Exam 2 Solution

1. The correspondence is $j = \frac{n_+ + n_-}{2}$ and $m = \frac{n_+ - n_-}{2}$, so:

(a) $J_+ |n_+, n_-\rangle = \hbar a_+^\dagger a_- |n_+, n_-\rangle = \hbar \sqrt{n_+ + 1}\sqrt{n_-}\, |n_+ + 1, n_- - 1\rangle$

$= \hbar \sqrt{j+m+1}\sqrt{j-m}\, |j' = j, m' = m+1\rangle$ since $\boxed{\begin{array}{l} n_+ = j+m \\ n_- = j-m \end{array}}$

or $\boxed{J_+ |j, m\rangle = \sqrt{j-m}\sqrt{j+m+1}\,\hbar |j, m+1\rangle}$

written in same form as in problems 2 & 4

$J_- |n_+, n_-\rangle = \hbar a_-^\dagger a_+ |n_+, n_-\rangle = \hbar \sqrt{n_- + 1}\sqrt{n_+}\, |n_+ - 1, n_- + 1\rangle$

or $\boxed{J_- |j, m\rangle = \sqrt{j+m}\sqrt{j-m+1}\,\hbar |j, m-1\rangle}$

(b) $\boxed{K_+ |j, m\rangle =} a_+^\dagger a_-^\dagger |n_+, n_-\rangle = \sqrt{n_+ + 1}\sqrt{n_- + 1}\, |n_+ + 1, n_- + 1\rangle$

$= \boxed{\sqrt{j+m+1}\sqrt{j-m+1}\, |j+1, m\rangle}$

$\boxed{K_- |j\, m\rangle =} a_+ a_- |n_+, n_-\rangle = \sqrt{n_+}\sqrt{n_-}\, |n_+ - 1, n_- - 1\rangle$

$= \boxed{\sqrt{j+m}\sqrt{j-m}\, |j-1, m\rangle}$

2. (a) $L_+ |j, j\rangle = 0 \Rightarrow -i\hbar e^{+i\phi}\left(+i\frac{\partial}{\partial\theta} - \cot\theta \frac{\partial}{\partial\phi}\right) Y_{\ell m}(\theta, \phi) = 0$ for $m = \ell$

$\Rightarrow i\frac{\partial}{\partial\theta} Y_{\ell m} = \cot\theta \cdot i m Y_{\ell m}$ with $m = \ell = 1$

$\Rightarrow \frac{\partial}{\partial\theta} Y_{11} = \frac{\cos\theta}{\sin\theta} Y_{\ell m}$ so $\boxed{Y_{11} = c \sin\theta \cdot e^{i\phi}}$

solution to ϕ equation

(b) $\int_0^\pi d\theta \int_0^{2\pi} \sin\theta\, d\phi \cdot |c|^2 \sin^2\theta = |c|^2 \cdot 2\pi \int_0^\pi d\theta \sin^3\theta$

$\Rightarrow 1 = |c|^2 \cdot 2\pi \int_0^\pi d\theta \sin\theta\,(1 - \cos^2\theta)$

$= |c|^2 \cdot 2\pi \left(\left[-\cos\theta\right]_0^\pi + \left[\frac{\cos^3\theta}{3}\right]_0^\pi \right)$

$= |c|^2 \cdot 2\pi \left(2 - \frac{2}{3}\right)$

$\Rightarrow \boxed{|c|^2 = \frac{3}{8\pi}}$ and a convention is $c = -\sqrt{\frac{3}{8\pi}}$

(c) $L_- |j, j\rangle = \sqrt{(j+j)(j-j+1)}\,\hbar |j, j-1\rangle \Rightarrow L_- |1, 1\rangle = \sqrt{2}\hbar |1, 0\rangle$

$\Rightarrow \boxed{Y_{1,0} =} \frac{1}{\sqrt{2}\hbar}\left(-i\hbar e^{-i\phi}\right)\left(-i\frac{\partial}{\partial\theta} - \cot\theta\frac{\partial}{\partial\phi}\right) c \sin\theta \cdot e^{i\phi}$

$= \frac{c}{\sqrt{2}}\left(-i e^{-i\phi}\right)\left(-i\cos\theta - i\cos\theta\right) e^{i\phi}$

$= \boxed{-c\sqrt{2}\cos\theta} = \sqrt{\frac{3}{4\pi}} \cos\theta$ with convention above

3. $$\begin{pmatrix} e^{-i(\alpha+\gamma)/2}\cos\frac{\beta}{2} & -e^{-i(\alpha-\gamma)/2}\sin\frac{\beta}{2} \\ e^{i(\alpha-\gamma)/2}\sin\frac{\beta}{2} & e^{i(\alpha+\gamma)/2}\cos\frac{\beta}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \cos\frac{\phi}{2} - i n_z \sin\frac{\phi}{2} & -i n_x \sin\frac{\phi}{2} - n_y \sin\frac{\phi}{2} \\ -i n_x \sin\frac{\phi}{2} + n_y \sin\frac{\phi}{2} & \cos\frac{\phi}{2} + i n_z \sin\frac{\phi}{2} \end{pmatrix}$$

Take trace: $2\cos\left(\frac{\alpha+\gamma}{2}\right)\cos\frac{\beta}{2} = 2\cos\frac{\phi}{2}$ (I)

$$\boxed{\phi = 2\cos^{-1}\left[\cos\left(\frac{\alpha+\gamma}{2}\right)\cos\frac{\beta}{2}\right]}$$

Sum and difference of off-diagonal elements:

$$-2i\, n_x \sin\frac{\phi}{2} = 2i \sin\left(\frac{\alpha-\gamma}{2}\right)\sin\frac{\beta}{2}$$

$$\Rightarrow \boxed{n_x = \sin\left(\frac{\gamma-\alpha}{2}\right)\sin\frac{\beta}{2} \Big/ \sin\frac{\phi}{2}}$$

$$2 n_y \sin\frac{\phi}{2} = 2\cos\left(\frac{\alpha-\gamma}{2}\right)\sin\frac{\beta}{2}$$

$$\Rightarrow \boxed{n_y = \cos\left(\frac{\gamma-\alpha}{2}\right)\sin\frac{\beta}{2} \Big/ \sin\frac{\phi}{2}}$$

Difference of diagonal elements:

$$2i\, n_z \sin\frac{\phi}{2} = 2i\sin\left(\frac{\alpha+\gamma}{2}\right)\cos\frac{\beta}{2}$$

$$\Rightarrow \boxed{n_z = \sin\left(\frac{\alpha+\gamma}{2}\right)\cos\frac{\beta}{2} \Big/ \sin\frac{\phi}{2}}$$

Check: $\sin^2\frac{\phi}{2}\,(n_x^2 + n_y^2 + n_z^2)$

$$= \sin^2\frac{\beta}{2} + \sin^2\left(\frac{\alpha+\gamma}{2}\right)\cos^2\frac{\beta}{2}$$

$$= \sin^2\frac{\beta}{2} + \cos^2\frac{\beta}{2}\left[1 - \cos^2\left(\frac{\alpha+\gamma}{2}\right)\right]$$

$$= 1 - \cos^2\frac{\phi}{2} \quad \text{from Eq. (I) above}$$

$$= \sin^2\frac{\phi}{2}$$

$$\Rightarrow n_x^2 + n_y^2 + n_z^2 = 1$$

4. (a) selection rules arise from Wigner-Eckart theorem:

$$\langle \alpha' j' m' | T_q^{(k)} | \alpha j m \rangle = \langle j k ; m q | j k ; j' m' \rangle \frac{\langle \alpha' j' \| T^{(k)} \| \alpha j \rangle}{\sqrt{2j+1}}$$

with Clebsch-Gordan coefficient equal to zero unless

$m' = m + q$ (not relevant here, since $m' = m = j$ and $q = 0$

$|j - k| \leq j' \leq j + k$.

Here we have $k = 2$, $j' = j$:

$|j - 2| \leq j$.

This can only be satisfied if $j \geq 1$ $(2 - j \leq j \Rightarrow 2 \leq 2j \Rightarrow 1 \leq j)$.

so $\boxed{Q = 0 \text{ if } j = 0 \text{ or } \tfrac{1}{2}}$.

For the proton, e.g., $j = \frac{1}{2}$ and $Q = 0$.

(b) Since $q = \pm 1$, nonzero if $\boxed{\Delta m \equiv m' - m = \pm 1}$.

In addition,

$$|j - k| \leq j' \leq j + k$$

$$\Rightarrow |j - 1| \leq j' \leq j + 1$$

$j' \leq j + 1 \Rightarrow \boxed{\Delta j \equiv j' - j \leq 1}$

<u>If $j \geq 1$</u>, $|j-1| \leq j' \Rightarrow j - 1 \leq j' \Rightarrow \underline{-1 \leq j' - j \equiv \Delta j}$.

If $j < 1$, $|j-1| \leq j' \Rightarrow 1 - j \leq j' \Rightarrow 1 \leq j + j'$

so $j = 0 \Rightarrow 1 \leq j' \Rightarrow j' = 1$ and $j = \frac{1}{2} \Rightarrow \frac{1}{2} \leq j' \Rightarrow 0 \leq \Delta j$

$\therefore \boxed{j \geq 1 \Rightarrow \Delta j = \pm 1, 0}$ (cannot be $\pm\frac{1}{2}$ because $\Delta m = \pm 1$)

$\boxed{j = \tfrac{1}{2} \Rightarrow \Delta j = 1, 0}$ $\boxed{j = 0 \Rightarrow \Delta j = 1}$

(c) Do the algebra your own favorite way! One way:

$J_x = \frac{1}{2}(J_+ + J_-)$, $J_y = \frac{1}{2i}(J_+ - J_-)$; let $T = T_q^{(k)}$

$$[J_x, [J_x, T]] = J_x [J_x, T] - [J_x, T] J_x = J_x^2 T - 2 J_x T J_x + T J_x^2$$
$$= \frac{1}{4}\left[(J_+ + J_-)^2 T - 2(J_+ + J_-) T (J_+ + J_-) + T (J_+ + J_-)^2\right]$$
$$[J_y, [J_y, T]] = -\frac{1}{4}\left[(J_+ - J_-)^2 T - 2(J_+ - J_-) T (J_+ - J_-) + T (J_+ - J_-)^2\right]$$

When we add these two expressions, $J_+^2 T$ etc. terms cancel:

$$[J_x,[J_x,T]] + [J_y,[J_y,T]]$$

$$= \frac{1}{2}\{(J_+J_- + J_-J_+)T - 2(J_+TJ_- + J_-TJ_+) + T(J_+J_- + J_-J_+$$

$$= \frac{1}{2}\{J_+[J_-,T] + J_-[J_+,T] + [T,J_+]J_- + [T,J_-]J_+\}$$

$$= \frac{1}{2}\{[J_+,[J_-,T]] + [J_-,[J_+,T]]\}$$

$$= \frac{\hbar}{2}\{\sqrt{(k+q)(k-q+1)}\,[J_+,T_{q-1}] + \sqrt{(k-q)(k+q+1)}\,[J_-,T_{q+}$$

$$= \frac{\hbar^2}{2}\{\sqrt{(k+q)(k-q+1)}\sqrt{[k-(q-1)][k+(q-1)+1]}\,T_q$$

$$+ \sqrt{(k-q)(k+q+1)}\sqrt{[k+(q+1)][k-(q+1)+1]}\,T_q\}$$

$$= \frac{\hbar^2}{2}\{(k+q)(k-q+1) + (k-q)(k+q+1)\}T_q$$

$$= \frac{\hbar^2}{2}\{[(k^2-q^2)+(k+q)] + [(k^2-q^2)+(k-q)]\}T_q$$

$$= \hbar^2[k(k+1)-q^2]\,T_q$$

Also, $[J_z,[J_z,T_q]] = [J_z,\hbar q T_q] = \hbar^2 q^2 T_q$.

$$\therefore \boxed{\vec{J}^2\{T_q^{(k)}\} = \hbar^2 k(k+1)\,T_q^{(k)}}$$

(as $\vec{J}^2|jm\rangle = \hbar^2 j(j+1)|jm\rangle$)