Exam 1 Solution

1.(a) $\langle 0|p_0\rangle = \langle 0|e^{-ip_0\hat{x}/\hbar}|0\rangle$

$= \langle 0|e^{-ib(\hat{a}^\dagger+\hat{a})}|0\rangle \quad , \quad b \equiv \frac{p_0}{\hbar}\sqrt{\frac{\hbar}{2m\omega}}$

$= \langle 0|e^{-ib\hat{a}^\dagger}e^{-ib\hat{a}}e^{-\underbrace{[-ib\hat{a}^\dagger,-ib\hat{a}]}_{=+[\hat{a},\hat{a}^\dagger]b^2=b^2}/2}$

$= \langle 0|e^{-ib\hat{a}^\dagger}e^{-ib\hat{a}}e^{-b^2/2}|0\rangle$

$= e^{-b^2/2}$ since $e^{-ib\hat{a}}|0\rangle = (1-ib\hat{a}+\ldots)|0\rangle = |0\rangle$ $[\hat{a}|0\rangle = 0]$

and $\langle 0|0\rangle = 1$

$= e^{-p_0^2/4m\hbar\omega}$

so $\boxed{P =} |\langle 0|p_0\rangle|^2 = \boxed{e^{-p_0^2/2m\hbar\omega}}$

(b) $C_0 = \int_{-\infty}^{\infty} dx\, \psi_0^* e^{-ip_0x/\hbar}\psi_0$

$= \left(\frac{m\omega}{\hbar\pi}\right)^{1/2}\int_{-\infty}^{\infty} dx\, e^{-ax^2+bx}$, $a \equiv \frac{m\omega}{\hbar}$ and $b \equiv -i\frac{p_0}{\hbar}$

[different than in (a)]

$= \left(\frac{m\omega}{\hbar\pi}\right)^{1/2}\sqrt{\frac{\pi}{a}}\, e^{b^2/4a}$

$= \left(\frac{m\omega}{\hbar\pi}\right)^{1/2}\left(\frac{\hbar\pi}{m\omega}\right)^{1/2} e^{-p_0^2/4m\hbar\omega}$

$= e^{-p_0^2/4m\hbar\omega}$

so $\boxed{P =} |C_0|^2 = \boxed{e^{-p_0^2/2m\hbar\omega}}$

2. (a) $-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi + \mathcal{V}_0\,\delta(x)\psi = E\psi$

$\int_{-\epsilon}^{+\epsilon} dx$ gives $-\frac{\hbar^2}{2m}\left[\frac{d\psi}{dx}\right]_{-\epsilon}^{\epsilon} + \mathcal{V}_0\psi = 0$ [since ψ is continuous]

or $-\frac{\hbar^2}{2m}(-\kappa\psi - \kappa\psi) + \mathcal{V}_0\psi = 0$ [since $\psi = Ae^{-\kappa|x|}$]

so $\boxed{\kappa =} - \frac{m\mathcal{V}_0}{\hbar^2} = \boxed{\frac{m}{\hbar^2}|\mathcal{V}_0|}$

(b) $1 = \int_{-\infty}^{\infty} dx\, A^2 e^{-2\kappa|x|} = 2A^2\int_0^{\infty} dx\, e^{-2\kappa x} = 2A^2\left[\frac{e^{-2\kappa x}}{-2\kappa}\right]_0^{\infty} = \frac{A^2}{\kappa}$

$\Rightarrow A = \sqrt{\kappa} \Rightarrow \boxed{\psi(x) = \sqrt{\kappa}\, e^{-\kappa|x|}}$

3.(a) $\left[\hat{x}_i\hat{p}_i, \frac{\hat{p}_i^2}{2m}\right] = -\frac{1}{2m}\left[\hat{p}_i^2, \hat{x}_i\hat{p}_i\right]$

$= -\frac{1}{2m}\left[\hat{p}_i^2, \hat{x}_i\right]\hat{p}_i - \frac{1}{2m}\hat{x}_i\underbrace{\left[\hat{p}_i^2, \hat{p}_i\right]}_{=0}$

$= \frac{1}{2m}\left[\hat{x}_i, \hat{p}_i^2\right]\hat{p}_i$

$= \frac{1}{2m}\left[\hat{x}_i, \hat{p}_i\right]\hat{p}_i^2 + \frac{1}{2m}\hat{p}_i\underbrace{\left[\hat{x}_i, \hat{p}_i\right]}_{=i\hbar}\hat{p}_i$

$= \frac{1}{m}i\hbar\,\hat{p}_i^2$

$[\hat{x}_i\hat{p}_i, V(\hat{\vec{x}})] = -[V(\hat{\vec{x}}), \hat{x}_i\hat{p}_i]$

$= -\underbrace{[V(\hat{\vec{x}}), \hat{x}_i]}_{=0}\hat{p}_i - \hat{x}_i\underbrace{[V(\hat{\vec{x}}), \hat{p}_i]}_{=+i\hbar\frac{\partial V(\hat{\vec{x}})}{\partial\hat{x}_i}}$

$= -i\hbar\,\hat{x}_i\frac{\partial V(\hat{\vec{x}})}{\partial \hat{x}_i}$

Then $[\hat{x}_i\hat{p}_i, \hat{H}] = i\hbar\frac{\hat{p}_i^2}{m} - i\hbar\,\hat{x}_i\frac{\partial V(\hat{\vec{x}})}{\partial\hat{x}_i}$

and $[\hat{\vec{x}}\cdot\hat{\vec{p}}, \hat{H}] = i\hbar\frac{\hat{\vec{p}}^2}{m} - i\hbar\,\hat{\vec{x}}\cdot\vec{\nabla}V(\hat{\vec{x}})$

components are $\frac{\partial}{\partial\hat{x}_i}$

and $\frac{d}{dt}(\hat{\vec{x}}\cdot\hat{\vec{p}}) = \frac{1}{i\hbar}[\hat{\vec{x}}\cdot\hat{\vec{p}}, \hat{H}]$

$= \frac{\hat{\vec{p}}^2}{m} - \hat{\vec{x}}\cdot\vec{\nabla}V(\hat{\vec{x}})$

sum over i implied, $\hat{x}_i \to x_i$ in notation

Take expectation value.

(b) $0 = 2\langle\hat{T}\rangle - \langle x_i\underbrace{\frac{\partial}{\partial x_i}\left(-k\frac{e^2}{r}\right)}_{=-ke^2\left(-\frac{x_i}{r^3}\right)}\rangle$ if worked out

$= 2\langle\hat{T}\rangle + \langle\hat{V}\rangle \Rightarrow \boxed{\langle\hat{V}\rangle = -2\langle\hat{T}\rangle}$

4. (a) $$\boxed{\langle \hat{x} \rangle} = \sqrt{\frac{1}{a^2\pi}} \int_{-\infty}^{\infty} dx\, e^{-(x-x_0)^2/a^2}\, x$$

$$= \sqrt{\frac{1}{a^2\pi}} \underbrace{\int_{-\infty}^{\infty} d(x-x_0)\, e^{-(x-x_0)^2/a^2} (x-x_0)}_{=0 \text{ since odd integrand}} + \sqrt{\frac{1}{a^2\pi}} \underbrace{\int_{-\infty}^{\infty} d(x-x_0)\, e^{-(x-x_0)^2/a^2}\, x_0}_{= x_0 \sqrt{a^2\pi}}$$

$$\boxed{= x_0}$$

$$\boxed{(\Delta x)^2} = \frac{1}{\sqrt{a^2\pi}} \int_{-\infty}^{\infty} dx\, e^{-(x-x_0)^2/a^2} (x - \langle x \rangle)^2$$

$$= \frac{1}{\sqrt{a^2\pi}} \int_{-\infty}^{\infty} d(x-x_0)\, e^{-(x-x_0)^2/a^2} (x-x_0)^2$$

$$= \frac{1}{\sqrt{a^2\pi}} \frac{a^2}{2} a\sqrt{\pi}$$

$$\boxed{= \frac{a^2}{2}}$$

(b) $$\boxed{\langle p \rangle} = \frac{1}{\sqrt{a^2\pi}} \int_{-\infty}^{\infty} dx\, e^{-(x-x_0)^2/2a^2} e^{-ip_0x/\hbar}$$

$$\times \underbrace{\left(-i\hbar \frac{d}{dx}\right) e^{ip_0x/\hbar}\, e^{-(x-x_0)^2/2a^2}}_{= -i\hbar\left[i\frac{p_0}{\hbar} - 2\frac{x-x_0}{2a^2}\right] e^{ip_0x/\hbar}\, e^{-(x-x_0)^2/2a^2}}$$

$$= \frac{1}{\sqrt{a^2\pi}} \left[\int_{-\infty}^{\infty} dx\, e^{-(x-x_0)^2/a^2}\, p_0 + \underbrace{\text{integral odd in } (x-x_0)}_{=0}\right]$$

$$= \frac{1}{\sqrt{a^2\pi}} \cdot \sqrt{a^2\pi}\, p_0$$

$$\boxed{= p_0}$$

$$x' \equiv x - x_0$$

$$\langle (\hat{p} - p_0)^2 \rangle = \frac{1}{\sqrt{a^2\pi}} \int_{-\infty}^{\infty} dx'\, e^{-x'^2/2a^2} e^{-ip_0x'/\hbar} \underbrace{\left(-i\hbar\frac{d}{dx'} - p_0\right)^2 e^{ip_0x'/\hbar} e^{-x'^2/2a^2}}_{\equiv f}$$

$$f = \left(-i\hbar\frac{d}{dx'} - p_0\right)\left(+i\hbar\frac{x'}{a^2}\right) e^{ip_0x'/\hbar} e^{-x'^2/2a^2}$$

$$= \frac{\hbar^2}{a^2} e^{ip_0x'/\hbar}\left(1 - x'\cdot\frac{x'}{a^2}\right) e^{-x'^2/2a^2}$$

$$\boxed{\text{some steps skipped or shortened}}$$

so $$\boxed{\langle (p - p_0)^2 \rangle} = \frac{1}{\sqrt{a^2\pi}} \frac{\hbar^2}{a^2}\left[\underbrace{\int_{-\infty}^{\infty} dx'\, e^{-x'^2/a^2}}_{=\sqrt{a^2\pi}} - \frac{1}{a^2}\underbrace{\int_{-\infty}^{\infty} dx'\, x'^2 e^{-x'^2/a^2}}_{=\frac{a^2}{2}a\sqrt{\pi}}\right]$$

$$\parallel$$
$$(\Delta p)^2$$

$$= \frac{\hbar^2}{a^2}\left(1 - \frac{1}{2}\right)$$

$$\boxed{= \frac{\hbar^2}{2a^2}}$$

(c) Then $\Delta x = \frac{a}{\sqrt{2}}$, $\Delta p = \frac{\hbar}{\sqrt{2}a}$, and $\boxed{\Delta p\, \Delta x} = \frac{\hbar}{\sqrt{2}a} \cdot \frac{a}{\sqrt{2}} \boxed{= \frac{1}{2}\hbar}$.