilon=hf=\hbar\omega$$

each. (Planck 1900)

21.3.2. Photo-electric effect.



One shines light on a metal plate. The metal plate is on one plate of a parallel plate capacitor, see figure. We can apply voltage to the plates of the capacitor and measure current. We also can control the frequency/wavelength and intensity of the light.

- For this experimental set up
  - If no light shines, there will be no current.
  - If we shine the light, the light may kick some electrons out of the metal plate. Then if we apply the negative terminal of the battery (opposite to what is show on the figure) to the plate with the metal, then if electrons are kicked out by the light we will measure the current. The current is proportional to the number of electrons "kicked out" by the light per second.
  - If we now apply the positive battery terminal to the metal plate only the electrons which have large enough kinetic energy to overcome the electric field inside the capacitor will reach the upper plate. This way measuring the current we will be able to measure the kinetic energy of the "kicked out" electrons.
- Classical expectations:
  - Classically the light is a wave, hence the energy of the "kicked out" electrons should only depend on intensity — the electric field in light. One can think of it as a wave in the ocean coming to a send beach. It hits the send and the velocities of the send particles will depend on how big the wave is (or how large the intensity is).
  - Classically the number of kicked out electrons should depend on how often we kick them i.e. of the light frequency. The number of flying send particle at a given waves' intensity, should depend on how often the waves hit the beach — on frequency of the waves.
- However, the experiment shows exactly the opposite:
  - The number of knocked out electrons (per unit time) depends on intensity and does not depend on frequency.
  - the energy E of the knocked out electrons does not depend on the light's intensity at all. It only depends on light's frequency ω,

$$E = \begin{cases} 0, & \text{if } \hbar\omega < \mathcal{A} \\ \hbar\omega - \mathcal{A}, & \text{if } \hbar\omega > \mathcal{A} \end{cases} .$$

    With the same constant $\hbar$ as in the Planck's formula!
  - The threshold energy $\mathcal{A}$ does not depend on light and only depends on the material.

This was another puzzle which does not match with classical theory.

The explanation was given in 1905 by A. Einstein (he got a Nobel prize for it in 1921) in the paper "On a Heuristic Point of View about the Creation and Conversion of Light".

- The light is a stream of "particles of light".
- Each particle has the energy $\hbar\omega$.
- The intensity of the light is how many light particles crosses a given cross-section in a given time — particle flux in the stream.
- An electron absorbs the light particle and acquires the energy $\hbar\omega$. It has to overcome the crystal attraction, it loses the energy $\mathcal{A}$ by doing so. So it emerges outside with the energy $E = \hbar\omega - \mathcal{A}$.
- The number of kicked out electrons (if they can overcome the attraction) depends on how many light particles hit the material per unit time — the light intensity.

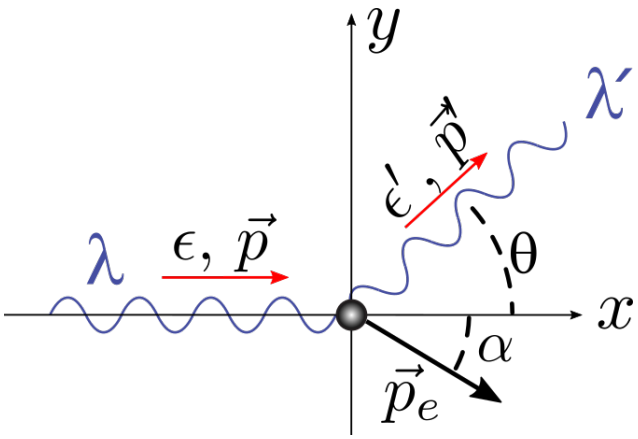
Einstein's explanation was absolutely heuristic. He did not say why, or what those light particles were. He only said that if light is a stream of particles, then the photoelectric effect has a very natural explanation.

But everyone new and could prove, that the light is a wave!

Current uses:

- Solar batteries.
- ARPES.

21.3.3. Compton scattering



This is the scattering of X-rays off almost free electrons. Compton earned the 1927 Nobel Prize for the discovery, the experiment was done in 1923.

- Experiment: X ray of large energy, electrons are free.
- The incoming X-ray has wave-length λ,
- The scattered X-ray were observed at different angles and their wavelength λ' was measured.
- The results of the experiment were well described by the formula

$$\lambda' - \lambda = \lambda_e(1 - \cos\theta), \qquad \lambda_e \approx 2.4 \times 10^{-12} m$$

where angle θ is the angle of the scattered light.

This was again contrary to what one would expect from the classical waves.

- Energy and momenta are conserved.
- In a classical wave the intensity is the measure of the electric field.
- Electrons interact with the electric field.
- One concludes that the larger the intensity the more electron will accelerate.
- The more it accelerates the more energy it takes from the light.
- The more energy light uses the less intense it would become.
- There should be change of frequency due to relativistic Doppler effect only.

This is clearly contradicts to the observed experiment. Where does the length λ_e come from? There is nothing of this kind in the Maxwell equations.

However, lets assume, that the light is a stream of particles. A single scattering event is a scattering of a light particle of an electron. Energy and momentum are conserved.

- We take the light particle to have energy
$$\epsilon = fh = ch/\lambda.$$
- Maxwell equations (or the theory of relativity) tell us that the momentum of such a particle must be
$$p = \epsilon/c = h/\lambda.$$
- Then momentum and energy conservation laws give
$$\vec{p} = \vec{p}_e + \vec{p}', \qquad \epsilon = \epsilon' + \frac{p_e^2}{2m}$$
where m is the mass of the electron.
- Writing momentum conservation in components we get:
$$\begin{aligned} x\text{-component:} \qquad & \tfrac{h}{\lambda} = \tfrac{h}{\lambda'}\cos(\theta) + p_e\cos(\alpha) \\ y\text{-component:} \qquad & 0 = \tfrac{h}{\lambda'}\sin(\theta) - p_e\sin(\alpha) \\ \text{energy:} \qquad & \tfrac{ch}{\lambda} = \tfrac{ch}{\lambda'} + \tfrac{p_e^2}{2m} \end{aligned}$$
Expressing p_e^2 from the first two equations and using it in the third we get
$$\frac{1}{2m}\left(\frac{h^2}{\lambda^2} + \frac{h^2}{\lambda'^2} - 2\frac{h^2}{\lambda\lambda'}\cos(\theta)\right) = \frac{ch}{\lambda} - \frac{ch}{\lambda'}.$$
In the case $\lambda \approx \lambda'$ (more precisely $\lambda' - \lambda \ll \lambda$) this simplifies to
$$\lambda' - \lambda = \frac{h}{cm}(1 - \cos\theta), \qquad \text{so} \quad \lambda_e = \frac{h}{cm}.$$

The mysterious λ_e appears to have the same Planck's constant h! And its numerical value coincide with the experiment!

- Surprisingly many of the puzzling experimental results can be explained by considering particles as waves and waves as particles.
- All heuristic explanations have the same(!) Planck's constant h. The units of h is energy $\cdot$ sec.
- There is no fundamental parameter of the meaning, units, and numerical value anywhere in the classical physics.

Historical facts:

- J. J. Thomson's measurement of e/m — 1897. Cathode ray deflected by magnetic field experiment established that electricity is carried by particles.
- Einstein's photo-electric explanation — 1905.
- Robert Millikan measured e — 1909, oil drop experiment.
- Rutherford's experiment — 1911, α particles scattered of a gold film, experiment established the structure of an atom.
- Bregg's reflection — 1912, coherent X-rays scattered by a crystal experiment established crystal structure.

LECTURE 22
Beginnings of the Quantum Mechanics.

- Evaluations end on August 8.

22.1. What we learned so far:

From the black body radiation, photo-electric effect and Compton scattering:

- Light is a stream of particles.
- The energy of each particle depends only on frequency

$$\epsilon = \hbar\omega, \qquad \text{or} \qquad \epsilon = hf.$$

- Intensity of light is how many particles cross a given area during a given time – the current of particles.
- The particles move with the speed of light.
- From Maxwell equations we know that for such particles the dispersion relation is $\epsilon = pc$.
- So a light particle — a photon — has the momentum

$$p = \frac{\epsilon}{c} = \frac{\hbar\omega}{c} = \frac{hf}{c}.$$

Or, as the wave-length λ is given by $\lambda = \frac{c}{f}$

$$p = \frac{h}{\lambda}.$$

From this analysis a new fundamental constant appeared — the Planck constant — h or $\hbar = \frac{h}{2\pi}$.

$$h \approx 6.62607015 \times 10^{-34}\text{J·s} = 4.135667696... \times 10^{-15}\text{eV·s}$$
$$\hbar \approx 1.054571817 \times 10^{-34}\text{J·s} = 6.582119569 \times 10^{-16}\text{eV·s}$$

There is no $\hbar$ (or h) in classical physics!

This picture of light as a stream of particles (photons) is used in many technological advances

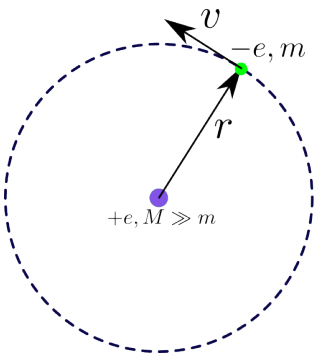
- Lasers.
- Solar batteries.
- ARPES.

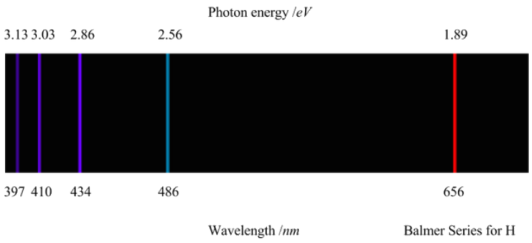
However, we still have not answered the following questions:

- Why atoms are stable? After Rutherford experiments (1911) we know the structure of the atoms. But according to classical mechanics such atoms could not be stable.
- Why atomic emission spectrum is discrete?

22.2. Bohr atom (1913).

We want to consider how atoms emit the light. Remember, the light is a bunch of particles, it is also a wave.





We consider the simplest of atoms — the hydrogen atom.

- Electron of charge $-e$ and mass m is moving around a center — nucleus — with the charge $+e$ and mass M. (Mass of the nucleus is much larger than the mass of the electron $M \gg m$.)
- There are two conserved quantities: energy E, and angular momentum L. We want to express energy in terms of angular momentum.
  - For a circular orbit we have
$$\frac{ke^2}{r^2} = \frac{mv^2}{r} \quad \Longrightarrow \quad mv^2 = \frac{ke^2}{r}$$
  - The angular momentum

$$L = mvr, \qquad \text{or} \qquad L^2 = m^2v^2r^2 = ke^2mr, \qquad r = \frac{L^2}{mke^2}, \qquad v = \frac{L}{mr} = \frac{ke^2}{L}$$

  - Energy and frequency $\omega = 2\pi f = 2\pi\frac{v}{2\pi r}$ can now be expressed in terms of angular momentum:
$$E = \frac{mv^2}{2} - \frac{ke^2}{r} = -\frac{1}{2}\frac{k^2e^4m}{L^2}, \qquad \omega = \frac{v}{r} = \frac{mk^2e^4}{L^3}.$$
- According to Maxwell the frequency of light emitted by a hydrogen atom must equal to the frequency of the rotation of the electron — this is "light as a wave" picture.
- When light is emitted the electron must change its orbit so the total energy of electron + light is conserved. The energy of the emitted "Einstein photon" $\hbar\omega$ must be equal to the difference in the energies of the two electron's states — the total energy is conserved! This is the "light as a particle" picture.
- Assume that the change of the electron's energy is small.
$$dE = \frac{dE}{dL}dL = \frac{k^2e^4m}{L^3}dL = \omega dL$$
(in fact $\omega = \dot{\phi} = \frac{\partial H(L,\phi)}{\partial L}$ – Hamiltonian equation.)

- This change of electron's energy dE must be equal to the energy of the emitted photon $\hbar\omega$. We then have

$$dE = \omega dL = \hbar\omega, \quad \text{or} \quad dL = \hbar.$$

- So the angular momentum L can only change by $\hbar$.

$$L = \hbar n + L_0, \qquad n = 1, 2 \dots$$

Assuming $L_0 = 0$ we get

$$L = \hbar n, \qquad n = 1, 2 \dots$$

These are the only "allowed" values of angular momentum. Using these we can compute

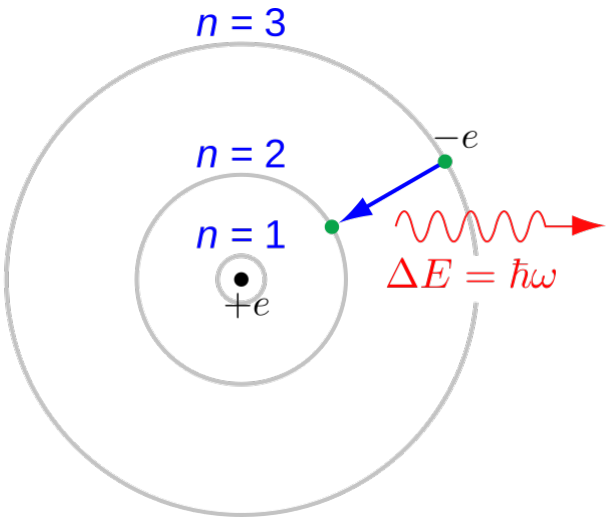
$$E_n = -\frac{1}{2}\frac{k^2 e^4 m}{\hbar^2}\frac{1}{n^2} = -\frac{13.6}{n^2}\text{eV}, \qquad r_n = \frac{\hbar^2}{mke^2}n^2 = a_B n^2, \quad a_B = 0.0529\text{nm}.$$

These are the only "allowed" values of energy, and sizes of the atom.

$$E_n = -\frac{1}{2}\frac{k^2 e^4 m}{\hbar^2}\frac{1}{n^2}$$

$$r_n = \frac{\hbar^2}{mke^2}n^2$$

$$L_n = \hbar n$$



From this picture it is not clear why there is only a discrete set of "allowed" energies for an electron. The calculation only states that in order for the "light as a wave" and "light as a particle" pictures to be consistent an atom must have only a discrete set of "allowed" energies. Simultaneously, this picture of an atom explains the two puzzles: the stability of the atom — there is a minimal "allowed" energy; the discrete atomic spectra — the light is emitted when an electron transitions from one "state" to another with lower energy

$$\hbar\omega = E_n - E_{k<n}.$$

So this spectrum is discrete and describes the experimental observation very well.

- Still the question remains: what is the nature of the electrons (and hence all other particles) that would lead to the above result.

22.3. de Brolie's idea (1924).

Let's return back to photons.

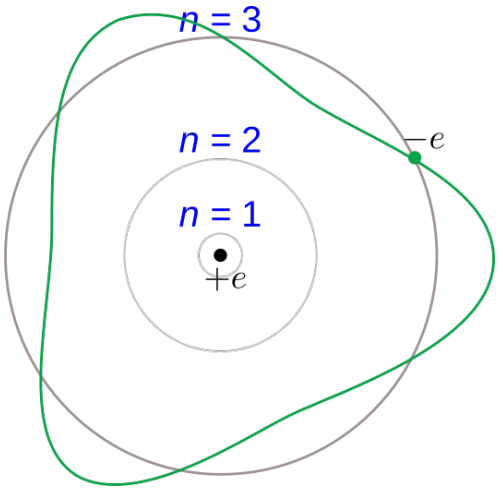
- They are particles of light with energy $\epsilon = \hbar\omega = hf$ per particle.
- As they are moving with the speed of light we know that $f\lambda = c$, or $f = c/\lambda$. So we can express the photon's energy as
$$\epsilon = \hbar\omega = hf = \frac{hc}{\lambda}.$$
- Due to Lorenz invariance we also know that for any particle moving with the speed of light (only particles with zero rest mass can do that) we must have
$$\epsilon = cp,$$
where p is the momentum of the particle.
- Comparing the two expressions for the particle's energy we see, that
$$cp = \frac{hc}{\lambda}, \quad \text{or} \quad p = \frac{h}{\lambda}.$$

Now let's see what the Bohr's model of atom tells us.

- According to Bohr
$$L = pr = n\hbar, \qquad \text{or} \qquad 2\pi rp = nh.$$
- If we now assume that the electron is a wave with the wavelength $\lambda = \frac{h}{p}$ — exactly the same as for the photons, then the Bohr quantization rule becomes
$$\frac{2\pi r}{\lambda} = n,$$
which is the condition for the constructive interference.

$L_n = \hbar n$

$\lambda = h/p$



- From this point of view the Bohr's quantization is very natural.

So the de Brolie's idea was that EVERY particle with energy ϵ and momentum p is a WAVE with frequency ω and wavelength λ (or wave number k) given by
$$f = \frac{\epsilon}{h} \quad \text{or} \quad \omega = \frac{\epsilon}{\hbar}, \qquad \lambda = \frac{h}{p} \quad \text{or} \quad k = \frac{p}{\hbar},$$
where, I remind $k = \frac{2\pi}{\lambda}$ is the wave number.

22.4. Particles are Waves!

Wave of what? This question is asked very often. Somehow, when we talk about particles no one asks "particle of what?" For example, if I say "electron is a particle" no one asks "particle of what?" however, if I say electron is a wave the standard question is "wave of what?". The reason for this is clear, when we say something is a particle we imply certain properties – trajectories or time evolution, momentum, interactions with other elements etc. This intuitive understanding of what to expect from "a particle" makes the question "particle of what?" irrelevant, because:

- Given the system and the initial state (initial position and velocity) we know (intuitively, or we can compute) the particle's state in a later time.
- Given the state (position and velocity) of a particle in we know (intuitively, or we can compute) what will be the result of different observations — we say that we know observables.

So the question "wave of what?" in fact means "what properties does this wave have?", or "what do we measure?" and "How does it evolve with time?" and "How does it interact with other elements?". These questions are non-trivial and are the central question of the quantum mechanics.

So the question "wave of what?" boils down to two questions

- Given the system and the initial wave how the wave will involve in time?
- If we know the wave at some moment of time how do we know/compute observables?

The question of what to measure we will address later.

The question we ask is if we know the wave at the initial time and we have a description of our system what will be that wave at a later time? (Notice, that this is exactly the same as with particles: if we have initial conditions and have a description of our system what will be the position of the particle at a later time?)

Time evolution of a wave should be described by a wave equation. The major tool for finding this wave equation is the realization, that classical mechanics works well and our new description of the world should not contradict the classical mechanics where classical mechanics works well.

LECTURE 23
Particles as waves. The Schrödinger equation.

- Evaluations end on August 8.

This lecture presents a series of plausible arguments which lead to the Schrödinger equation. By no means it is a "proof" of the Schrödinger equation or Quantum mechanics. The only proof of the matter that can exist is experiment. Quantum mechanics has been thoroughly tested and is been tested all the time. We know that it works!

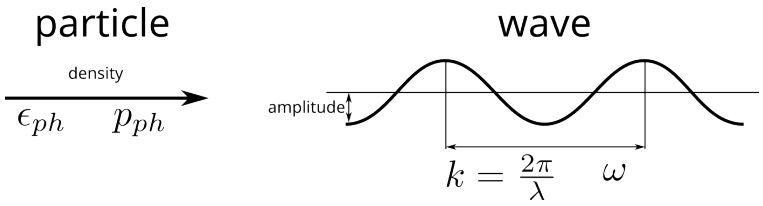
23.1. Particles and waves, approach to Quantum Mechanics.

In order to describe particles as waves we need:

- Find the equation which will give us the time evolution of a wave.
- Find how to compute observables if we know the wave.

In the limit of large masses, energies, momenta the new mechanics must reproduce the Newtonian mechanics.

23.1.1. Light.



- Maxwell equations tell us that light is a wave.
- Planck/Einstein idea was that light is a stream of particles — photons.

Let's see how particle-wave duality works for photons.

Notice, that the Maxwell equations in vacuum is a wave equation which give the time evolution of el.-m.

- The energy of each photon is

$$\epsilon_{ph} = \hbar\omega_{ph} = hf_{ph}.$$

- The photon's dispersion relation from the Maxwell equations/Lorenz invariance must be

$$\epsilon_{ph} = cp_{ph}.$$

- This gives us the relation between the e.-m. wave's wave number $k_{ph} = \frac{2\pi}{\lambda_{ph}}$ and the photon's momentum

$$p_{ph} = \frac{\epsilon_{ph}}{c} = \frac{hf_{ph}}{c} = \frac{h}{\lambda_{ph}} = \hbar k_{ph}.$$

- So for photon as a particle that have energy ϵ_{ph} and momentum p_{ph} we have

$$\omega_{ph} = \frac{\epsilon_{ph}}{\hbar}, \quad k_{ph} = \frac{p_{ph}}{\hbar}.$$

- Intensity of light is how many particles cross a unit area per unit time. According to Maxwell equations this is proportional to the speed of light times average of el.-m. wave amplitude squared $I \sim c\overline{\vec{E}^2}$.
- As each photon moves with the speed of light it means that in the picture of photons the density of photons in the stream is the average of square of the wave's amplitude.

Let us summarize:

- A particle — a photon — is also a wave.
- If we talk about a photon as a particle, then it has energy ϵ_{ph} and momentum p_{ph}.
- If we talk about photon as a wave it has frequency ω_{ph} and wave number k_{ph}.
- These two descriptions are related by

$$\epsilon_{ph} = \hbar\omega_{ph}, \qquad p_{ph} = \hbar k_{ph}.$$

- The intensity of light is proportional to $c\overline{\vec{E}^2}$ in terms of waves and to the density of photons in terms of particles.

Now if you think about light as a stream of particles (photons) the only difference between these particles and any other particles is the dispersion relation:

- for photons it is given by the Lorenz invariance $\epsilon_{ph} = cp_{ph}$;
- for a classical and non-relativistic particle it is given by Newton's laws $\epsilon = \frac{p^2}{2m}$.

23.1.2. Arbitrary particles.

According to de Brolie a particle is a wave

- A particle has a momentum p_e and energy ϵ_e.
- According to de Brolie a particle as a wave has a wavelength

$$\lambda_e = \frac{h}{p_e}.$$

- So for the wave number we have

$$k_e = \frac{p_e}{\hbar}$$

- Noticing that this is the same expressions as for the photons we can make a hypotheses that:
  - A particle wave has a frequency

$$\omega_e = \frac{\epsilon_e}{\hbar}.$$

  - An average of the square of the waves amplitude give the density of particles!

– The equation which gives the time evolution of a free particle wave must give the correct dispersion relation

$$\epsilon_e = \frac{p_e^2}{2m}, \qquad \omega_e = \frac{\hbar}{2m} k_e^2$$

So we need to look for a wave equation which will give us the correct dispersion relation!

23.2. The wave equation.

We want to find the simplest possible way to construct the description of particles as waves. If a particle has kinetic energy E, then the wave must have the frequency $\omega = E/\hbar$. If the particle's momentum is p, then the wave's wave number must be $k = p/\hbar$.

As we want to consider only the simplest case — the free particle, the wave equation must give the correct dispersion relation $E = \frac{p^2}{2m}$.

Another very important requirement comes from the fact that the average square of the wave amplitude gives the density of particles. We want to find the wave equation which will give us the uniform density (for free particles!) or the square of the wave amplitude must not depend on the coordinate.

23.2.1. Wave equation in zero dimensional world.

An oscillator.

$$\ddot{f} + \omega^2 f = 0$$

There are two linearly independent solutions

$$f_1(t) = \cos(\omega t) \qquad \text{and} \qquad f_2(t) = \sin(\omega t).$$

Any linear combination of these is also a solution. In particular

$$f(t) = \cos(\omega t) + i \sin(\omega t) = e^{i\omega t}$$

is a solution. This solution has the property that

$$|f|^2 = 1$$

at all times.

We can look at this oscillator as a zero dimensional wave.

Notice, that in order to have the "uniform" solution we had to use the complex function!

23.2.2. Wave equation in $1D$.

In $1D$ the standard wave equation is

$$\frac{\partial^2 f}{\partial t^2} - v^2 \frac{\partial^2 f}{\partial x^2} = 0$$

The simplest solutions are

$$f_+(x,t) = e^{i\omega x/v + i\omega t}, \qquad f_-(x,t) = e^{i\omega x/v - i\omega t}$$

for any ω. Any linear combination of these functions is also a solution.

- Both solutions describe the waves propagating with the velocity v.
- The velocity does not depend on ω.

- The period and the wavelength of the both waves are
$$T = \frac{2\pi}{\omega}, \qquad \lambda = \frac{2\pi v}{\omega} = Tv.$$
- f_+ propagates to the left, f_- propagates to the right.
- Both solutions have the property that
$$|f_\pm|^2 = 1$$
at all times and everywhere in space.

We still have two problems:

(a) our wave equation describes the propagation both to the left and to the right, a particle should propagate only one way;

(b) all the waves propagate with the same velocity independently of the wave number (in the particle picture this means that the particle's velocity is independent of momentum)

We need to take care of these two problems. Let's start with the first one.

The wave equation can be written as
$$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - v\frac{\partial}{\partial x}\right) f = 0$$
Looking at each factor separately we see that
$$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right) f_- = 0$$
$$\left(\frac{\partial}{\partial t} - v\frac{\partial}{\partial x}\right) f_+ = 0$$
So the equation
$$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right) f = 0$$
describes a wave propagating to the right only. Notice, that the reason for that is that the equation is first order in time derivative. This is good, but it still all the waves propagate with the same velocity v.

Notice again, that in order to have the "uniform" solution we had to use the complex function f!

23.3. Schrödinger equation (1926).

The propagation of the electromagnetic wave of frequency ω and wave number k is given by $e^{ikx-i\omega t}$. For the el.-m. wave the velocity is always c, so $\omega = ck$. For matter wave we do not have such a restriction. However, for the both el.-m. and matter waves we have $p = \hbar k$ and $E = \hbar\omega$, so we write
$$\Psi(x,t) = e^{ikx-i\omega t} = e^{ipx/\hbar - iEt/\hbar}$$
For a classical particle we must have $E = \frac{p^2}{2m}$, the wave Ψ then must satisfy the following equation
$$\left[i\hbar\frac{\partial}{\partial t} - \frac{1}{2m}\left(-i\hbar\frac{\partial}{\partial x}\right)^2\right]\Psi = 0.$$

(one can check that the function $\Psi(x,t) = e^{ipx/\hbar - iEt/\hbar}$ satisfies the above equation if and only if $E = \frac{p^2}{2m}$.)

Let's write this equation in the form

$$i\hbar\frac{\partial\Psi}{\partial t} = \frac{1}{2m}\left(-i\hbar\frac{\partial}{\partial x}\right)^2\Psi,$$

and let's look at the operator

$$\hat{p} = -i\hbar\frac{\partial}{\partial x}.$$

If we act on a wave function by this operator we get $\hat{p}\Psi = p\Psi$. So this is an operator of momentum. Using this notation we get

$$i\hbar\frac{\partial\Psi}{\partial t} = \frac{\hat{p}^2}{2m}\Psi.$$

Comparing this to the Hamiltonian for the free moving particle $H = \frac{p^2}{2m}$, one can write

$$i\hbar\frac{\partial\Psi}{\partial t} = \hat{H}\Psi, \qquad \hat{H} = \frac{\hat{p}^2}{2m} + U(x).$$

where we made the simplest conjecture that if the particle is not free, but instead is in potential $U(x)$, then we just substitute the full Hamiltonian instead of free particle Hamiltonian.

The operator

$$\hat{H} = \frac{\hat{p}^2}{2m} + U(x)$$

is called the Hamiltonian operator or simply the Hamiltonian. The above equation is the Srödinger equation.

- Double slit experiment (1961, 1974).

LECTURE 24
Wave function. Observables.

- Homework.
- Evaluations end on August 8.

We will work only in $1D$.

In the previous lectures we discussed that in order for the wave description to make sense we must answer two questions

(a) What is the time evolution of a wave in a given system? What it means is that if we have a description of our system — the Hamiltonian, and the wave function at the initial time $\Psi(x, t = 0)$, then we must be able to say what will be the wave function at any later time $\Psi(x, t > 0)$.

(b) How do we compute the observables if we know the wave function $\Psi(x, t)$? For example, if we know the wave function of an electron at some moment of time and we measure the position of the electron how do we compute the result of such a measurement? or momentum? or energy? etc.

- Of course, after it is all constructed the experiment is the judge if this description is the correct physical description.

The Schrödinger equation give the answer to the first question: What is the time evolution of the wave function. The time evolution of a wave function is given by the Schrödinger equation (we will only discuss $1D$ case.)

$$i\hbar\frac{\partial\Psi}{\partial t} = \hat{H}\Psi, \qquad \hat{H} = \frac{\hat{p}^2}{2m} + U(x), \qquad \int_{-\infty}^{\infty} |\Psi(x,t)|^2 dx = 1.$$

We also need the initial state — $\Psi(x, t = 0)$. The Hamiltonian $\hat{H}$ describes our system. If we know the initial state and the Hamiltonian then the Schrödinger equation gives the state $\Psi(x, t)$ at any later time.

- Mathematically, the last condition defines the space of functions we are working on. It is called L^2, or Hilbert space. The condition $\int_{-\infty}^{\infty} |\Psi(x,t)|^2 dx = 1$ is called normalization condition, or just normalization.
- NORMALIZATION!! You must normalize the wave functions. Always.

This lecture we will answer the second question: How to know/compute the observables if we know the wave function $\Psi(x, t)$.

24.1. Wave function.

- Interpretation. Probability Density Amplitude.
  - Average density of photons is proportional to the intensity of light $\sim \overline{\vec{E}^2}$.
  - Total number of photons $\sim \int_{-\infty}^{\infty} \vec{E}^2 dx$.
  - Total number of particles $\int_{-\infty}^{\infty} |\Psi(x,t)|^2 \, dx$.
  - As we consider only one particle, then we must have
$$\int_{-\infty}^{\infty} |\Psi(x,t)|^2 \, dx = 1$$
at all times.
  - One can look at this statement from the "particles" point of view. As we have only one particle me have to interpret $|\Psi(x,t)|^2 \, dx$ as the probability to find the particle in the interval $(x, x+dx)$ at time t.
  - With this interpretation the statement
$$\int_{-\infty}^{\infty} |\Psi(x,t)|^2 \, dx = 1$$
means that the probability to find the particle SOMEWHERE is 1.
- Srödinger equation is linear in Ψ and homogeneous, so Ψ is defined up to a multiplicative factor — its overall amplitude.
- This means that the overall factor must be computed from the normalization of the wave function.
- But the integral $\int_{-\infty}^{\infty} |\Psi(x,t)|^2 \, dx$ must exists, or, in other words, converge.
- This normalization condition must be independent of time if the number of particles is conserved in the system.
- NORMALIZATION!! You MUST normalize the wave functions. ALWAYS.

24.2. Wave function. Observables.

- Coordinate. The probability to find a particle in the interval $(x, x+dx)$ is $|\Psi(x,t)|^2 dx$. So the average result of the measurement of the particle position is given by:
$$\bar{x}(t) = \int_{-\infty}^{\infty} |\Psi(x,t)|^2 x dx = \int_{-\infty}^{\infty} \Psi^*(x,t) x \Psi(x,t) dx = \int_{-\infty}^{\infty} \Psi^*(x,t) \hat{x} \Psi(x,t) dx, ,$$
where $\hat{x}$ is an operator of coordinate — it just multiplies the function it acts on by x: $\hat{x}\Psi(x,t) \equiv x\Psi(x,t)$.
- Momentum. Momentum operator $\hat{p} = -i\hbar \frac{\partial}{\partial x}$, so the average momentum is
$$\bar{p} = \int_{-\infty}^{\infty} \Psi^* \hat{p} \Psi dx.$$
It will always be real! despite the fact that there is i in the definition of $\hat{p}$. In order to see it, let's compute $\bar{p}^*$.

$$\bar{p}^* = \left(\int_{-\infty}^{\infty} \Psi^* \hat{p} \Psi dx \right)^* = \left(\int_{-\infty}^{\infty} \Psi^* \left(-i\hbar \frac{\partial}{\partial x} \right) \Psi dx \right)^* = i\hbar \int_{-\infty}^{\infty} \Psi \frac{\Psi^*}{\partial x} dx = -i\hbar \int_{-\infty}^{\infty} \Psi^* \frac{\Psi}{\partial x} dx = \bar{p},$$

(I used integration by parts in the fourth equality.) So $\bar{p}^* = \bar{p}$, it means that $\bar{p}$ is real.

- Energy. Energy is the average value of the Hamiltonian, so
$$\bar{E} = \int_{-\infty}^{\infty} \Psi^* \hat{H} \Psi dx.$$
- For any other measurable property $\mathcal{O}$ there is a corresponding quantum mechanical operator $\hat{\mathcal{O}}$. The result of the measurement is the operator average
$$\bar{\mathcal{O}} = \int_{-\infty}^{\infty} \Psi^* \hat{\mathcal{O}} \Psi dx$$
- If one measures average $\overline{\mathcal{O}} = \int_{-\infty}^{\infty} \Psi^* \hat{\mathcal{O}} \Psi dx$ and $\overline{\mathcal{O}^2} = \int_{-\infty}^{\infty} \Psi^* \hat{\mathcal{O}}^2 \Psi dx$ then in general
$$\overline{\mathcal{O}^2} \neq \overline{\mathcal{O}}^2.$$
This is quantum noise.
- Noise. The good (but not the only) measure of Quantum mechanical noise in the measurement $\hat{\mathcal{O}}$ is
$$(\Delta\mathcal{O})^2 = \overline{\left(\hat{\mathcal{O}} - \bar{\mathcal{O}}\right)^2} = \overline{\mathcal{O}^2} - \bar{\mathcal{O}}^2$$

Generalities.

- What operator average is measured by a given experiment is always the first question in analysis of the experiments in quantum mechanics.
- Notice, that the wave mechanics itself is fully deterministic: If we know the Hamiltonian and we know the wave function in the initial moment, then we know the wave function at any later moment unambiguously.
- The quantum noise or uncertainty appears ONLY at measurement, when we "project" the quantum world onto classical measurements.

24.3. Particle-wave equivalence

- Particles as waves.
- Heisenberg uncertainty principle: $\Delta x \Delta p \geq \hbar/2$ (Heisenberg 1927).
  - We try to localize a free particle withing an interval Δx.
  - The boundary conditions demand $\frac{\Delta x}{\lambda} = n/2$. As $p = \frac{2\pi\hbar}{\lambda}$, for the lowest momentum ($n = 1$) we have $p\Delta x = \pi\hbar$.
  - We see, that we cannot localize a particle without changing its momentum!
  - A particle which is localized in a small volume will have large(!) kinetic energy.
- Waves as particles: Notice, that the wave function $e^{\frac{i}{\hbar}(px - Et)}$ can be written as $e^{\frac{i}{\hbar}\int(p\dot{x} - E)dt} = e^{\frac{i}{\hbar}S}$, where S is the classical Action (R. Feynman 1948).
- To classical. If we take $\hbar \to 0$, then only the stationary point (minimum) of the Action will contribute, so the trajectory of the classical particle is the one which is given by the minimum of the Action! This is the Hamilton principle!

LECTURE 25
Time independent Schrödinger equation.

- Evaluations end on August 8.

25.1. What we have learned so far.

We will work only in $1D$.

- A particle is described by a wave function $\Psi(x,t)$.
- The time evolution of the wave function is given by Schrödinger equation

$$i\hbar\frac{\partial\Psi}{\partial t}=\hat{H}\Psi, \qquad \hat{H}=\frac{\hat{p}^2}{2m}+U(x,t), \qquad \int_{-\infty}^{\infty}\Psi^*(x,t)\Psi(x,t)dx=1.$$

- If we know the initial state — $\Psi(x,t=0)$ and the Hamiltonian $\hat{H}$, then the Schrödinger equation gives the state $\Psi(x,t)$ at any later time.
- If we know the particle's wave function, $\Psi(x,t)$ and we want to compute an observable $\mathcal{O}$, then we need to know the quantum mechanical operator $\hat{\mathcal{O}}$ corresponding to this observable. The average value of the observable at time t in the experiment is given by

$$\bar{\mathcal{O}}=\int_{-\infty}^{+\infty}\Psi^*(x,t)\hat{\mathcal{O}}\Psi(x,t)dx.$$

25.2. Time independent Schrödinger equation.

In the case of time independent Hamiltonian (this means that the potential energy depends only on the coordinate, and does not depend on time t) the classical problem has an exact solution discussed in the lecture 6.3.

In quantum mechanics the problem also simplifies in this case.

If the Hamiltonian does not depend on time, then we can look for the solution of the Schrödinger equation

$$i\hbar\frac{\partial\Psi}{\partial t}=\hat{H}\Psi, \qquad \int_{-\infty}^{\infty}\Psi^*(x,t)\Psi(x,t)dx=1.$$

in the form

$$\Psi(x,t)=e^{-iEt/\hbar}\psi(x),$$

Then we have

$$\hat{H}\psi = E\psi, \qquad \int \psi^*(x)\psi(x)dx = 1$$

The equation $\hat{H}\psi = E\psi$ is a second order (remember, $\hat{H}$ has $\hat{p}^2$ inside: $\hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + U(x)$), linear homogeneous differential equation. For any E it has two linearly independent solutions. However, if we are looking for the solutions that satisfy the normalization condition $\int \psi^*\psi dx = 1$, then we find that such solutions exist only for real E and in many cases only for a discrete set of E.

- Energy as an eigen-value of the Hamiltonian.
  - There is a set of functions $\psi_n(x)$ and corresponding set of numbers E_n, such that
  $$\hat{H}\psi_n(x) = E_n\psi_n(x), \qquad \int \psi_n^*(x)\psi_n(x)dx = 1.$$
  - From linear algebra we know that if $\hat{H}$ is hermitian `https://en.wikipedia.org/wiki/Hermitian_matrix`, then all E_n are real, and
  $$\int \psi_n^*(x)\psi_{n'}(x)dx = \delta_{n,n'}.$$
  - The functions $\psi_n(x)$ are called states.
- Quantum numbers = enumeration of the eigen functions.
- Eigen functions = Complete basis in the space of functions.
- Bra-ket notations $|\psi_n\rangle \equiv \psi_n(x)$, and $\langle\psi_n| \equiv \psi^*(x)$. And for any two functions $\chi(x)$ and $\xi(x)$
$$\langle\chi|\xi\rangle \equiv \int_{-\infty}^{\infty} \chi^*(x)\xi(x)dx.$$
- Normalization. $\langle\psi_{n'}|\psi_n\rangle = \int_{-\infty}^{\infty} \psi_{n'}^*(x)\psi_n(x)dx = \delta_{n,n'}$.

In the bra-ket notations the time-independent Schrödinger equation has the following form

$$\hat{H}|\psi_n\rangle = E_n|\psi_n\rangle, \qquad \langle\psi_{n'}|\psi_n\rangle = \delta_{n,n'},$$

which looks like a standard linear algebra eigen value/eigen vector equation. We then can use the whole power of linear algebra!

- The functions $\psi_n(x)$ are ortho-normal basis functions of our functional space (space of normalizable functions).
- The scalar product of any two functions $\chi(x)$ and $\xi(x)$ from our space is defined as $\langle\chi|\xi\rangle$.
- The bases $\psi_n(x)$ is complete. This means that ANY function $\xi(x)$ from our space can be presented as a liner combination of the bases functions $\xi(x) = \sum_n a_n\psi_n(x)$, where a_n are some (complex) numbers.

If at initial time we have $\Psi(x, 0)$, then we can write

$$\Psi(x,0) = \sum_n a_n\psi_n(x), \qquad \text{or} \qquad |\Psi(t=0)\rangle = \sum_n a_n|\psi_n\rangle, \qquad \text{or} \qquad a_n = \langle\psi_n|\Psi(t=0)\rangle$$

Notice, that the last equation follows from the second by a simple operation $\langle\psi_{n'}|\Psi(t=0)\rangle = \sum_n a_n\langle\psi_{n'}|\psi_n\rangle = \sum_n a_n\delta_{n,n'} = a_{n'}$.

The time evolution of an eigen function is simple

$$|\psi_n\rangle \to |\psi_n\rangle e^{-iE_nt/\hbar}$$

so

$$|\Psi(t)\rangle = \sum_n a_n e^{-iE_n t/\hbar}|\psi_n\rangle, \quad \text{or} \quad \Psi(x,t) = \sum_n a_n e^{-iE_n t/\hbar}\psi_n(x).$$

We see, that if the Hamiltonian does not depend on time the set of eigenvalues and eigenfunctions of the Hamiltonian operator solves the problem — we can compute the wave function at all times.

In particular if the initial state $\Psi(x, t = 0)$ coincides with one of the states $\psi_n(x)$, then the particle will stay in this state forever, and the only dynamics is the time-depended overall phase.

In order to compute a quantum mechanical average for some time independent operator $\hat{\mathcal{O}}$ at arbitrary time we can use

$$\bar{\mathcal{O}} = \int_{-\infty}^{\infty} \Psi^*(x,t)\hat{\mathcal{O}}\Psi(x,t)dx =$$
$$\langle\Psi(t)|\hat{\mathcal{O}}|\Psi(t)\rangle = \sum_n e^{iE_n t/\hbar} a_n^* \langle\psi_n|\hat{\mathcal{O}} \sum_m a_m e^{-iE_m t/\hbar}|\psi_m\rangle = \sum_n \sum_m e^{i(E_n - E_m)t} a_n^* \langle\psi_n|\hat{\mathcal{O}}|\psi_m\rangle a_m$$

Similar to the matrix manipulations. Numbers $\langle\psi_n|\hat{\mathcal{O}}|\psi_m\rangle$ are called matrix elements of the operator $\hat{\mathcal{O}}$.

25.3. A bit of terminology.

- The state ψ_0 of the lowest energy is called the "ground state".
- All other states are called "excited states".
- Allowed energies are called "energy levels".
- If there is only one state with some specific energy, we say the the energy level is "non-degenerate".
- If there are more then one state on a particular energy level, then we say, that the energy level is "degenerate". The number of states on an energy level is called "degree of degeneracy", or we just say "double degenerate" or "triple degenerate" for the degeneracy degrees 2 and 3 respectively.
- This also means that one should be careful with the enumeration. The enumerations of energy levels and the enumeration of the states can be different.
- Quantum numbers are the numbers that are used to enumerate the states.
- Small interaction with something external to our system (say weak interaction with electromagnetic wave in case of a charged particle) is called perturbation.
- There are two different cases time independent perturbation and time dependent perturbation.
- Time independent perturbations change the wave functions of the states and the states energies. In many cases they lift the degeneracy of the energy levels.
- A time dependent perturbation can induce transitions of the particle to another state: say a particle on the ground state can absorbed of a photon with energy $\epsilon = \hbar\omega = E_n - E_0$ and transition to a state with energy E_n. Notice, that this means that only photons with some very specific energies can be absorbed. The same is with emission. A particle in the ground state cannot emit a photon.
- A set of energy levels can be either discrete of continuous. Typically a part of spectrum is discrete and part is continuous.

- States can be either localized or extended. (Normalization of extended states is a separate problem.)
- Localized states typically have discrete energy levels. Extended states typically have continuous spectrum.

25.4. Light absorption. Transparency.

Classically, when a radio wave of frequency f hits a metal, the electrons in the metal move with the same frequency and emit the radio wave of the same frequency. This is how a radio wave is reflected by a metal. From this picture it is difficult to explain such things as transparency of the materials in a certain range of frequencies.

In quantum mechanics the picture is the following:

- Electrons in atoms, or molecules are localized.
- Their energy spectrum is discrete.
- They are in the ground state (at zero temperature).
- Light is a time-dependent perturbation.
- If the frequency of light is such, that $\hbar\omega$ equals to the energy difference between one of the excited states and the energy of the ground state, then the photons of this light will be absorbed.
- Otherwise, the photons just go through — transparency.
- After the photon was absorbed the electron of the atom/molecule is in the excited state (in fact the whole atom/molecule is in the excited state.)
- The electron (atom/molecule) in the excited state can emit a photon or some other excitation by transitioning back to the ground state.

In the rest of the class we consider several quantum mechanical problems. All of them are $1D$ and are defined by the potential. So classically all forces are conservative, then knowing the potential energy $U(x)$ we can relatively easily solve these problem. Moreover, just by looking at the form of the potential energy we can figure out how the classical motion will look like. Quantum mechanics has some surprises even for these simple problems.

LECTURE 26
Discrete spectrum. Classically prohibited region. Tunneling. Band structure.

- Evaluations end on August 8.

26.1. Time independent Schrödinger equation.

If the Hamiltonian operator $\hat{H}$ does not depend on time, then the solution of the full Schrödinger equation

$$i\hbar\frac{\partial\Psi}{\partial t} = \hat{H}\Psi$$

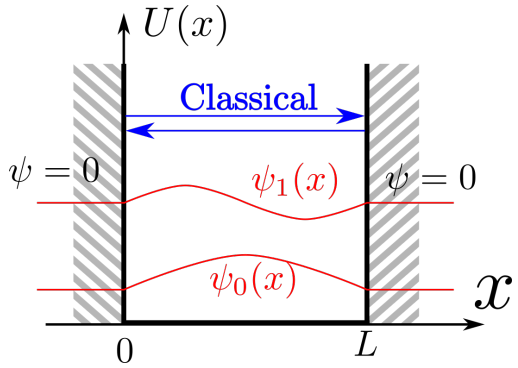
as well as all the observables can be constructed from the solution of the time-independent Schrödinger equation.

$$\hat{H}|\psi_n\rangle = E_n|\psi_n\rangle, \qquad \langle\psi_{n'}|\psi_n\rangle = \delta_{n,n'}$$

or in the language of differential equations

$$-\frac{\hbar^2}{2m}\psi_n''(x) + U(x)\psi_n(x) = E_n\psi_n(x), \qquad \int_{-\infty}^{+\infty}\psi_{n'}^*(x)\psi_n(x)dx = \delta_{n',n}.$$

26.2. Particle in the infinite square well potential.



- The potential:

$$U(x) = \begin{cases} 0 & \text{for } 0 < x < L \\ \infty & \text{for } x < 0 \text{ and } x > L \end{cases}$$

- Classical picture: any energy, particle is localized within the well.
- The time independent Schrödinger equation is

$$-\frac{\hbar^2}{2m}\psi'' + U(x)\psi = E\psi.$$

Considering three regions $x < 0$, $0 < x < L$, and $L < x$ separately we find the solution

$$\psi(x) = C\begin{cases} 0, & \text{for } x < 0 \\ \sin(kx), \quad \text{or} \quad \cos(kx), & \text{for } 0 < x < L \\ 0, & \text{for } L < x \end{cases}, \qquad \text{where } k^2 = \frac{2mE}{\hbar^2}.$$

It must be zero outside of the well, as in equation it is multiplied by a very large potential.

- Boundary conditions: the wave function must be continuous:

$$\psi(x = 0) = 0, \qquad \psi(x = L) = 0.$$

so the solutions in $0 < x < L$ are

$$\psi(x) = C\sin(k_n x), \qquad k_n L = \pi n, \quad n = 1, 2 \ldots$$

- Normalization constant C is found from

$$1 = \int_{-\infty}^{\infty} \psi^*(x)\psi(x)dx = |C|^2 \int_0^L \sin^2\left(\frac{\pi n x}{L}\right) dx = |C|^2 \frac{L}{2}, \qquad C = \sqrt{\frac{2}{L}}.$$

- As $k^2 = \frac{2mE}{\hbar^2}$ we find that the energy spectrum is discrete:

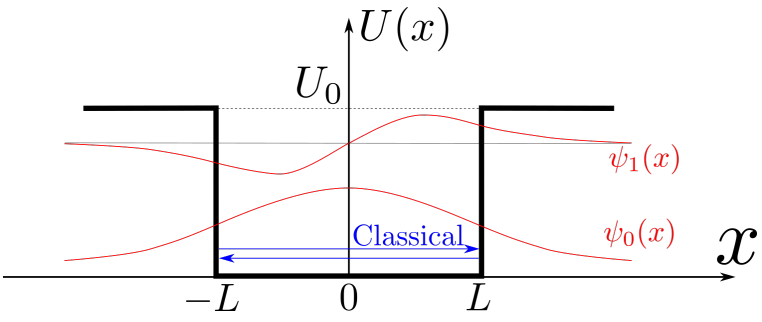
$$E_n = \frac{\hbar^2 k_n^2}{2m} = \frac{\hbar^2\pi^2}{2mL^2}n^2.$$

- Particle is localized within the well, but it can only have a discrete (infinite) set of energies. The set is very dense in the classical limit.
- The ground state corresponds to $n = 1$ (for $n = 0$ the wave function $\psi = 0$ everywhere and the function is not-normalizable). The ground state energy is $E_{\text{ground state}} == \frac{\hbar^2\pi^2}{2mL^2}$

Building intuition.

- When a wave is confined to some region of space the energy spectrum is discrete.
- if we squeeze the wave further (it means we decrease L) the energy (the kinetic!!! energy) of each state will go up.

26.3. Particle in the finite square well potential.



Classically, if the particle's energy E is less then U_0 the particle is completely confined inside the potential well. In Quantum mechanics the situation is different.

(Using intuition.) The potential outside of the well is finite. We just learned, that the kinetic energy decreases if the wave "spreads out". "Spreading out", however, will increase the potential energy. So one expects a balance: the wave "spreads out" outside of the well just a bit. And this is exactly what happens. So even when the energy E is less than U_0 we still have a finite probability to find the particle outside of the well.

- Consider a potential

$$U(x) = \begin{cases} 0 & \text{for } |x| < L \\ U_0 & \text{for } |x| > L \end{cases}$$

- I am interested only in solutions for $E < U_0$.
- Classical: the particle can have any energy $0 < E < U_0$; the particle is completely localized in $-L < x < L$ region.
- The time independent Schrödinger equation is

$$-\frac{\hbar^2}{2m}\psi'' + U(x)\psi = E\psi.$$

- As the potential energy function is not continuous, we need to consider the Schrödinger equation in the regions $x < -L$, $-L < x < L$, and $L < x$ separately.

Case $x < -L$: The Schrödinger equation reads

$$-\frac{\hbar^2}{2m}\psi'' = -(U_0 - E)\psi.$$

As $E < U_0$ this differential equation has two linearly independent solutions $\psi = e^{\kappa x}$ and $\psi = e^{-\kappa x}$, where $\kappa = \sqrt{\frac{2m(U_0-E)}{\hbar^2}}$. The second solution grows exponentially when $x \to -\infty$, so it is not normalizable (the integral from $-\infty$ to $-L$ of $|\psi|^2$ will not converge) and must be discarded.

Case $-L < x < L$: The Schrödinger equation reads

$$-\frac{\hbar^2}{2m}\psi'' = E\psi.$$

This is an oscillator equation, two linearly independent solutions are $\psi = \sin(kx)$ and $\psi = \cos(kx)$, where $k = \sqrt{\frac{2mE}{\hbar^2}}$. Both solutions are normalizable, as the integral from $-L$ to L of $|\psi|^2$ converges for both of them. So we must keep both.

Case $L < x$: The Schrödinger equation reads

$$-\frac{\hbar^2}{2m}\psi'' = -(U_0 - E)\psi.$$

As $E < U_0$ this differential equation has two linearly independent solutions $\psi = e^{\kappa x}$ and $\psi = e^{-\kappa x}$, where $\kappa = \sqrt{\frac{2m(U_0-E)}{\hbar^2}}$. The first solution grows exponentially when $x \to \infty$, so it is not normalizable (the integral from L to $+\infty$ of $|\psi|^2$ will not converge) and must be discarded.

- The normalizable solutions are (C is overall normalization factor which must be found at the end by the requirement $\int_{-\infty}^{\infty}|\psi|^2dx=1$)

$$\psi(x)=C\begin{cases}A_-e^{\kappa x}, & \\ \sin(kx), \quad \text{or} \quad \cos(kx), & \text{for } -L<x<L \\ A_+e^{-\kappa x}, & \text{for } L<x\end{cases}, \qquad k=\sqrt{\frac{2mE}{\hbar^2}}, \qquad \kappa=\sqrt{\frac{2m(U_0-E)}{\hbar^2}}$$

- Symmetry. As the Hamiltonian is symmetric with respect to $x\to -x$ the solutions are either symmetric $\psi(-x)=\psi(x)$ or antisymmetric $\psi(-x)=-\psi(x)$. These symmetric and antisymmetric solutions are

$$\psi_s(x)=C\begin{cases}Ae^{\kappa x} & \text{for } x<-L \\ \cos(kx) & \text{for } -L<x<L \\ Ae^{-\kappa x} & \text{for } x>L\end{cases}, \qquad \psi_a(x)=C\begin{cases}-Ae^{\kappa x} & \text{for } x<-L \\ \sin(kx) & \text{for } -L<x<L \\ Ae^{-\kappa x} & \text{for } x>L\end{cases},$$

where

$$k=\sqrt{\frac{2mE}{\hbar^2}}, \qquad \kappa=\sqrt{\frac{2m(U_0-E)}{\hbar^2}}=\sqrt{k_u^2-k^2}, \qquad k_u\equiv\sqrt{\frac{2mU_0}{\hbar^2}}.$$

- Matching the solutions at $x=L$.
  - The wave function ψ must be continuous.
  $$\psi(x=L-\epsilon)=\psi(x=L+\epsilon), \quad \text{at the limit } \epsilon\to 0.$$
  - In addition, let's integrate the Schrödinger equation over x from $L-\epsilon$ to $L+\epsilon$. We have
  $$-\frac{\hbar^2}{2m}\left(\psi'(L+\epsilon)-\psi'(L-\epsilon)\right)+\int_{L-\epsilon}^{L+\epsilon}U(x)\psi(x)dx=E\int_{L-\epsilon}^{L+\epsilon}\psi(x)dx.$$
  Taking a limit $\epsilon\to 0$ we have
  $$\psi'(L+0)=\psi'(L-0)$$
  So ψ' must also be continuous at the points $x=\pm L$ (and thus everywhere).
- So we need to match the value of ψ and ψ' from both sides for $x=L$, so we have (left column for the symmetric, right for antisymmetric)

$$\begin{aligned} Ae^{-\kappa L}&=\cos(kL) & Ae^{-\kappa L}&=\sin(kL) \\ -\kappa Ae^{-\kappa L}&=-k\sin(kL) & -\kappa Ae^{-\kappa L}&=k\cos(kL)\end{aligned}$$

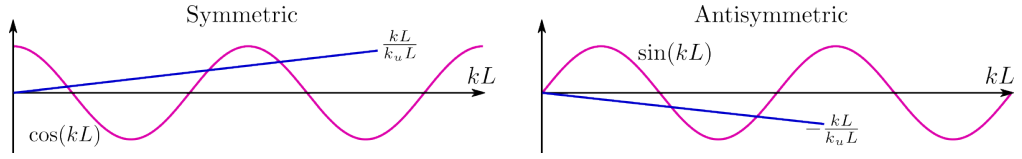
Dividing the equation we get

$$k\tan(kL)=\kappa \qquad k\cot(kL)=-\kappa,$$

which can be written as

$$\cos(kL)=\frac{k}{k_u}, \qquad \sin(kL)=-\frac{k}{k_u}, \qquad \text{where } k_u=\sqrt{\frac{2mU_0}{\hbar^2}}.$$

This means that k, and thus E, cannot be arbitrary, but instead must satisfy above equations. The equations are transcendental, but we can look at them graphically.

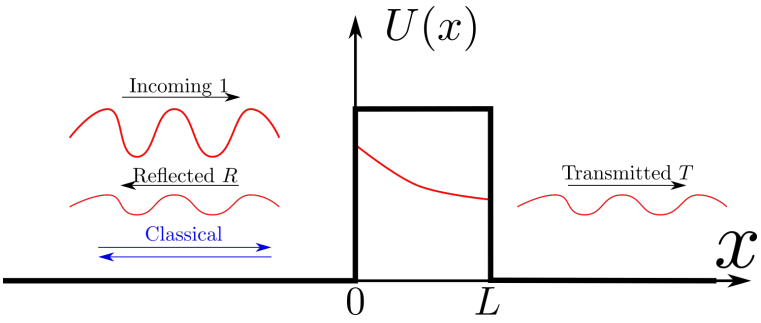


- These equations have a discrete set of solutions. So there energy spectrum for $E < U_0$ is discrete. (The energy spectrum for $E > U_0$ is continuous.)
- No matter how small U_0 is there is always at least one symmetric localized solution!
- Unlike classical case the particle can be found outside the well.
- The transition in behavior at $E \approx U_0$ is not as sharp in Quantum mechanics. Classically the behaviors of a particles with the energies E just below U_0 and just above U_0 are very different and this transition between the two behaviors is very sharp as E crosses U_0. In Quantum mechanics the wave function of a particle with the energy just below U_0 spreads out of the well to the distance $\sim 1/\kappa \sim 1/\sqrt{U_0 - E}$. This distance diverges as $E \to U_0$. So the transition from the localized state at $E < U_0$ to the fully extended state at $E > U_0$ happens smoothly.

(Building intuition.)

- The quantum mechanical wave function leaks out to classically prohibited region. In that region the wave function decays.
- Things do not change abruptly in quantum mechanics.

26.4. Tunneling.



In classical mechanics the particle coming from the left is simply reflected back by the barrier.

Use intuition.

If one keeps in mind that particles are waves, then the tunneling (transition through a potential barrier) is not very surprising effect. Waves do not change suddenly. So imagine a wave which hits a finite barrier. The barrier is the classically prohibited region, so the wave will leak into inside the barrier and will start decaying. If the barrier has finite length the wave will still be finite when it reaches the other side. So it will penetrate through and propagate further. So when a wave hits a barrier some of it will penetrate through and some of it will reflect back. This is exactly what happens.

- Transition through a square potential bump.

$$U(x) = \begin{cases} 0 & \text{for } x < 0 \\ U_0 & \text{for } 0 < x < L \\ 0 & \text{for } x > L \end{cases} .$$

- We are interested at energies $0 < E < U_0$.

- We look for the solution in the form

$$\psi(x) = \begin{cases} e^{ipx/\hbar} + Re^{-ipx/\hbar} & \text{for } x < 0 \\ A_+ e^{\kappa x/\hbar} + A_- e^{-\kappa x/\hbar} & \text{for } 0 < x < L \\ Te^{ipx/\hbar} & \text{for } x > L \end{cases},$$

  where R and T are reflection and transition amplitudes respectively and

$$\frac{p^2}{2m} = E, \qquad \frac{\kappa^2}{2m} = U_0 - E$$

- At the points $x = 0$ and $x = L$ we must match the value of the wave function and its derivatives from the left and the right. So we have four linear conditions/equations and four unknowns T, R, A_+, and A_-!
- The answer is

$$|T|^2 = \frac{4p^2\kappa^2}{(p^2+\kappa^2)^2 \sinh^2(\kappa L/\hbar) + 4p^2\kappa^2}, \qquad |R|^2 = 1 - |T|^2.$$

- Using the definitions of p and κ it can be written

$$|T|^2 = \frac{4E(U_0 - E)}{4E(U_0 - E) + U_0^2 \sinh^2\left(\sqrt{\frac{U_0 - E}{\epsilon}}\right)}, \qquad \epsilon = \frac{\hbar^2}{2mL^2}$$

- Limits of large $L \gg \hbar/\kappa$ and $\kappa \gg p$ (or $U_0 \gg E$).

$$\kappa \approx \sqrt{2mU_0}, \qquad |T|^2 \approx \frac{16E}{U_0} e^{-\kappa L/\hbar}$$

- This is under barrier transition — tunneling.
- As things do not change abruptly, there is also over barrier reflection.

Scanning tunneling microscopy.

26.5. Particle in the δ-function attractive potential.

For self-study.

- I want to consider a potential

$$U(x) = -U_0 \delta(x).$$

- I am interested only in localized state, so $E < 0$.
- The Schrödinger equation is

$$-\frac{\hbar^2}{2m}\psi'' - U_0\delta(x)\psi = -|E|\psi$$

- Let's integrate this equation over x from $-\epsilon$ to ϵ, we get

$$-\frac{\hbar^2}{2m}\left(\psi'(\epsilon) - \psi'(-\epsilon)\right) - U_0\psi(0) = -|E| \int_\epsilon^\epsilon \psi(x)dx.$$

  Taking the limit $\epsilon \to 0$ we see that

$$\psi'(+0) - \psi'(-0) = -\frac{2mU_0}{\hbar^2}\psi(0)$$

  So the function ψ' must have a jump (discontinuity at $x = 0$)

- The solutions are

$$\psi = \begin{cases} Ae^{\kappa x} & \text{for } x < 0 \\ Ae^{-\kappa x} & \text{for } x > 0 \end{cases}, \tag{26.1}$$

where

$$\kappa = \sqrt{\frac{2m|E|}{\hbar^2}}$$

- Then

$$\psi'(+0) = -\kappa A, \qquad \psi'(-0) = \kappa A, \qquad \psi(0) = A$$

- Using the condition for matching the derivatives we get

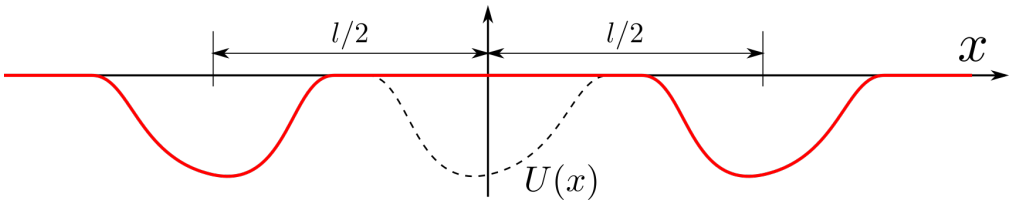
$$2\kappa = \frac{2mU_0}{\hbar^2}, \qquad |E| = \frac{U_0^2}{2m\hbar^2}$$

- Although the potential is very short range the particle can be found in the finite region $-1/\kappa < x < 1/\kappa$, or

$$-\frac{\hbar^2}{mU_0} < x < \frac{\hbar^2}{mU_0}.$$

26.6. Particle in two far away potential wells.

For self-study.



- The Hamiltonian is

$$\hat{H} = \frac{\hat{p}^2}{2m} + U_L(x) + U_R(x), \qquad U_R = U(x - l/2), \quad U_L = U(-x - l/2), \qquad l \gg \frac{\hbar^2}{mU_0}$$

(it also means, that $U_R(x) = U_L(-x)$.)

- The condition $l \gg \frac{\hbar^2}{mU_0}$ means that the distance between the wells are much larger than the spread of the wave function.
- If the two wells are far away from each other, then the overlap of the wave functions is small.
- Let's define two functions $|\psi_L\rangle$ and $|\psi_R\rangle$ which are the solutions of the Schrödinger equation for the left and right potential wells — $\hat{H}_L \equiv \frac{\hat{p}^2}{2m} + U_L(x)$ and $\hat{H}_R \equiv \frac{\hat{p}^2}{2m} + U_R(x)$ — correspondingly. As the wells are identical we consider the solution with one particular energy E_0, the same for both wells.

$$\hat{H}_L|\psi_L\rangle = \left(\frac{\hat{p}^2}{2m} + U_L\right)|\psi_L\rangle = E_0|\psi_L\rangle, \qquad \langle\psi_L|\psi_L\rangle = 1$$

$$\hat{H}_R|\psi_R\rangle = \left(\frac{\hat{p}^2}{2m} + U_R\right)|\psi_R\rangle = E_0|\psi_R\rangle, \qquad \langle\psi_R|\psi_R\rangle = 1$$

We also notice, that

$$|\langle\psi_R|\psi_L\rangle| \ll 1.$$

- Let's look for the solution in the form

$$|\psi\rangle = a_L|\psi_L\rangle + a_R|\psi_R\rangle.$$

- The Schrödinger equation now reads.

$$a_L E|\psi_L\rangle + a_R E|\psi_R\rangle = a_L\hat{H}|\psi_L\rangle + a_R\hat{H}|\psi_R\rangle = a_L\hat{H}_L|\psi_L\rangle + a_L U_R|\psi_L\rangle + a_R\hat{H}|\psi_R\rangle + a_R U_L|\psi_R\rangle,$$

or using the definitions of ψ_L and ψ_R

$$a_L E|\psi_L\rangle + a_R E|\psi_R\rangle = a_L E_0|\psi_L\rangle + a_L U_R|\psi_L\rangle + a_R E_0|\psi_R\rangle + a_R U_L|\psi_R\rangle$$

- Multiplying this equation by $\langle\psi_L|$ and $\langle\psi_R|$ we get

$$\begin{aligned} E a_L &= (E_0 + \langle\psi_L|U_R|\psi_L\rangle)\, a_L + \langle\psi_L|U_L|\psi_R\rangle a_R \\ E a_R &= (E_0 + \langle\psi_R|U_L|\psi_R\rangle)\, a_R + \langle\psi_R|U_R|\psi_L\rangle a_L. \end{aligned}$$

- We expect $E \approx E_0$, so we ignored the terms of the kind $(E - E_0)\langle\psi_L|\psi_R\rangle$, as they are of the second order.
- From the symmetry of the problem we see, that

$$\langle\psi_L|U_R|\psi_L\rangle = \langle\psi_R|U_L|\psi_R\rangle, \qquad \langle\psi_R|U_R|\psi_L\rangle = \langle\psi_L|U_L|\psi_R\rangle$$

- Introducing $\tilde{E}_0 = E_0 + \langle\psi_L|U_R|\psi_L\rangle$, $\Delta = -\langle\psi_R|U_R|\psi_L\rangle$, and a vector $\begin{pmatrix} a_L \\ a_R \end{pmatrix}$ we have

$$E\begin{pmatrix} a_L \\ a_R \end{pmatrix} = \begin{pmatrix} \tilde{E}_0 & -\Delta \\ -\Delta & \tilde{E}_0 \end{pmatrix}\begin{pmatrix} a_L \\ a_R \end{pmatrix}.$$

- So E is just an eigenvalue of the simple 2×2 matrix. The result is

$$E_\pm = \tilde{E}_0 \pm |\Delta|.$$

- A single degenerate energy level is split in two levels: symmetric and antisymmetric combinations.
- Interaction splits degeneracy.
- This is also the origin of the covalent bond in chemistry.
- In the symmetric potential the ground state is always symmetric.

26.7. Strong periodic potential. (Tight binding model.)

For self-study.

- The potential is

$$U(x) = \sum_{n=-\infty}^{\infty} U_0(x - nl).$$

- We again assume that l is much grater than the spread of a wave function for a single well.

$$\left(\frac{\hat{p}^2}{2m} + U_0(x)\right)|\psi(x)\rangle = E_0|\psi(x)\rangle, \qquad \langle\psi(x)|\psi(x)\rangle = 1$$

- We look at the solution in the form

$$|\psi\rangle = \sum_{n=-\infty}^{\infty} a_n|\psi(x - nl)\rangle$$

- We then have

$$\sum_{n=-\infty}^{\infty} a_n E|\psi(x-nl)\rangle = \sum_{n=-\infty}^{\infty} a_n \left(\frac{\hat{p}^2}{2m} + U_0(x-nl) + \sum_{n'=-\infty;n'\neq n}^{\infty} U_0(x-n'l) \right) |\psi(x-nl)\rangle$$

$$\sum_{n=-\infty}^{\infty} a_n E|\psi(x-nl)\rangle = \sum_{n=-\infty}^{\infty} a_n E_0|\psi(x-nl)\rangle + \sum_{n=-\infty}^{\infty} a_n \sum_{n'=-\infty;n'\neq n}^{\infty} U_0(x-n'l)|\psi(x-nl)\rangle$$

$$\sum_{n=-\infty}^{\infty} a_n (E-E_0)|\psi(x-nl)\rangle = \sum_{n=-\infty}^{\infty} a_n \sum_{n'=-\infty;n'\neq n}^{\infty} U_0(x-n'l)|\psi(x-nl)\rangle$$

Now I multiply this equation by $\langle\psi(x-n''l)|$ from the left and get (again, separating the diagonal terms)

$$a_{n''}(E-E_0)\langle\psi(x-n''l)|\psi(x-n''l)\rangle + \sum_{n=-\infty;n\neq n''}^{\infty} a_n(E-E_0)\langle\psi(x-n''l)|\psi(x-nl)\rangle$$

$$= a_{n''} \sum_{n'=-\infty;n'\neq n''}^{\infty} \langle\psi(x-n''l)|U_0(x-n'l)|\psi(x-n''l)\rangle$$

$$+ \sum_{n=-\infty;n\neq n''}^{\infty} a_n \sum_{n'=-\infty;n'\neq n}^{\infty} \langle\psi(x-n''l)|U_0(x-n'l)|\psi(x-nl)\rangle$$

The second term in the LHS is small, as for $n \neq n''$ the overlap $\langle\psi(x-n''l)|\psi(x-nl)\rangle$ is small and so is $E-E_0$. Using $\langle\psi(x-n''l)|\psi(x-n''l)\rangle = 1$, and denoting

$$\tilde{E}_0 \equiv E_0 + \sum_{n'=-\infty;n'\neq n''}^{\infty} \langle\psi(x-n''l)|U_0(x-n'l)|\psi(x-n''l)\rangle,$$

$$= E_0 + \sum_{k=-\infty;k\neq 0}^{\infty} \langle\psi(x)|U_0(x+kl)|\psi(x)\rangle$$

we will get the equation in the form

$$Ea_{n''} = \tilde{E}_0 a_{n''} + \sum_{n=-\infty;n\neq n''}^{\infty} a_n \sum_{n'=-\infty;n'\neq n}^{\infty} \langle\psi(x-n''l)|U_0(x-n'l)|\psi(x-nl)\rangle.$$

Now I shift the definition of n by $n = n'' + k$, and also $n' = n'' + k'$, and shifting x under the integration $x - n''l \to x$, we have

$$Ea_{n''} = \tilde{E}_0 a_{n''} + \sum_{k=-\infty;k\neq 0}^{\infty} a_{n''+k} \sum_{k'=-\infty;k'\neq k}^{\infty} \langle\psi(x)|U_0(x-k'l)|\psi(x-kl)\rangle$$

The function $\psi(x)$ decays exponentially with distance. So in the sum all terms will decay exponentially with $|k|$. It means, that with good approximation we can keep only the terms with $k = \pm 1$ and we have

$$Ea_{n''} = \tilde{E}_0 a_{n''} + a_{n''+1} \sum_{k'=-\infty;k'\neq 1}^{\infty} \langle\psi(x)|U_0(x-k'l)|\psi(x-l)\rangle$$

$$+ a_{n''-1} \sum_{k'=-\infty;k'\neq -1}^{\infty} \langle\psi(x)|U_0(x-k'l)|\psi(x+l)\rangle.$$

The largest contribution to the sums come from the terms $k' = 0$. Keeping only these terms we get

$$Ea_{n''} = \tilde{E}_0 a_{n''} + a_{n''+1}\langle\psi(x)|U_0(x)|\psi(x-l)\rangle + a_{n''-1}\langle\psi(x)|U_0(x)|\psi(x+l)\rangle$$

so denoting

$$\Delta \equiv -\langle\psi(x)|U_0(x)|\psi(x-l)\rangle = -\langle\psi(x)|U_0(x)|\psi(x+l)\rangle$$

(the last equality is correct if $U_0(x) = U_0(-x)$) we will get

$$-\Delta a_{n-1} + \tilde{E}_0 a_n - \Delta a_{n+1} = Ea_n$$

or

$$\begin{pmatrix} . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & 0 & -\Delta & \tilde{E}_0 & -\Delta & 0 & 0 & 0 & . \\ . & 0 & 0 & -\Delta & \tilde{E}_0 & -\Delta & 0 & 0 & . \\ . & 0 & 0 & 0 & -\Delta & \tilde{E}_0 & -\Delta & 0 & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \end{pmatrix} \begin{pmatrix} . \\ a_{n-3} \\ a_{n-2} \\ a_{n-1} \\ a_n \\ a_{n+1} \\ a_{n+2} \\ a_{n+3} \\ . \end{pmatrix} = E \begin{pmatrix} . \\ a_{n-3} \\ a_{n-2} \\ a_{n-1} \\ a_n \\ a_{n+1} \\ a_{n+2} \\ a_{n+3} \\ . \end{pmatrix}.$$

- We look for the solution in the form $a_n = ae^{ipln/\hbar}$, so

  $$-\Delta e^{ipl(n-1)/\hbar} + \tilde{E}_0 e^{ipln/\hbar} - \Delta e^{ipl(n-1)/\hbar} = Ee^{ipln/\hbar},$$

  which gives

  $$E(p) = \tilde{E}_0 - 2\Delta\cos(pl/\hbar), \qquad -\pi\hbar/l < p < \pi\hbar/l.$$

  So a single energy level is split into a band.
- p is quasi-momentum. In particular, for small p

  $$E(p) \approx \tilde{E}_0 - 2\Delta + \frac{p^2}{2(\hbar^2/2l^2\Delta)}.$$

  So it behaves as a normal particle with the "effective" mass $m^* = \hbar^2/2l^2\Delta$.

26.8. Density of states.

- Density of states: Discrete spectrum to continuous.
- Tunneling current as a measure of the density of states (STM).

LECTURE 27
Wave function. Wave packet.

- Homework.
- Evaluations end on August 8.

Srödinger equation.

$$i\hbar\frac{\partial\Psi}{\partial t} = \hat{H}\Psi, \qquad \hat{H} = \frac{\hat{p}^2}{2m} + U(x).$$

Momentum operator

$$\hat{p} = -i\hbar\frac{\partial}{\partial x}.$$

27.1. A bit of math.

27.1.1. Gaussian integral

$$\int_{-\infty}^{\infty} e^{-\frac{x^2}{2\alpha^2}}dx = \alpha\sqrt{2\pi}.$$

It can be derived as the following: Let's denote

$$I = \int_{-\infty}^{\infty} e^{-x^2}dx.$$

And consider I^2

$$I^2 = \int_{-\infty}^{\infty} e^{-x^2}dx\int_{-\infty}^{\infty} e^{-y^2}dy = \iint_{-\infty}^{\infty} e^{-(x^2+y^2)}dxdy = \int_0^{2\pi} d\phi\int_0^{\infty} e^{-r^2}rdr = \pi\int_0^{\infty} e^{-r^2}d(r^2) = \pi$$

So we have

$$\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}.$$

27.1.2. Complete the square.

If we have a quadratic polynomial $ax^2 + bx + c$ we can always rewrite it as

$$ax^2 + bx + c = a\left(x^2 + 2\frac{b}{2a}x\right) + c = a\left(x^2 + 2\frac{b}{2a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2}\right) + c$$

$$= a\left(x^2 + 2\frac{b}{2a}x + \frac{b^2}{4a^2}\right) - \frac{b^2}{4a} + c = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c.$$

This is by the way the origin of the formula for the roots of a quadratic equation.

27.1.3. Gaussian integral with linear term.

Let's now consider an integral

$$\int_{-\infty}^{\infty} e^{-\frac{x^2}{2\alpha^2}-\beta x}dx.$$

First we complete the square in the exponent

$$\int_{-\infty}^{\infty} e^{-\frac{x^2}{2\alpha^2}-\beta x}dx = \int_{-\infty}^{\infty} e^{-\frac{1}{2\alpha^2}\left(x+\alpha^2\beta\right)^2+\frac{1}{2}\alpha^2\beta^2}dx = e^{\frac{1}{2}\alpha^2\beta^2}\int_{-\infty}^{\infty} e^{-\frac{1}{2\alpha^2}\left(x+\alpha^2\beta\right)^2}dx$$

Now changing the variable $x+\alpha^2\beta = y$ we get

$$\int_{-\infty}^{\infty} e^{-\frac{x^2}{2\alpha^2}-\beta x}dx = e^{\frac{1}{2}\alpha^2\beta^2}\int_{-\infty}^{\infty} e^{-\frac{y^2}{2\alpha^2}}dy = e^{\frac{1}{2}\alpha^2\beta^2}\alpha\sqrt{2\pi}.$$

27.2. A particle as a wave packet.

Let's consider a free particle $U(x) = 0$.

- A wave $e^{ipx/\hbar}$ has a definite momentum p, but is everywhere $-\infty < x < \infty$.
- It's eigen energy is E_p.
- A particle is localized in space.
- Srödinger equation is linear in Ψ.

A particle must be represented by wave packet.

We know how a plane wave evolves with time

$$e^{ipx/\hbar} \to e^{ipx/\hbar - iE_p t/\hbar}.$$

How does the wave packet — the particle — evolves with time?

27.2.1. Evolution of a wave packet.

Let's assume that we know that at initial time $t = 0$ the wave function is given by $\Psi(x, 0)$, we want to know what will be the wave function at time t.

In order to do that we need to present $\Psi(x, 0)$ as a collection of a plane waves — the wave packet.

$$\Psi(x,0) = \int_{-\infty}^{\infty} a_p e^{ipx/\hbar}\frac{dp}{2\pi\hbar}, \qquad a_p = \int_{-\infty}^{\infty}\Psi(x,0)e^{-ipx/\hbar}dx$$

After a time t a wave $e^{ipx/\hbar}$ becomes $e^{ipx/\hbar - iE_p t/\hbar}$. So

$$\Psi(x,t) = \int a_p e^{ipx/\hbar - iE_p t/\hbar}\frac{dp}{2\pi\hbar}.$$

Let's see how it works for a classical free particle $E_p = \frac{p^2}{2m}$.

27.2.1.1. *Wave packet spreading.*

Let's assume, that we have started with the initial wave-function

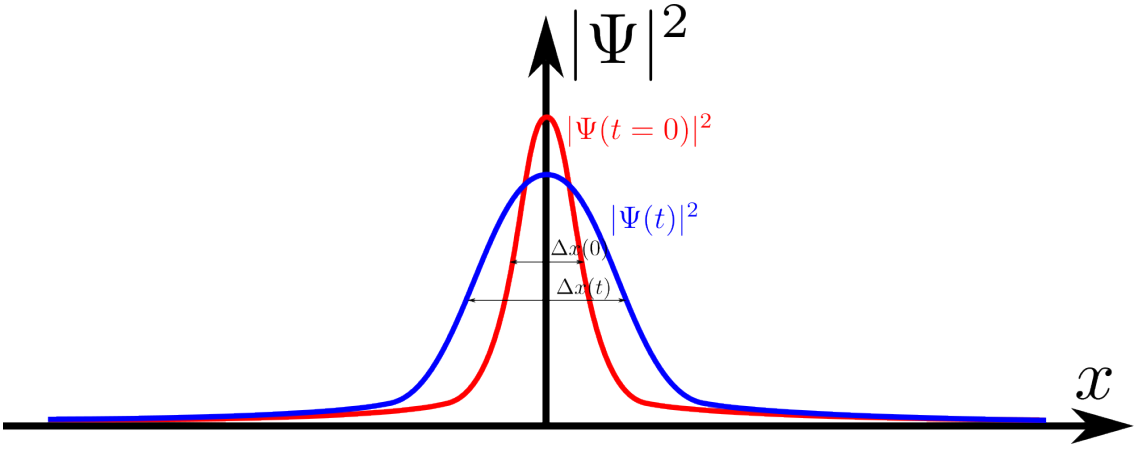
$$\Psi(x,0) = Ce^{-x^2/4\alpha^2}$$

and $|\Psi(x,0)|^2 = C^2 e^{-x^2/2\alpha^2}$, so that

$$\Delta x = \alpha.$$

First we must compute C from the normalization condition

$$1 = \int_{-\infty}^{\infty}|\Psi(x,0)|^2 dx = |C|^2\int_{-\infty}^{\infty} e^{-x^2/2\alpha^2}dx = |C|^2\sqrt{2\pi}\alpha$$



then

$$a_p = \int_{-\infty}^{\infty} \Psi(x,0) e^{-ipx/\hbar} dx = C \int_{-\infty}^{\infty} e^{-x^2/4\alpha^2 - ipx/\hbar} dx = C \int_{-\infty}^{\infty} e^{-\frac{1}{4\alpha^2}\left(x^2 + 2ipx\frac{2\alpha^2}{\hbar} - p^2\frac{4\alpha^4}{\hbar^2}\right) - p^2\frac{\alpha^2}{\hbar^2}} dx =$$

$$Ce^{-p^2\frac{\alpha^2}{\hbar^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{4\alpha^2}\left(x + 2ip\frac{\alpha^2}{\hbar}\right)^2} dx = 2C\alpha\sqrt{\pi} e^{-p^2\frac{\alpha^2}{\hbar^2}}$$

Notice, that this function is peaked at $p = 0$ and the "spread" is

$$\Delta p = \hbar/2\alpha, \qquad \Delta x \Delta p = \hbar/2.$$

According to the prescription

$$\Psi(x,t) = \int_{-\infty}^{\infty} a_p e^{ipx/\hbar - iE_p t/\hbar} \frac{dp}{2\pi\hbar} = \int_{-\infty}^{\infty} C\alpha\sqrt{2\pi} e^{-p^2\frac{\alpha^2}{\hbar^2} + ipx/\hbar - p^2\frac{it}{2m\hbar}} \frac{dp}{2\pi\hbar} =$$

$$C\alpha\sqrt{2\pi} \int_{-\infty}^{\infty} e^{-p^2\left(\frac{\alpha^2}{\hbar^2} + it/2m\hbar\right) + ipx/\hbar} \frac{dp}{2\pi\hbar} = C\alpha\sqrt{2\pi} \int_{-\infty}^{\infty} e^{-\frac{p^2}{4\left(\frac{4\alpha^2}{\hbar^2} + \frac{2it}{m\hbar}\right)^{-1}} + ipx/\hbar} \frac{dp}{2\pi\hbar} =$$

$$2C\alpha\frac{1}{\hbar}\left(\frac{4\alpha^2}{\hbar^2} + \frac{2it}{m\hbar}\right)^{-1/2} e^{-\frac{x^2}{4\hbar^2\left(\frac{\alpha^2}{\hbar^2} + \frac{it}{2m\hbar}\right)}} = \frac{C}{\sqrt{1 + \frac{it\hbar}{2m\alpha^2}}} e^{-\frac{x^2}{4\left(\alpha^2 + \frac{it\hbar}{2m}\right)}}$$

So we see that

$$|\Psi(x,t)|^2 = \frac{C^2}{\sqrt{1 + \left(\frac{t\hbar}{2m\alpha^2}\right)^2}} e^{-\frac{x^2}{2\left(\alpha^2 + \left(\frac{t\hbar}{2m\alpha}\right)^2\right)}}$$

So we see, that the particle is still at the center on average, but the width/spread is given by ($\alpha = \Delta x(t=0)$)

$$\Delta x(t) = \sqrt{[\Delta x(0)]^2 + \left[\frac{t\hbar}{2m\Delta x(0)}\right]^2}$$

We now can compute how much time it would take for a $1g$ marble initially localized with a precision $0.1mm$ to disperse so that $\Delta x(t) = 10\Delta x(0)$. The answer is $t \approx 2 \times 10^{24} s$ – by far longer than the life-time of our Universe.

27.2.1.2. *Group velocity.*

Let's construct a wave packet with a momentum p_0 on average at $t = 0$. We want this packet to be very sharply peaked at p_0.

$$\Psi(x,0) = C\int_{-\infty}^{\infty} e^{-\frac{(p-p_0)^2}{4\alpha^2}} e^{ipx/\hbar} dp$$

where we assume that the $\alpha \sim \Delta p$ is small.

At time t the wave packet will be

$$\Psi(x,t) = C\int_{-\infty}^{\infty} e^{-\frac{(p-p_0)^2}{4\alpha^2}} e^{ipx/\hbar - iE_p t/\hbar} dp$$

As α is small, only $p \sim p_0$ contribute to the integral, so we can write

$$\Psi(x,t) \approx Ce^{ip_0 x/\hbar - iE_{p_0}t/\hbar}\int_{-\infty}^{\infty} e^{-(p-p_0)^2\left(\frac{1}{4\alpha^2}+i\frac{1}{2\hbar}\frac{\partial^2 E_p}{\partial p_0^2}t\right)+\frac{i}{\hbar}(p-p_0)\left(x-\frac{\partial E}{\partial p_0}t\right)} dp$$

So we see, that

$$|\Psi(x,t)|^2 = f\left(x - \frac{\partial E}{\partial p_0}t, t\right)$$

So we see, that the wave packet is moving with the "group" velocity

$$v = \frac{\partial E}{\partial p_0},$$

as it should according to the Hamiltonian equations.

27.3. Relativistic quantum mechanics.

- Srödinger equation is not Lorenz invariant – it is non-relativistic.
- The relativistic quantum mechanics is described by Dirac equation `https://en.wikipedia.org/wiki/Paul_Dirac`.
- Every particle has an antiparticle.

LECTURE 28
Commutators. Quantum harmonic oscillator.

For self-study.

- Evaluations end on August 8.

Quantum harmonic oscillator.

- Hermitian operators. Observables.
- x as an operator.
- $[\hat{p}, \hat{x}] = -i\hbar$.
- Hamiltonian for a harmonic oscillator $\hat{H} = \frac{\hat{p}^2}{2m} + \frac{k\hat{x}^2}{2} = \frac{\hat{p}^2}{2m} + m\omega^2\frac{\hat{x}^2}{2}$.
- Operators $\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i}{m\omega}\hat{p}\right)$ and $\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i}{m\omega}\hat{p}\right)$.
- $[\hat{a}, \hat{a}^\dagger] = 1$, and $\hat{H} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + 1/2\right)$.
- The Schrödinger equation $\hat{H}|\psi\rangle = E|\psi\rangle$ becomes

$$\hbar\omega\hat{a}^\dagger\hat{a}|\psi\rangle = (E - \hbar\omega/2)\,|\psi\rangle$$

- A function $|0\rangle$ such that $\hat{a}|0\rangle = 0$ and $\langle 0|0\rangle = 1$ exists.

$$|0\rangle = \left(\frac{m\omega}{\pi\hbar}\right)e^{-\frac{m\omega}{2\hbar}x^2}, \qquad E_0 = \frac{1}{2}\hbar\omega$$

- Consider a function/state $|1\rangle = \hat{a}^\dagger|0\rangle$. Let's act on it by an operator $\hbar\omega\hat{a}^\dagger\hat{a}$

$$\hbar\omega\hat{a}^\dagger\hat{a}|1\rangle = \hbar\omega\hat{a}^\dagger\hat{a}\hat{a}^\dagger|0\rangle = \hbar\omega\hat{a}^\dagger\left(\hat{a}^\dagger\hat{a} + 1\right)|0\rangle = \hbar\omega\hat{a}^\dagger\hat{a}^\dagger\hat{a}|0\rangle + \hbar\omega\hat{a}^\dagger|0\rangle = \hbar\omega\hat{a}^\dagger|0\rangle = \hbar\omega|1\rangle.$$

So we see, that the function $|1\rangle$ is an eigen function of our Hamiltonian and

$$E_1 = \hbar\omega + \frac{1}{2}\hbar\omega.$$

- Normalization

$$\langle 1|1\rangle = \langle 0|\hat{a}\hat{a}^\dagger|0\rangle = \langle 0|1 + \hat{a}^\dagger\hat{a}|0\rangle = \langle 0|0\rangle = 1$$

- For a state $|n\rangle = \frac{\left(\hat{a}^\dagger\right)^n}{\sqrt{n!}}|0\rangle$ we have

$$\hbar\omega\hat{a}^\dagger\hat{a}|n\rangle = n\hbar\omega|n\rangle, \qquad \langle n|n\rangle = 1,$$

so

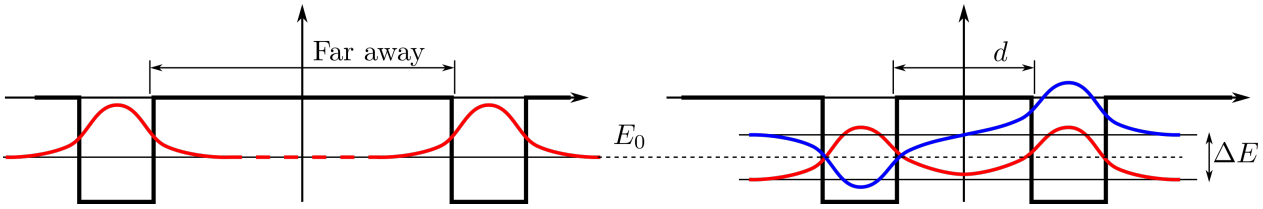
$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega.$$

- Also $\langle n|m\rangle = 0$, for $n \neq m$, and
$$\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle, \qquad \hat{a}|n\rangle = \sqrt{n}|n-1\rangle$$
- $\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a} + \hat{a}^\dagger\right)$, and $\hat{p} = i\sqrt{\frac{m\omega\hbar}{2}}\left(\hat{a}^\dagger - \hat{a}\right)$, so
$$\langle n|\hat{x}|n\rangle = 0, \qquad \langle n|\hat{p}|n\rangle = 0$$
and
$$\langle n|\hat{x}^2|n\rangle = \frac{\hbar}{2m\omega}\langle n|\hat{a}^\dagger\hat{a}+\hat{a}\hat{a}^\dagger|n\rangle = \frac{\hbar}{2m\omega}\langle n|2\hat{a}^\dagger\hat{a}+1|n\rangle = (n+1/2)\frac{\hbar}{m\omega}, \qquad \langle n|\hat{p}^2|n\rangle = (n+1/2)m\omega\hbar$$
- Coherent states. For any α we construct a state:
$$|\alpha\rangle = e^{-|\alpha|^2/2}e^{\alpha\hat{a}^\dagger}|0\rangle, \qquad \langle\alpha|\alpha\rangle = 1, \qquad \langle\alpha|\hat{a}^\dagger\hat{a}|\alpha\rangle = |\alpha|^2.$$
This set of such states is overcomplete $\langle\alpha|\alpha'\rangle \neq 0$, for $\alpha \neq \alpha'$. The time evolution of these states describes the motion of a particle.

LECTURE 29
Many-particle states. Identical particles.

- Evaluations.
- Exam procedure.

29.1. Double well potential. Bloch theorem.



- Double well potential.
- Periodic potential.

$$-\frac{\hbar^2}{2m}\psi''(x) + U(x)\psi(x) = E\psi(x)$$

or

$$\psi''(x) = -\left(\frac{2mE}{\hbar^2} - \frac{2m}{\hbar^2}U(x)\right)\psi(x).$$

By changing the notations $x \to t$, $\psi \to x$, and $\frac{2mE}{\hbar^2} = \omega_0^2$ this equation is

$$\ddot{x} = -\left(\omega_0^2 - \frac{2m}{\hbar^2}U(t)\right)x = -\Omega^2(t)x.$$

This is an oscillator with parameters periodically depending on time – parametric resonance. The difference is that the wave function must be normalizable.

– Bloch theorem `https://en.wikipedia.org/wiki/Bloch_wave`.

– Band structure! `https://en.wikipedia.org/wiki/Electronic_band_structure`

29.2. Many-particle states. Identical particles.

- Identical particles `https://en.wikipedia.org/wiki/Identical_particles`. Bosons and fermions.

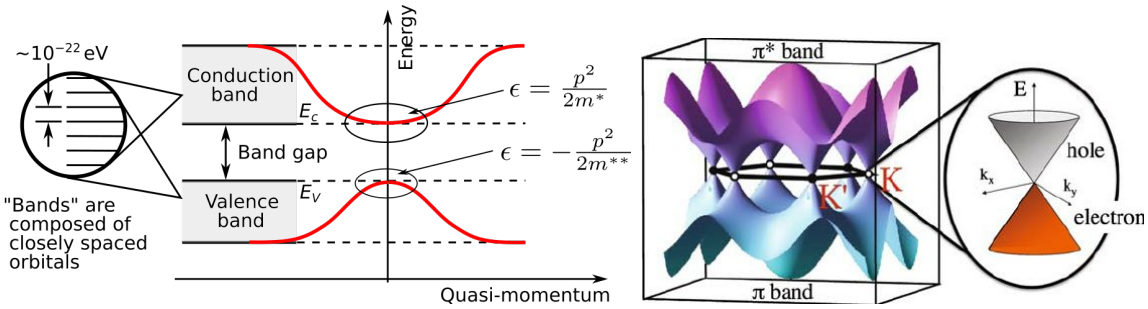


Figure 1. The picture on the left is taken from Wikipedia. The picture on the right is the bans structure of graphene taken from Eur. Phys. J. B**72**, 1–24 (2009).

  - Consider two identical non-interacting(!) particles. They are in some quantum mechanical state. This state is a function of two coordinates of the two particles $\vec{r}_1$ and $\vec{r}_2$: $\psi(\vec{r}_1, \vec{r}_2)$.
  - But the particles are identical. So no matter what we measure we will not be able to distinguish between them.
  - So if we interchange the two particles $\vec{r}_1 \leftrightarrow \vec{r}_2$, the only thing that may happen is that the wave function acquires a phase
$$\psi(\vec{r}_1, \vec{r}_2) \to e^{i\phi}\psi(\vec{r}_2, \vec{r}_1).$$
  - But then if we interchange them again, we must again acquire the same phase.
$$\psi(\vec{r}_1, \vec{r}_2) \to e^{i\phi}\psi(\vec{r}_2, \vec{r}_1) \to e^{2i\phi}\psi(\vec{r}_1, \vec{r}_2).$$
  - So we returned exactly where we were before, so $e^{2i\phi} = 1$, or $e^{i\phi} = \pm 1$.
  - As in $3D$ every path of exchange can be smoothly deformed into any other, we then conclude, that there are two different types of particles

$\psi(\vec{r}_1, \vec{r}_2) \to \psi(\vec{r}_2, \vec{r}_1)$ — bosons, $\quad \psi(\vec{r}_1, \vec{r}_2) \to -\psi(\vec{r}_2, \vec{r}_1)$ — fermions.

  - This distinction has a very significant consequences. The two particle wave function for bosons can have a form $\psi(\vec{r}_1, \vec{r}_2) = \psi_0(\vec{r}_1)\psi_0(\vec{r}_2)$, so two particles can be in the same quantum mechanical state. For the fermions it is impossible! So no two fermions can be in the same state!!!!!

- Bosons `https://en.wikipedia.org/wiki/Boson`.
  - Bose-Einstein condensate `https://en.wikipedia.org/wiki/Bose%E2%80%93Einstein_condensate`.
  - Superfluidity (1937, Pyotr Kapitsa and independently by John F. Allen and Don Misener) `https://en.wikipedia.org/wiki/Superfluidity`.
- Electrons are fermions.
  - Metals and insulators. Response to the electric field.
  - Semiconductors `https://en.wikipedia.org/wiki/Semiconductor`.
  - Electrons and holes `https://en.wikipedia.org/wiki/Electron_hole`.
  - LED `https://en.wikipedia.org/wiki/Light-emitting_diode`.
  - Lasers `https://en.wikipedia.org/wiki/Laser`.
  - Fermi-surface `https://en.wikipedia.org/wiki/Fermi_surface`.
  - Superconductivity (1911, Heike Kamerlingh Onnes) `https://en.wikipedia.org/wiki/Superconductivity`.
- Relativistic Quantum mechanics.

  - Dirac equation (1928) https://en.wikipedia.org/wiki/Dirac_equation.
  - Positrons https://en.wikipedia.org/wiki/Positron.
  - Energy and momentum are conserved!
- Inner Life of the Cell: https://youtu.be/wJyUtbn0O5Y
- Closing remarks. More is different https://science.sciencemag.org/content/177/4047/393.

Simple rules — complex behavior.