Physics 624 Exam 1 Solution

1. probability amplitude $= \langle Z=2 \text{ ground state} \,|\, Z=1 \text{ ground state}\rangle$

$$= \int d^3r \, \frac{1}{\sqrt{\pi}}\left(\frac{2}{a_0}\right)^{3/2} e^{-2r/a_0} \cdot \frac{1}{\sqrt{\pi}}\left(\frac{1}{a_0}\right)^{3/2} e^{-r/a_0}$$

$$= \frac{1}{\pi}\,\frac{2^{3/2}}{a_0^3}\int_0^\infty dr \cdot 4\pi r^2 e^{-3r/a_0}$$

$$= \frac{8\sqrt{2}}{a_0^3}\cdot\left(\frac{a_0}{3}\right)^3 \underbrace{\int_0^\infty du \; u^2 e^{-u}}_{=2} \quad , \quad u \equiv \frac{3}{a_0} r$$

$$= \frac{16\sqrt{2}}{27}$$

$$\therefore \boxed{\text{probability} = \left(\frac{16}{27}\right)^2 \cdot 2} \approx 70\%$$

2. (a) $F = -\dfrac{dV_t}{dx} \Rightarrow V_t = -\int dx\, F + \text{constant} = -F_0\, x\, e^{-t/\tau}$

(b) $\boxed{P_{0\to1}(t)} = \left| \dfrac{1}{i\hbar}\displaystyle\int_0^t dt'\, e^{i\omega t} \underbrace{\langle 1 | (-F_0 x e^{-t/\tau}) | 0\rangle}_{= -F_0\sqrt{\frac{\hbar}{2m\omega}}\, e^{-t/\tau}} \right|^2$

$$= \frac{F_0^2}{2m\hbar\omega}\left|\int_0^t dt'\, e^{(i\omega-\frac{1}{\tau})t'}\right|^2$$

$$= \frac{F_0^2}{2m\hbar\omega}\left|\left[\frac{e^{(i\omega-\frac{1}{\tau})t'}}{i\omega-\frac{1}{\tau}}\right]_0^t\right|^2$$

$$= \frac{F_0^2}{2m\hbar\omega}\;\frac{e^{(i\omega-\frac{1}{\tau})t}-1}{i\omega-\frac{1}{\tau}}\cdot\frac{e^{(-i\omega-\frac{1}{\tau})t}-1}{-i\omega-\frac{1}{\tau}} \qquad [|z|^2 = zz^*]$$

$$= \frac{F_0^2}{2m\hbar\omega}\;\frac{e^{-2t/\tau} - e^{-t/\tau}(e^{i\omega t}+e^{-i\omega t})+1}{\omega^2+\frac{1}{\tau^2}}$$

$$\boxed{= \frac{F_0^2}{2m\hbar\omega}\;\frac{1-2e^{-t/\tau}\cos(\omega t)+e^{-2t/\tau}}{\omega^2+\frac{1}{\tau^2}}}$$

As $t\to\infty$,

$$P_{0\to1}(t) \to \boxed{\frac{F_0^2}{2m\hbar\omega}\;\frac{1}{\omega^2+\frac{1}{\tau^2}}}$$

which is a finite constant because F falls rapidly to zero.

3. (a) $\boxed{a_n(t) =} \frac{1}{i\hbar} e^{-i\mathcal{E}_n t/\hbar} \int_{-\infty}^{t} dt'\, e^{-i(\mathcal{E}_n - \mathcal{E}_0)t'/\hbar} \langle n | (-\mu_z E_0 (e^{i\omega t} + e^{-i\omega t})) | 0 \rangle$

$$= -\frac{E_0}{i\hbar} e^{-i\mathcal{E}_n t/\hbar} \langle n|\mu_z|0\rangle \left(\int_{-\infty}^{t} dt'\, e^{i(\omega_{n0}+\omega)t'} + \int_{-\infty}^{t} dt'\, e^{i(\omega_{n0}-\omega)t'} \right)$$

$$\rightarrow \boxed{\frac{E_0}{\hbar} e^{-i\mathcal{E}_n t/\hbar} \langle n|\mu_z|0\rangle \left(\frac{e^{i(\omega_{n0}+\omega)t}}{\omega_{n0}+\omega} + \frac{e^{i(\omega_{n0}-\omega)t}}{\omega_{n0}-\omega} \right)}$$

(b) With $\langle n|\mu_z|n\rangle = 0$ and $a_0 \approx e^{-i\mathcal{E}_0 t/\hbar}$

$$\boxed{\langle \psi_t|\mu_z|\psi_t\rangle =} \left(\langle 0|a_0^* + \sum_{n\neq 0} \langle n|a_n^* \right) \mu_z \left(a_0|0\rangle + \sum_{n\neq 0} a_n|n\rangle \right)$$

$$= \boxed{\sum_{n\neq 0} \left(e^{i\mathcal{E}_0 t/\hbar} a_n \langle 0|\mu_z|n\rangle + e^{-i\mathcal{E}_0 t/\hbar} a_n^* \langle n|\mu_z|0\rangle \right)}$$

(c) $e^{i\mathcal{E}_0 t/\hbar} a_n$ has the factor

$$e^{i(\mathcal{E}_0 - \mathcal{E}_n)t/\hbar} e^{i\omega_{n0} t} = 1, \text{ so}$$

$$\langle \psi_t|\mu_z|\psi_t\rangle = \frac{E_0}{\hbar} \sum_{n\neq 0} |\langle n|\mu_z|0\rangle|^2 \left(\frac{e^{i\omega t}}{\omega_{n0}+\omega} + \frac{e^{-i\omega t}}{\omega_{n0}-\omega} \right)$$

$$+ \text{ complex conjugate}$$

$$= \frac{2E_0}{\hbar} \cos(\omega t) \sum_{n\neq 0} |\langle n|\mu_z|0\rangle|^2 \frac{(\omega_{n0}-\omega) + (\omega_{n0}+\omega)}{\omega_{n0}^2 - \omega^2}$$

$$= \frac{4E_0}{\hbar} \cos(\omega t) \sum_{n\neq 0} |\langle n|\mu_z|0\rangle|^2 \frac{\omega_{n0}}{\omega_{n0}^2 - \omega^2}$$

Then $\boxed{\alpha(\omega) = \frac{2}{\hbar} \sum_{n\neq 0} \frac{\omega_{n0} |\langle n|\mu_z|0\rangle|^2}{\omega_{n0}^2 - \omega^2}}$

4. This is an outline and more steps should be shown in some parts.

(a) $P_{\vec{k}\lambda} = \frac{\partial L}{\partial \dot{Q}_{\vec{k}\lambda}} = \dot{Q}_{\vec{k}\lambda}$, $H = \sum_{\vec{k}\lambda} P_{\vec{k}\lambda} \dot{Q}_{\vec{k}\lambda} - L = \frac{1}{2} \sum_{\vec{k}\lambda} (P_{\vec{k}\lambda}^2 + \omega_{\vec{k}\lambda}^2 Q_{\vec{k}\lambda}^2)$

(b) $[Q_{\vec{k}\lambda}, P_{\vec{k}'\lambda'}] = i\hbar \delta_{\vec{k}\vec{k}'} \delta_{\lambda\lambda'}$, $[Q_{\vec{k}\lambda}, Q_{\vec{k}'\lambda'}] = 0$, $[P_{\vec{k}\lambda}, P_{\vec{k}'\lambda'}] = 0$

(c) $[a_{\vec{k}\lambda}, a^\dagger_{\vec{k}'\lambda'}] = \frac{1}{2\hbar\omega_{\vec{k}\lambda}} \omega_{\vec{k}\lambda} (-2i) [Q_{\vec{k}\lambda}, P_{\vec{k}'\lambda'}] = \delta_{\vec{k}\vec{k}'} \delta_{\lambda\lambda'}$

Similarly, or with other reasoning, $[a_{\vec{k}\lambda}, a_{\vec{k}'\lambda'}] = 0$, $[a^\dagger_{\vec{k}\lambda}, a^\dagger_{\vec{k}'\lambda'}] = 0$

(d) After algebra, with $Q_{\vec{k}\lambda}$ and $P_{\vec{k}\lambda}$ written in terms of $a_{\vec{k}\lambda}$ and $a^\dagger_{\vec{k}\lambda}$,

$$H = \frac{1}{2} \sum_{\vec{k}\lambda} (a^\dagger_{\vec{k}\lambda} a_{\vec{k}\lambda} + a_{\vec{k}\lambda} a^\dagger_{\vec{k}\lambda}) \hbar\omega_{\vec{k}\lambda} = \sum_{\vec{k}\lambda} (N_{\vec{k}} + \tfrac{1}{2}) \hbar\omega$$

(e) $i\hbar \frac{d}{dt} a_{\vec{k}\lambda}(t) = [a_{\vec{k}\lambda}, H] = [a_{\vec{k}\lambda}, \sum_{\vec{k}'\lambda'} (a^\dagger_{\vec{k}'\lambda'} a_{\vec{k}'\lambda'} + \tfrac{1}{2}) \hbar\omega_{\vec{k}'\lambda'}] = [a_{\vec{k}\lambda}, a^\dagger_{\vec{k}\lambda} a_{\vec{k}\lambda}] \hbar\omega_{\vec{k}\lambda}$

$[a, a^\dagger a] = a a^\dagger a - a^\dagger a a = (a^\dagger a + 1) a - a^\dagger a a = a$

$\therefore i\hbar \frac{d}{dt} a_{\vec{k}\lambda} = a_{\vec{k}\lambda} \hbar\omega \Rightarrow \frac{da_{\vec{k}\lambda}}{a_{\vec{k}\lambda}} = -i\omega\, dt \Rightarrow \ln a_{\vec{k}\lambda} = -i\omega t + \text{const.}$

$\Rightarrow a_{\vec{k}\lambda}(t) = a_{\vec{k}\lambda}(0) e^{-i\omega t}$

(f) Baym (13-13) with $A_{\vec{k}\lambda} = \sqrt{\frac{2\pi\hbar c^2}{\omega}}\, a_{\vec{k}\lambda}$ or (13-67)