1. (15) A particle is in an eigenstate of the angular momentum operator $\hat{L}_Z$:

$$\hat{L}_Z|m\rangle = m\hbar|m\rangle \ .$$

Calculate the expectation values of $\hat{L}_X$ and $\hat{L}_Y$, $\langle m|\hat{L}_X|m\rangle$ and $\langle m|\hat{L}_Y|m\rangle$.

[Hint: One method involves using the commutation relations for the angular momentum operators.]

$$[\hat{L}_y, \hat{L}_z] = i\hbar \hat{L}_x \quad , \quad [\hat{L}_z, \hat{L}_x] = i\hbar \hat{L}_y$$

since $[\hat{L}_i, \hat{L}_j] = i\hbar\, \varepsilon_{ijk} \hat{L}_k$ with $1,2,3 \leftrightarrow x,y,z$

$$\therefore \boxed{\langle m|\hat{L}_x|m\rangle =} \frac{1}{i\hbar}\left(\langle m|\hat{L}_y \hat{L}_z|m\rangle - \langle m|\hat{L}_z \hat{L}_y|m\rangle\right)$$

$$= \frac{1}{i\hbar}\left(m\hbar \langle m|\hat{L}_y|m\rangle - m\hbar\langle m|\hat{L}_y|m\rangle\right)$$

$$= \boxed{0}$$

Similarly, $\boxed{\langle m|\hat{L}_y|m\rangle = 0}$.

[Another method is to use

$$\hat{L}_x = \frac{1}{2}(\hat{L}_+ + \hat{L}_-) \quad , \quad \hat{L}_y = \frac{1}{2i}(\hat{L}_+ - \hat{L}_-) \ .]$$

2. An electron in a hydrogen atom is in the state

$$|\psi\rangle = A\left(3|1,0,0\rangle + |2,1,1\rangle - |2,1,0\rangle + |2,1,-1\rangle\right)$$

where the eigenstates are labeled $|n,l,m\rangle$.

You may use what you know about the energies and other properties of the eigenstates.

(a) (5) Calculate the normalization constant A (for the state to normalized to one).

(b) (5) Calculate the expectation value of the energy, as a dimensionless constant times the energy $E_1 = -\dfrac{ke^2}{2a_0}$ of the ground state.

(c) (5) Calculate the expectation value of the orbital angular momentum operator $\hat{L}^2$.

(d) (5) Calculate the expectation value of $\hat{L}_Z$.

(a) $1 = \langle\psi|\psi\rangle = |A|^2 (9+1+1+1) = |A|^2 \cdot 12$

$\Rightarrow \boxed{A = \frac{1}{\sqrt{12}}}$ if chosen real & positive

(b) $\boxed{\langle\hat{H}\rangle =} \langle\psi|H|\psi\rangle$

$= \frac{1}{12}(9E_1 + E_2 + E_2 + E_2)$, $E_2 = \frac{E_1}{2^2} = \frac{1}{4}E_1$

$= \frac{1}{12}(9 + \frac{3}{4})E_1$

$= \boxed{\frac{13}{16}E_1}$

(c) $\boxed{\langle\hat{L}^2\rangle =} \langle\psi|\hat{L}^2|\psi\rangle$

$= \frac{1}{12}(9\cdot 0 + 2 + 2 + 2)\hbar^2$ $[\, l=1 \Rightarrow l(l+1) = 2 \,]$

$= \boxed{\frac{1}{2}\hbar^2}$

(d) $\boxed{\langle\hat{L}_z\rangle =} \frac{1}{12}(9\cdot 0 + 1 + 0 - 1)\hbar$

$= \boxed{0}$

3. (15) Show that

$$p_r = -i\hbar\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)$$

is a Hermitian operator. Assume that the functions are finite at $r=0$ and that they $\to 0$ as $r\to\infty$.

$$I = \int d^3r\ \psi_1^*(\vec{r})\left[-i\hbar\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)\right]\psi_2(\vec{r})$$

$$= -i\hbar\int d\Omega\ \underbrace{\int_0^\infty dr\ r^2\ \psi_1^*(\vec{r})\frac{\partial}{\partial r}\psi_2(\vec{r})}_{\equiv I'} - i\hbar\int d^3r\left[\frac{1}{r}\psi_1^*(\vec{r})\right]^*\psi_2(\vec{r})$$

with $\int d\Omega = \int_0^\pi d\theta\ \sin\theta\int_0^{2\pi}d\phi$

$$I' = \int_0^\infty \underbrace{r^2\psi_1^*}_{u}\ \underbrace{d\psi_2}_{dv}$$

$$= \underbrace{\left[r^2\psi_1^*\psi_2\right]_0^\infty}_{\Rightarrow 0 \text{ assuming } \psi_1^* \,\&\, \psi_2 \to 0 \text{ fast enough as } r\to\infty} - \int_0^\infty \psi_2\, d(r^2\psi_1^*)$$

$$= -\int_0^\infty \psi_2\left(r^2 d\psi_1^* + 2r\,dr\cdot\psi_1^*\right)$$

$$= -\int_0^\infty dr\ r^2\left(\frac{\partial\psi_1^*}{\partial r}+\frac{2}{r}\psi_1^*\right)\psi_2$$

Then $$I = +i\hbar\int d\Omega\int_0^\infty dr\ r^2\left[\left(\frac{\partial\psi_1^*}{\partial r}+\frac{2}{r}\psi_1^*\right)\psi_2 - \frac{1}{r}\psi_1^*\psi_2\right]$$

$$= \int d^3r\left(-i\hbar\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)\psi_1\right)^*\psi_2 \quad .$$

I.e., for $\hat{p}_r = -i\hbar\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)$, we have shown that

$$\int d^3r\ \psi_1^*(\vec{r})\,\hat{p}_r\,\psi_2(\vec{r}) = \int d^3r\left(\hat{p}_r\,\psi_1(\vec{r})\right)^*\psi_2(\vec{r}) \quad .$$

4. Slow neutrons with momentum $\hbar\vec{k} = \hbar k\hat{z}$, pointing in the z direction, are scattered off a diatomic molecule. (Here $\hat{z}$ is the unit vector along the z axis.) The molecule has atoms centered at $y-b$ and $y+b$, and it is modeled by the potential

$$V(\mathbf{r}) = a\delta(y-b)\delta(x)\delta(z) + a\delta(y+b)\delta(x)\delta(z) \ .$$

(a) (20) Calculate the scattering amplitude in the (first-order) Born approximation, as a function of the polar angle θ and the azimuthal angle ϕ for the scattered wavevector $\vec{k}'$ (with your answer also depending on a, b, $\hbar$, and the neutron mass m, of course).

Recall that $k'_y = k'\sin\theta\sin\phi$.

(b) (3) Calculate the differential scattering cross-section $\frac{d\sigma}{d\Omega}$ as a function of these same quantities.

(c) (2) What most obviously demonstrates that this is a quantum-mechanical and not a classical result?

(a) $$\boxed{f(\vec{k}, \vec{k}') =} - \frac{m}{2\pi\hbar^2}\int d^3r'\, e^{i(\vec{k}-\vec{k}')\cdot\vec{r}'}\, V(\vec{r}')$$

[with $\psi_{\vec{k}}(\vec{r}') \to e^{i\vec{k}\cdot\vec{r}'}$ in Born approx.]

$$= -\frac{ma}{2\pi\hbar^2}\left(e^{+ik'_y b} + e^{-ik'_y b}\right)$$

$$= -\frac{ma}{\pi\hbar^2}\cos(k'_y b)$$

$$= \boxed{-\frac{ma}{\pi\hbar^2}\cos(kb\sin\theta\sin\phi)}$$

(b) $$\boxed{\frac{d\sigma}{d\Omega} = \left(\frac{ma}{\pi\hbar^2}\right)^2 \cos^2(kb\sin\theta\sin\phi)}$$

(c) clearly interference, also $\hbar$ dependence

5. A particle moves in a central potential $V(r)$. The potential is short-range, and this means, as usual, that

$$V(r) \to 0 \quad \text{and} \quad \psi(r) \to \text{constant} \times \frac{e^{ikr}}{r} \quad \text{as} \quad r \to \infty$$

where k is real for a scattering state and imaginary for a bound state.

But here we are given that an energy eigenstate of the particle has precisely the form

$$\psi(r) = A\frac{e^{-\alpha r} + e^{-\beta r}}{r}$$

for all r, with $\beta > \alpha$.

(a) (5) What is the angular momentum quantum number ℓ for this state? Explain.

(b) (10) Determine the energy of this state.

(c) (10) Calculate the potential $V(r)$ that produced this state.

(a) can calculate, but no θ or ϕ dependence (f(r) only) $\Rightarrow \boxed{\ell = 0}$

(b) As $r \to \infty$,

$$\left[-\frac{\hbar^2}{2m}\frac{1}{r}\frac{d^2}{dr^2} r\right]\psi(r) = E\,\psi(r) \qquad \text{with } \psi(r) \to A\frac{e^{-\alpha r}}{r}$$

[since $\beta > \alpha$]

$$\Rightarrow -\frac{\hbar^2}{2m}\frac{d^2}{dr^2} e^{-\alpha r} = E\,e^{-\alpha r}$$

$$\Rightarrow \boxed{E = -\frac{\hbar^2}{2m}\alpha^2}$$

(c) For general r,

$$\left[-\frac{\hbar^2}{2m}\frac{1}{r}\frac{d^2}{dr^2} r + V(r)\right]\psi(r) = E\,\psi(r)$$

$$\Rightarrow -\frac{\hbar^2}{2m}\underbrace{\frac{d^2}{dr^2}(e^{-\alpha r} + e^{-\beta r})}_{(\alpha^2 e^{-\alpha r} + \beta^2 e^{-\beta r})} + V(r)(e^{-\alpha r} + e^{-\beta r}) = -\frac{\hbar^2}{2m}\alpha^2(e^{-\alpha r} + e^{-\beta r})$$

$$\Rightarrow \boxed{V(r) =} -\frac{\hbar^2}{2m}\alpha^2 + \frac{\hbar^2}{2m}\frac{\alpha^2 e^{-\alpha r} + \beta^2 e^{-\beta r}}{e^{-\alpha r} + e^{-\beta r}}$$

$$= \frac{\hbar^2}{2m}\frac{\cancel{-\alpha^2 e^{-\alpha r}} - \alpha^2 e^{-\beta r} + \cancel{\alpha^2 e^{-\alpha r}} + \beta^2 e^{-\beta r}}{e^{-\alpha r} + e^{-\beta r}}$$

$$= \boxed{-\frac{\hbar^2}{2m}\frac{(\beta^2 - \alpha^2)\,e^{-\beta r}}{e^{-\alpha r} + e^{-\beta r}}}$$