Physics 606, Quantum Mechanics, Exam 1

NAME________________________________

Please show all your work.

(You are graded on your work, with partial credit where it is deserved.)

All problems are, of course, nonrelativistic.

Vectors and matrices are in boldface.

According to a future homework problem, the commutation relation between $\hat{x}_i$ and $\hat{p}_i$ implies that

$$\left[\hat{p}_i, F\left(\hat{\mathbf{r}}\right)\right] = -i\hbar \frac{\partial F}{\partial \hat{x}_i} \qquad \text{and} \qquad \left[\hat{x}_i, G\left(\hat{\mathbf{p}}\right)\right] = i\hbar \frac{\partial G}{\partial \hat{p}_i} \quad .$$

Possibly useful:

$$\int_{-\infty}^{\infty} du\, e^{-au^2+bu+c} = \sqrt{\frac{\pi}{a}}\, e^{\frac{b^2}{4a}+c}$$

$$\int_{-\infty}^{\infty} du\, u\, e^{-au^2+bu+c} = \frac{d}{db}\int_{-\infty}^{\infty} du\, e^{-au^2+bu+c} = \frac{b}{2a}\sqrt{\frac{\pi}{a}}\, e^{\frac{b^2}{4a}+c}$$

$$\int_{-\infty}^{\infty} du\, u^2\, e^{-au^2+bu+c} = \frac{d^2}{db^2}\int_{-\infty}^{\infty} du\, e^{-au^2+bu+c} = \frac{1}{2a}\sqrt{\frac{\pi}{a}}\, e^{\frac{b^2}{4a}+c} + \left(\frac{b}{2a}\right)^2 \sqrt{\frac{\pi}{a}}\, e^{\frac{b^2}{4a}+c}$$

$$\left[\hat{A}, \hat{B}\hat{C}\right] = \left[\hat{A}, \hat{B}\right]\hat{C} + \hat{B}\left[\hat{A}, \hat{C}\right]$$

1. (25) A harmonic oscillator is initially in its ground state $|0\rangle$, when an instantaneous force imparts to it a momentum p_0. This means that the new state is $|p_0\rangle = e^{-ip_0\hat{x}/\hbar}|0\rangle$.

   We wish to calculate the probability $P = |\langle 0|p_0\rangle|^2$ that the oscillator will be found in its ground state if a measurement is performed, in terms of p_0, $\hbar$, the particle mass m, and the angular frequency ω.

   Let us do the calculation two different ways, which must give the same answer.

   (a) Use

   $$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a}^\dagger + \hat{a}\right) \qquad \text{and} \qquad e^{\hat{A}+\hat{B}} = e^{\hat{A}}e^{\hat{B}}e^{-[\hat{A},\hat{B}]/2}$$

   to calculate P.

   (b) Now define $\psi_0(x)$ and $\psi_{p_0}(x)$ to be the wavefunctions for the ground state and the state after the momentum p_0 has been imparted, respectively. The ground state wavefunction is

   $$\psi_0(x) = \left(\frac{m\omega}{\hbar\pi}\right)^{1/4} e^{-m\omega x^2/2\hbar} \quad .$$

   Recall that $\psi(x) = \langle x|\psi\rangle$ and use

   $$\psi_{p_0}(x) = \sum_{n=0}^{\infty} c_n \psi_n(x) \quad , \quad c_n = \int_{-\infty}^{\infty} dx\, \psi_n^*(x)\psi_{p_0}(x)$$

   to calculate $P = |c_0|^2$, by doing the integral.

2. (25) In class we prepared for scattering problems in 3 dimensions by using a Green's function to treat both propagating and bound states for a 1 dimensional "atom" with potential

$$v(x) = v_0 \delta(x) \quad .$$

Here let us treat the bound state problem, for $v_0 < 0$, with a more direct approach.

Take the (trial) wavefunction to have the form $\psi(x) = Ae^{-\kappa|x|}$ where κ is real.

(a) By integrating the (time-independent) Schrödinger equation from $x = -\varepsilon$ to $x = +\varepsilon$, calculate κ in terms of v_0, the particle mass m, and $\hbar$.

(b) From the normalization condition on the wavefunction, calculate A in terms of κ.

3. (25) A particle in 3 dimensions has the Hamiltonian

$$\widehat{H} = \widehat{T} + \widehat{V} = \frac{\hat{\mathbf{p}}^2}{2m} + V(\hat{\mathbf{x}}) \quad .$$

(Here $\hat{\mathbf{x}}$ is the position vector, also called $\hat{\mathbf{r}}$.)

(a) By calculating $\left[\hat{\mathbf{x}}\cdot\hat{\mathbf{p}}\,,\widehat{H}\right]$ obtain

$$\frac{d}{dt}\left\langle \hat{\mathbf{x}}\cdot\hat{\mathbf{p}}\right\rangle = \left\langle \frac{\hat{\mathbf{p}}^2}{m}\right\rangle - \left\langle \hat{\mathbf{x}}\cdot\nabla V\right\rangle \quad .$$

(This is the quantum version of the virial theorem.)

(b) For an electron in a hydrogen atom which is in a stationary state, determine how the expectation values of the kinetic and potential energy, $\langle \widehat{T} \rangle$ and $\langle \widehat{V} \rangle$, are related.

4. (25) A particle has the wavefunction

$$\psi(x) = \left(\frac{1}{a^2\pi}\right)^{1/4} e^{ip_0 x/\hbar}\, e^{-(x-x_0)^2/2a^2}$$

(i.e., a plane wave times a Gaussian).

(a) Calculate the expectation value of the position $\langle \hat{x} \rangle$ and the uncertainty in position Δx.

(b) Calculate the expectation value of the momentum $\langle \hat{p} \rangle$ and the uncertainty in momentum Δp.

(c) Calculate the uncertainty product $\Delta p\, \Delta x$.