Technical Supplement (proofs of propositions) to:


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1 Proposition 1

In the model in which the executive has a local constituency, if $C_1 + C_2 < 1$, then the following is a subgame-perfect equilibrium (SPE):

(a) Executive 1 always proposes $[C_1 + \frac{1-\delta_2}{1-\delta_1\delta_2} \cdot (1 - C_1 - C_2), C_2 + \frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2} \cdot (1 - C_1 - C_2)]$, and always accepts any proposal $y$ such that $1 - C_2 \geq y_1 \geq C_1 + \frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2} \cdot (1 - C_1 - C_2)$.

(b) Executive 2 always proposes $[C_1 + \frac{\delta_1(1-\delta_2)}{1-\delta_1\delta_2} \cdot (1 - C_1 - C_2), C_2 + \frac{1-\delta_1}{1-\delta_1\delta_2} \cdot (1 - C_1 - C_2)]$, and always accepts any proposal $x$ such that $1 - C_1 \geq x_2 \geq C_2 + \frac{\delta_1(1-\delta_1)}{1-\delta_1\delta_2} \cdot (1 - C_1 - C_2)$.

PROOF:

We assume that if an agreement is reached that offers country $i$ at least $C_i$, executive $i$ distributes it among his legislators so as to achieve ratification. The only time this behavior is not required in a subgame-perfect equilibrium is when country $i$ receives just $C_i$ (or when the other country receives less than $C_j$, so that the agreement cannot be ratified in country $j$), so that executive $i$ is indifferent between achieving ratification in his country and not doing so, since his payoff will be 0 regardless. In order that the other executive be able to expect ratification in these situations, we assume that an executive distributes his country’s share of the pie so as to achieve ratification whenever possible.\(^1\)

Consider executive 1’s proposal in an even-numbered period $t^*$ ($t^* = 0, 2, 4, \ldots$). Executive 1 can adopt one of four different types of strategies: (1) make an offer that is accepted right away, (2) use a strategy such that agreement is first reached in a future even period, (3) use a strategy such that agreement is first reached in a future odd period, or (4) use a strategy such that agreement is never reached. If he adopts a strategy of type 1, he is best off proposing $x_2 = C_2 + \frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2} \cdot (1 - C_1 - C_2)$ in the current period, since that is

\(^1\)If ratification failure is costly for an executive, then this assumption is substantively justified as well.
the minimal proposal that executive 2 will accept (note that executive 2 is only accepting agreements that will be ratified in both legislatures). This proposal results in executive 1’s payoff being \(1 - \delta_2 \cdot (C_1 - C_2)\). If he uses a strategy of type 2, because executive 2’s acceptance rule doesn’t change over time and future payoffs are discounted, the best he can do using such a strategy is to offer \(x_2 = C_2 + \frac{\delta_2(1-\delta_1)}{1-\delta_1 \delta_2} \cdot (1 - C_1 - C_2)\) in period \(t = t^* + 2\). Since future payoffs are discounted, it is strictly better to make that proposal in the current period. Using a strategy of type 3, because executive 2’s proposal does not change over time and future payoffs are discounted, the best executive 1 can do is to accept executive 2’s proposal \(y_1 = C_1 + \frac{\delta_1(1-\delta_2)}{1-\delta_1 \delta_2} \cdot (1 - C_1 - C_2)\) in the next period, \(t = t^* + 1\), in which case it has a current discounted value of \(\frac{\delta_1^2(1-\delta_2)}{1-\delta_1 \delta_2} \cdot (1 - C_1 - C_2)\) for executive 1. Because \(\frac{1-\delta_2}{1-\delta_1 \delta_2} \cdot (1 - C_1 - C_2) > \frac{\delta_2(1-\delta_1)}{1-\delta_1 \delta_2} \cdot (1 - C_1 - C_2)\), executive 1 is strictly best off offering \(x_2 = C_2 + \frac{\delta_2(1-\delta_1)}{1-\delta_1 \delta_2} \cdot (1 - C_1 - C_2)\) in the current period (the fourth type of strategy would give him 0, and \(\frac{1-\delta_2}{1-\delta_1 \delta_2} \cdot (1 - C_1 - C_2) > 0\)). Since this is true for any even-numbered period \(t^*\), executive 1 proposes \(x_2 = C_2 + \frac{\delta_2(1-\delta_1)}{1-\delta_1 \delta_2} \cdot (1 - C_1 - C_2)\) whenever it is his turn to offer a proposal in any subgame-perfect equilibrium in which executive 2 plays the specified strategy.

Now consider executive 1’s acceptance rule: accept any proposal \(y\) such that \(1 - C_2 \geq y_1 \geq C_1 + \frac{\delta_1(1-\delta_2)}{1-\delta_1 \delta_2} \cdot (1 - C_1 - C_2)\). Executive 1 should accept any offer \(y = (y_1, y_2)\) that (i) gives him at least his (optimal) discounted continuation value for rejecting the offer, and (ii) will be accepted by both legislatures (since if he accepts an offer that is not ratified by both legislatures he will get 0, whereas he can ensure himself \(\frac{1-\delta_2}{1-\delta_1 \delta_2} \cdot (1 - C_1 - C_2) > 0\) in the next period by rejecting the current proposal and counter-offering \(x_2 = C_2 + \frac{\delta_2(1-\delta_1)}{1-\delta_1 \delta_2} \cdot (1 - C_1 - C_2)\) in the next period, which executive 2 accepts, and which both legislatures ratify). To ensure

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2 Note that \(1 - C_2 > C_1 + \frac{\delta_1(1-\delta_2)}{1-\delta_1 \delta_2} \cdot (1 - C_1 - C_2)\) can be simplified to obtain \(C_1 + C_2 < 1\), which is true by assumption; therefore, the acceptance rule makes sense.
that he gets at least his (optimal) discounted continuation value and that his own legislature ratifies the agreement, he requires that \( y_1 \geq D + C_1 \), where \( D \) is his (optimal) discounted continuation value. His (optimal) discounted continuation value for rejecting an offer is 
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\frac{\delta_1(1-\delta_2)}{1-\delta_1\delta_2} \cdot (1 - C_1 - C_2) \quad \text{(since we have already shown that it is optimal for him to offer \( x_2 = C_2 + \frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2} \cdot (1 - C_1 - C_2) \) in the next period, and executive 2 and the legislatures accept this proposal). Therefore, he requires that \( y_1 \geq C_1 + \frac{\delta_1(1-\delta_2)}{1-\delta_1\delta_2} \cdot (1 - C_1 - C_2) \). To ensure that country 2’s legislature ratifies the agreement, he requires that \( y_2 \geq C_2 \). Since \( y_1 + y_2 = 1 \), this is equivalent to \( 1 - y_1 \geq C_2 \), or \( 1 - C_2 \geq y_1 \).

For executive 2, the arguments below are exactly analogous.

Consider executive 2’s proposal in an odd-numbered period \( t' \) (\( t' = 1, 3, 5, \ldots \)). Executive 2 can adopt one of four different types of strategies: (1) make an offer that is accepted right away, (2) use a strategy such that agreement is first reached in a future odd period, (3) use a strategy such that agreement is first reached in a future even period, or (4) use a strategy such that agreement is never reached. If he adopts a strategy of type 1, he is best off proposing \( y_1 = C_1 + \frac{\delta_1(1-\delta_2)}{1-\delta_1\delta_2} \cdot (1 - C_1 - C_2) \) in the current period, since that is the minimal proposal that executive 1 will accept (note that executive 1 is only accepting agreements that will be ratified in both legislatures). This proposal results in executive 2’s payoff being 
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\frac{1-\delta_1}{1-\delta_1\delta_2} \cdot (1 - C_1 - C_2) \]. If he uses a strategy of type 2, because executive 1’s acceptance rule doesn’t change over time and future payoffs are discounted, the best he can do using such a strategy is to offer \( y_1 = C_1 + \frac{\delta_1(1-\delta_2)}{1-\delta_1\delta_2} \cdot (1 - C_1 - C_2) \) in period \( t = t' + 2 \). Since future payoffs are discounted, it is strictly better to make that proposal in the current period. Using a strategy of type 3, because executive 1’s proposal does not change over time and future payoffs are discounted, the best executive 2 can do is to accept executive
1's proposal $x_2 = C_2 + \frac{\delta_2(1-\delta_1)}{1-\delta_1 \delta_2} \cdot (1 - C_1 - C_2)$ in the next period, $t = t' + 1$, in which case it has a current discounted value of $\frac{\delta^2(1-\delta_1)}{1-\delta_1 \delta_2} \cdot (1 - C_1 - C_2)$ for executive 2. Because

$$\frac{1-\delta_1}{1-\delta_1 \delta_2} \cdot (1 - C_1 - C_2) > \frac{\delta^2(1-\delta_1)}{1-\delta_1 \delta_2} \cdot (1 - C_1 - C_2),$$

executive 2 is strictly best off offering $y_1 = C_1 + \frac{\delta_1(1-\delta_2)}{1-\delta_1 \delta_2} \cdot (1 - C_1 - C_2)$ in the current period (the fourth type of strategy would give him 0, and $\frac{1-\delta_1}{1-\delta_1 \delta_2} \cdot (1 - C_1 - C_2) > 0$). Since this is true for any odd-numbered period $t'$, executive 2 proposes $y_1 = C_1 + \frac{\delta_1(1-\delta_2)}{1-\delta_1 \delta_2} \cdot (1 - C_1 - C_2)$ whenever it is his turn to offer a proposal in any subgame-perfect equilibrium in which executive 1 plays the specified strategy.

Now consider executive 2's acceptance rule: accept any proposal $x$ such that $1 - C_1 \geq x_2 \geq C_2 + \frac{\delta_1(1-\delta_1)}{1-\delta_1 \delta_2} \cdot (1 - C_1 - C_2)$. Executive 2 should accept any offer $x = (x_1, x_2)$ that (i) gives him at least his (optimal) discounted continuation value for rejecting the offer, and (ii) will be accepted by both legislatures (since if he accepts an offer that is not ratified by both legislatures he will get 0, whereas he can ensure himself $\frac{1-\delta_1}{1-\delta_1 \delta_2} \cdot (1 - C_1 - C_2) > 0$ in the next period by rejecting the current proposal and counter-offering $y_1 = C_1 + \frac{\delta_1(1-\delta_2)}{1-\delta_1 \delta_2} \cdot (1 - C_1 - C_2)$ in the next period, which executive 1 accepts, and which both legislatures ratify). To ensure that he gets at least his (optimal) discounted continuation value and that his own legislature ratifies the agreement, he requires that $x_2 \geq D + C_2$, where $D$ is his (optimal) discounted continuation value. His (optimal) discounted continuation value for rejecting an offer is $\frac{\delta^2(1-\delta_1)}{1-\delta_1 \delta_2} \cdot (1 - C_1 - C_2)$ (since we have already shown that it is optimal for him to offer $y_1 = C_1 + \frac{\delta_1(1-\delta_2)}{1-\delta_1 \delta_2} \cdot (1 - C_1 - C_2)$ in the next period, and executive 1 and the legislatures accept this offer). Therefore, he requires that $x_2 \geq C_2 + \frac{\delta^2(1-\delta_1)}{1-\delta_1 \delta_2} \cdot (1 - C_1 - C_2)$. To ensure that country 1’s legislature ratifies the agreement, he requires that $x_1 \geq C_1$. Since $x_1 + x_2 = 1$, this is equivalent to $1 - x_2 \geq C_1$, or $1 - C_1 \geq x_2$.

Note that $1 - C_1 > C_2 + \frac{\delta^2(1-\delta_1)}{1-\delta_1 \delta_2} \cdot (1 - C_1 - C_2)$ can be simplified to obtain $C_1 + C_2 < 1$, which is true by assumption; therefore, the acceptance rule makes sense.
Therefore, the strategies in Proposition 1 do indeed comprise a SPE. Q.E.D.

2 Explanation of Figure 1

In this section, I discuss in more detail why the graphs in Figure 1 look like they do.

Consider the case of non-equal recognition probabilities with \( p_1 = 1/3 + \epsilon, p_2 = 1/3, \) and \( p_3 = 1/3 - \epsilon, \) for \( 0 < \epsilon < 1/6. \)

As discussed in case 4, \( v_1 = (\frac{1}{3} + \epsilon)[1 - \delta (\frac{1}{3} - \epsilon)], \) \( v_2 = \frac{1}{3}[1 - \delta (\frac{1}{3} - \epsilon)] + (\frac{1}{3} - \epsilon)\frac{\delta}{3} = \frac{1}{3}, \)
and \( v_3 = (\frac{1}{3} - \epsilon)[1 - \frac{\delta}{3}] + (\frac{2}{3} + \epsilon)\delta[\frac{1}{3} - \epsilon]. \)
Note that \( \frac{dv_1}{d\epsilon} = 1 + 2\delta\epsilon > 0, \) \( \frac{dv_2}{d\epsilon} = 0, \) and \( \frac{dv_3}{d\epsilon} = -1 - 2\delta\epsilon < 0. \)

Setting \( v_1 > v_2 \) and simplifying, we obtain \( 9\delta\epsilon^2 + 9\epsilon - \delta > 0. \) Setting \( v_1 > v_3, \) we obtain the same thing. Therefore, when \( f(\epsilon) \equiv 9\delta\epsilon^2 + 9\epsilon - \delta > 0, \) \( v_1 \) is the highest. Setting \( v_2 > v_3, \) we again obtain \( f(\epsilon) > 0. \) Therefore, when \( f(\epsilon) > 0, v_1 > v_2 > v_3. \)

Similarly, it can be shown that when \( f(\epsilon) < 0, v_3 > v_2 > v_1, \) and when \( f(\epsilon) = 0, v_1 = v_2 = v_3. \)

The curve \( y = f(\epsilon) \) \((y \text{ plotted against } \epsilon)\) is a convex (opening upwards) parabola with \( y\)-intercept of \(-\delta < 0. \) Therefore, the equation \( f(\epsilon) = 0 \) has a single positive root, call it \( \epsilon_{\text{critical}}, \) which by the quadratic formula is given by \( \epsilon_{\text{critical}} = \frac{-9 + \sqrt{81 + 36\delta^2}}{18}. \) For \( \epsilon > \epsilon_{\text{critical}}, f(\epsilon) > 0 \) and so \( v_1 > v_2 > v_3. \)
When \( \epsilon < \epsilon_{\text{critical}}, f(\epsilon) < 0 \) and so \( v_3 > v_2 > v_1. \) Therefore, for \( \epsilon < \epsilon_{\text{critical}}, \) the constraint \( C \) is given by \( C = \delta(v_1 + v_2), \) and for \( \epsilon > \epsilon_{\text{critical}}, C = \delta(v_2 + v_3). \)

This is why the graphs in Figure 1 look like they do.

Notice that Figure 1 suggests that \( \epsilon_{\text{critical}} \) is increasing in \( \delta. \) Setting \( \frac{d\epsilon_{\text{critical}}}{d\delta} > 0 \) and simplifying, we obtain \( \sqrt{81 + 36\delta^2} > 9, \) which is true for \( \delta > 0. \) Note that \( \lim_{\delta \to 1} \epsilon_{\text{critical}} = \frac{-9 + \sqrt{117}}{18} \approx 0.101, \) and the upper bound of \( \epsilon \) is \( 1/6 \approx 0.167. \) Also note that \( \lim_{\delta \to 0} \epsilon_{\text{critical}} = 0, \) and the lower bound of \( \epsilon \) is of course 0.
3 Robustness Check

To provide support for the idea that the relationship between the executive’s constraint and party 1’s recognition probability $p_1$ displayed in Figure 3 does not depend on the specific functional form that I used for the recognition probabilities, in Figures 4 and 5 I plot the constraint as a function of $p_1$ for two other functional forms. Figure 4 shows the constraint as a function of $1/3 < p_1 < 1$, where for $p_1 < 1/2$, the recognition probabilities are given by $p_2 = p_3 = \frac{1-p_1}{2}$. That is, as party 1 becomes larger, the other two parties become smaller at the same rate, and as $p_1 \to \frac{1}{2}$ (from below), the party composition of the legislature approaches $\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}\}$. In Figure 5, for $p_1 < 1/2$, the recognition probabilities are given by $p_2 = p_1$ and $p_3 = 1 - 2p_1$. That is, as party 1 becomes larger, party 2 becomes larger at the same rate, and as $p_1 \to \frac{1}{2}$ (from below), the party composition of the legislature approaches $\{\frac{1}{2}, \frac{1}{2}, 0\}$. The graphs have the same general shape in Figures 4 and 5 as they do in Figure 3. This suggests that the qualitative results do not depend on the specific functional form used for the recognition probabilities.\(^4\)

\(^4\)When two parties have the same recognition probabilities, as in Figures 4 and 5 for $p_1 < 1/2$, I use the unique symmetric subgame-perfect equilibrium in which the other party, if recognized in period $t = 0$, tosses a coin to decide which of the other two parties to offer pie to.
Figure 4: Constraint as a function of $p_1$, for four different values of $\delta$, as $p_1$ ranges from $1/3$ to 1. For $p_1 < \frac{1}{2}$, the recognition probabilities are given by $p_2 = p_3 = \frac{1-p_1}{2}$. Hence, as $p_1 \to \frac{1}{2}$ (from below), the party composition of the legislature approaches \{\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\}. 
Figure 5: Constraint as a function of $p_1$, for four different values of $\delta$, as $p_1$ ranges from $1/3$ to 1. For $p_1 < \frac{1}{2}$, the recognition probabilities are given by $p_2 = p_1$ and $p_3 = 1 - 2p_1$. Hence, as $p_1 \to \frac{1}{2}$ (from below), the party composition of the legislature approaches $\{\frac{1}{2}, \frac{1}{2}, 0\}$. 