# EXAM 1. Tuesday, March 21-23, 2023, take home. Due on Thursday, March 23, 12:45pm.

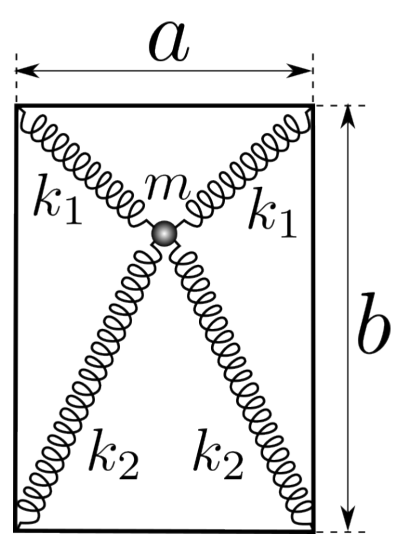
**Problem 1.** *1990-Fall-CM-U-3*
A large sphere of radius $R$ and mass $M$ has a mass density that varies according to the distance from the center, $r$:

$$\rho(r) = \begin{cases} \rho_0 \left[1 - (r/R)^n\right], & \text{if } r \le R; \\ 0, & \text{if } r > R. \end{cases},$$

where $n > 0$. A very small hole is drilled through the center of the sphere and a small object of mass $m$ is released from rest into the hole at the surface. How fast will the object be moving when it reaches the center of the sphere? Express your answer through $M$, $R$, $n$, and $G$.

**Problem 2.** *A frame and a ball*
A ball of mass $m$ is attached by massless springs to four corners of a $a$ by $b$ rectangular frame as shown on the figure. The springs' spring constants are $k_1$ and $k_2$. The springs have negligible equilibrium (un-stretched) length. Neglect the force of gravity. The ball can move in all three directions.

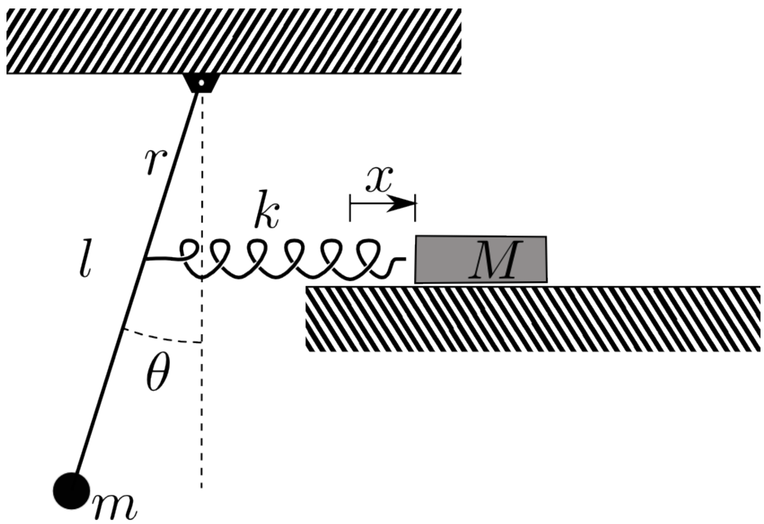
1. Find the equilibrium position of the ball.
2. Find the frequencies of small amplitude oscillations of the ball around the equilibrium.
3. Identify the type of motion associated with each frequency.



**Problem 3.** *2001-Spring-CM-U-2*

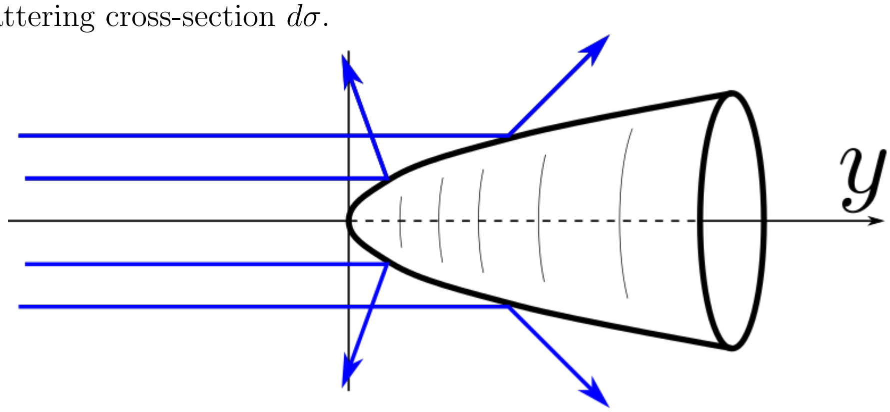
Consider the system, pictured below, which consists of a ball of mass $m$ connected to a massless rod of length $l$. This is then joined at point $r$ to a spring of spring constant $k$ connected to a block of mass $M$ which rests on a frictionless table. When $\theta = 0$ and $x = 0$ the spring is unstretched.

1. Write the Lagrangian for the system in terms of the coordinates $\theta$ and $x$ assuming small displacements of the pendulum
2. Write the equations of motion for the system.
3. Making the simplifying assumptions that $M = m$, $l = 2r$, and setting $k/m = g/l = \omega_0^2$, find the normal-mode frequencies of this system for small oscillations in terms of $\omega_0$
4. Assuming the same conditions, calculate the ratio of amplitudes (for each of the two masses) of the two normal modes of oscillation. In other words, find the normal modes.




**Problem 4.** *Paraboloid scattering.*

An immovable paraboloid (which was produced by rotating a parabola $y = \frac{x^2}{4R}$ around the $y$ axis), elastically scatters particles that are coming from $y = -\infty$. Find the differential scattering cross-section $d\sigma$.

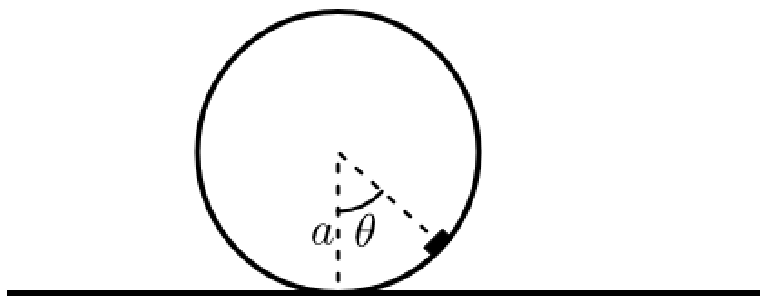


# EXAM 2. Final. Tuesday, May 9, 2023, 8:00-10 am

**Problem 1.** *1991-Fall-CM-U-3.*
A small block of mass $m$ is attached near the outer rim of a solid disk of radius $a$ which also has mass $m$. The disk rolls without slipping on a horizontal straight line.

1. Find the equation of motion for the angle $\theta(t)$ (measured with respect to the vertical as shown) for all $\theta$.

2. Find the system's small amplitude oscillation frequency about its stable equilibrium position.

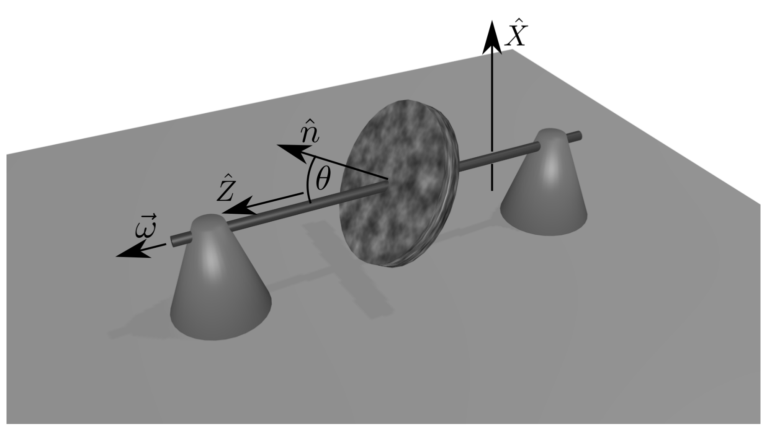



**Problem 2.** *Disc at an angle*
A disk is rigidly attached to an massless axle passing through its center so that the disc's symmetry axis $\hat{n}$ makes an angle $\theta$ with the axle. The moments of inertia of the disc relative to its center are $C$ about the symmetry axis $\hat{n}$ and $A$ about any direction $\hat{n}'$ perpendicular to $\hat{n}$. The axle spins with constant angular velocity $\vec{\omega} = \omega\hat{Z}$ ($\hat{Z}$ is a unit vector along the axle.)

1. What is the kinetic energy of the disc?

2. What is the angular momentum, $\vec{L}$, expressed in the internal frame of reference of principle axes of inertia.

3. Find the torque, $\vec{\tau}$, which must be exerted on the axle by the bearings which support it. Specify the components of $\vec{\tau}$ along the principle axes of inertia of the disc.



**Problem 3.** *Two rollers and a Pendulum.*

Find eigen frequencies for the system shown on the figure. Both rollers are solid uniform discs. There is no slipping. Both disks, the platform, and the pendulum bob have the same mass $m$. Use $k/m = g/R = \omega_0^2$.