Physics 624, Quantum II -- Exam 1

Please show all your work on the separate sheets provided (and be sure to include your name).

You are graded on your work on those pages, with partial credit where it is deserved.

All problems are, of course, nonrelativistic.

Vectors are marked with arrows, are in boldface, or are explicitly stated to be vectors.

For a hydrogen-like atom or ion with atomic number Z,

$$\psi_{n=1,\ell=0}(\mathbf{r}) = \frac{1}{\sqrt{\pi}}\left(\frac{Z}{a_0}\right)^{3/2} e^{-Zr/a_0} \quad .$$

You may use the result

$$a_n(t) = \langle n|\psi_t\rangle = \frac{1}{i\hbar} e^{-i\varepsilon_n t/\hbar} \int_{t_0}^{t} dt'\, e^{i(\varepsilon_n - \varepsilon_i)t'/\hbar} \langle n|V_{t'}|i\rangle \quad .$$

You may also use, for a harmonic oscillator,

$$\langle n'|x|n\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n}\,\delta_{n',n-1} + \sqrt{n+1}\,\delta_{n',n+1}\right) \quad .$$

1. (25) The hydrogen isotope tritium is an important component in both fission and fusion nuclear weapons. The fact that it undergoes beta decay to helium-3, with a half-life of about 12 years, is one of the few positive facts about nuclear weapons. Here we consider the electron in a tritium atom before and after this decay.

   The electron is initially in the ground state of a ${}^{3}_{1}\mathrm{H}$ atom, with 1 proton and 2 neutrons in the nucleus. Immediately after the fast decay, with $n^0 \rightarrow p^+ + e^- + \overline{\nu}_e$, it remains in the same state, but is now in a ${}^{3}_{2}\mathrm{He}^+$ ion, with 2 protons and 1 neutron in the nucleus.

   Calculate the probability that the electron will be found in the ground state of this ion immediately after the decay.

2. A 1-dimensional harmonic oscillator is in its ground state for $t < 0$. For $t \geq 0$ it is subjected to a time-dependent but spatially uniform force

$$F(t) = F_0 \, e^{-t/\tau} \quad .$$

(a) (5) What is the potential energy V_t due to this force, as a function of time, with $V_t = 0$ at $x = 0$?

(b) (15) Using time-dependent perturbation theory to first order, calculate the probability of finding the oscillator in its first excited state for $t > 0$.

Give your answer in terms of τ, F_0, $\hbar$, and ω and m for the harmonic oscillator, of course.

(c) (5) Obtain the limit of this probability as $t \to \infty$.

Also, explain why it is reasonable that this limit is a constant.

3. Here we will obtain the frequency-dependent polarizability $\alpha(\omega)$ for, e.g., an atom or molecule.

The system is initially in the ground state $|0\rangle$ (at $t \to -\infty$) and the electric field is

$$\mathbf{E}(t) = 2E_0\,\hat{z}\cos(\omega t) = E_0\,\hat{z}\left(e^{i\omega t} + e^{-i\omega t}\right)$$

where $\hat{z}$ is the unit vector in the z direction.

Please use the following simplifying assumption: Since the insertion of an adiabatic turn-on factor of $e^{\eta t}$ (with $\eta \to 0+$), would complicate the calculation, just assume that the contribution from the lower limit $t_0 \to -\infty$ can be taken to be 0 when an integral $\int_{-\infty}^{t} dt'\, f(t') = F(t) - F(-\infty) \to F(t)$ is performed below. (This assumption can easily be justified by using the slow turn-on factor.)

The perturbation in the Hamiltonian (in the Schrödinger picture) is

$$V_t = -\mu \cdot \mathbf{E}(t) = -\mu_z E(t)$$

and the relevant dipole matrix element is $\langle n|\mu_z|0\rangle$ (where $\mu = q\mathbf{R}$ is the dipole operator).
The state vector is

$$|\psi_t\rangle = a_0(t)|0\rangle + \sum_{n\neq 0} a_n(t)|n\rangle \quad .$$

(a) (10) Using first-order time-dependent perturbation theory (see front page of exam), calculate $a_n(t)$ in terms of E_0, $\hbar$, ω, $\langle n|\mu_z|0\rangle$, and

$$\omega_{n0} = \frac{\varepsilon_n - \varepsilon_0}{\hbar} \quad .$$

(b) (5) Using the expression above for $|\psi_t\rangle$, and keeping only terms up to first order in a_n for $n \neq 0$, obtain the time-dependent expectation value for the electric dipole moment,

$$\langle \mu_z(t)\rangle \equiv \langle \psi_t|\mu_z|\psi_t\rangle$$

in terms of the $a_n(t)$ and the $\langle n|\mu_z|0\rangle$ etc.
Assume that $a_0 \approx e^{-i\varepsilon_0 t/\hbar}$ (the unperturbed value) and that this system has no permanent dipole moment:

$$\langle 0|\mu_z|0\rangle = 0 \quad .$$

(c) (10) Using the results of parts (a) and (b), and

$$\langle \mu_z(t)\rangle = \alpha(\omega) E_z(t)$$

obtain a simple expression for $\alpha(\omega)$ in terms of the $\left|\langle n|\mu_z|0\rangle\right|^2$, ω_{n0}, $\omega_{n0}{}^2$, ω^2, and $\hbar$.

[For a classical charge treated as a harmonic oscillator with angular frequency ω_0, the polarizability is proportional to $\dfrac{1}{\omega_0{}^2 - \omega^2}$.]

4. The classical radiation field can be written in terms of modes (Fourier components) with frequencies $\omega_{\vec{k}\lambda}$ and canonical variables (generalized coordinates) $Q_{\vec{k}\lambda}$. The Lagrangian is

$$L = \frac{1}{2}\sum_{\vec{k}\lambda}\left(\dot{Q}_{\vec{k}\lambda}{}^2 - \omega_{\vec{k}\lambda}{}^2 Q_{\vec{k}\lambda}{}^2\right) \quad .$$

(a) (4) Obtain the conjugate momenta $P_{\vec{k}\lambda}$ and the Hamiltonian H from L.

(b) (4) After quantization, $Q_{\vec{k}\lambda}$ and $P_{\vec{k}'\lambda'}$ become Heisenberg operators. Write down their commutation relations.

(c) (4) One can then define the operators

$$a_{\vec{k}\lambda} = \frac{1}{\sqrt{2\hbar\omega_{\vec{k}\lambda}}}\left(\omega_{\vec{k}\lambda}Q_{\vec{k}\lambda} + i\,P_{\vec{k}\lambda}\right) \quad , \quad a^{\dagger}{}_{\vec{k}\lambda} = \frac{1}{\sqrt{2\hbar\omega_{\vec{k}\lambda}}}\left(\omega_{\vec{k}\lambda}Q_{\vec{k}\lambda} - i\,P_{\vec{k}\lambda}\right) \quad .$$

Calculate their commutation relations from those for $Q_{\vec{k}\lambda}$ and $P_{\vec{k}'\lambda'}$.

(d) (4) Calculate the expression for the Hamiltonian H in terms of the $a_{\vec{k}\lambda}$ and $a^{\dagger}{}_{\vec{k}\lambda}$. Then obtain it in terms of the number operators $N_{\vec{k}\lambda} = a^{\dagger}{}_{\vec{k}\lambda}a_{\vec{k}\lambda}$.

(e) (4) Using the Heisenberg equation of motion, obtain the Heisenberg operator $a_{\vec{k}\lambda}(t)$ as a function of time, $a_{\vec{k}\lambda}(0)$, and $\omega_{\vec{k}\lambda}$.

(f) (5) Write down the basic form for the quantized vector potential $\mathbf{A}(\mathbf{r},t)$ in terms of $a_{\vec{k}\lambda}(0)$, $a_{\vec{k}\lambda}{}^{\dagger}(0)$, the polarization vector λ, etc.

Do not worry about constants or the precise form of prefactors. Just write down the basic form.