Physics 606, Quantum Mechanics, Exam 2

Please show all your work.

(You are graded on your work, with partial credit where it is deserved.)

All problems are, of course, nonrelativistic.

Vectors here are marked with arrows.

$\hat{L}^2$ etc. here are operators in the coordinate representation.

Some potentially useful equations:

$$\left[-\frac{\hbar^2}{2m}\frac{1}{r}\frac{\partial^2}{\partial r^2}r+\frac{\hat{L}^2}{2mr^2}+V(r)\right]\psi(\vec{r})=E\psi(\vec{r})$$

$$\psi_{\vec{k}}(\vec{r})=e^{i\vec{k}\cdot\vec{r}}+\frac{e^{ikr}}{r}f(\vec{k}',\vec{k})$$

$$f(\vec{k}',\vec{k})=-\frac{m}{2\pi\hbar^2}\int d^3r'\,e^{-i\vec{k}'\cdot\vec{r}'}V(\vec{r}')\psi_{\vec{k}}(\vec{r}')$$

$$\frac{d\sigma}{\Omega}=\left|f(\vec{k}',\vec{k})\right|^2$$

1. (15) A particle is in an eigenstate of the angular momentum operator $\hat{L}_z$:

$$\hat{L}_z|m\rangle = m\hbar|m\rangle \quad .$$

Calculate the expectation values of $\hat{L}_x$ and $\hat{L}_y$, $\langle m|\hat{L}_x|m\rangle$ and $\langle m|\hat{L}_y|m\rangle$.

[Hint: One method involves using the commutation relations for the angular momentum operators.]

2. An electron in a hydrogen atom is in the state

$$|\psi\rangle = A\left(3|1,0,0\rangle + |2,1,1\rangle - |2,1,0\rangle + |2,1,-1\rangle\right)$$

where the eigenstates are labeled $|n,l,m\rangle$.

You may use what you know about the energies and other properties of the eigenstates.

(a) (5) Calculate the normalization constant A (for the state to normalized to one).

(b) (5) Calculate the expectation value of the energy, as a dimensionless constant times the energy $E_1 = -\frac{ke^2}{2a_0}$ of the ground state.

(c) (5) Calculate the expectation value of the orbital angular momentum operator $\hat{L}^2$.

(d) (5) Calculate the expectation value of $\hat{L}_z$.

3. (15) Show that

$$p_r = -i\hbar\left(\frac{\partial}{\partial r} + \frac{1}{r}\right)$$

is a Hermitian operator. Assume that the functions are finite at $r = 0$ and that they $\rightarrow 0$ as $r \rightarrow \infty$.

4. Slow neutrons with momentum $\hbar\vec{k} = \hbar k\hat{z}$, pointing in the z direction, are scattered off a diatomic molecule. (Here $\hat{z}$ is the unit vector along the z axis.) The molecule has atoms centered at $y-b$ and $y+b$, and it is modeled by the potential

$$V(\mathbf{r}) = a\delta(y-b)\delta(x)\delta(z) + a\delta(y+b)\delta(x)\delta(z) \quad .$$

(a) (20) Calculate the scattering amplitude in the (first-order) Born approximation, as a function of the polar angle θ and the azimuthal angle ϕ for the scattered wavevector $\vec{k}'$ (with your answer also depending on a, b, $\hbar$, and the neutron mass m, of course).

(b) Recall that $k'_y = k'\sin\theta\sin\phi$.

(c) (3) Calculate the differential scattering cross-section $\frac{d\sigma}{d\Omega}$ as a function of these same quantities.

(d) (2) What most obviously demonstrates that this is a quantum-mechanical and not a classical result?

5. A particle moves in a central potential $V(r)$. The potential is short-range, and this means, as usual, that

$$V(r) \to 0 \quad \text{and} \quad \psi(r) \to \text{constant} \times \frac{e^{ikr}}{r} \quad \text{as} \quad r \to \infty$$

where k is real for a scattering state and imaginary for a bound state.

But here we are given that an energy eigenstate of the particle has precisely the form

$$\psi(r) = A\frac{e^{-\alpha r} + e^{-\beta r}}{r}$$

for all r, with $\beta > \alpha$.

(a) (5) What is the angular momentum quantum number ℓ for this state? Explain.

(b) (10) Determine the energy of this state.

(c) (10) Calculate the potential $V(r)$ that produced this state.