Physics 624, Quantum II -- Exam 2

Please show all your work on the separate sheets provided (and be sure to include your name).

You are graded on your work on those pages, with partial credit where it is deserved.

(You receive credit for clear, well-organized, complete work, and an unjustified final answer receives zero credit.)

All problems are, of course, nonrelativistic.

Vectors are marked with arrows, are in boldface, or are explicitly stated to be vectors.

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1. Schwinger's model for angular momentum is based on

$$J_+ \equiv \hbar a_+^\dagger a_- \quad , \quad J_- \equiv \hbar a_-^\dagger a_+ \quad , \quad J_z \equiv \frac{1}{2}\hbar\left(a_+^\dagger a_+ - a_-^\dagger a_-\right) \quad , \quad J^2 = J_z^2 + \frac{1}{2}\left(J_+ J_- + J_- J_+\right) \quad .$$

where $a_\pm$ and $a_\pm^\dagger$ satisfy the usual commutation relations for two independent harmonic oscillators.

After the results are calculated for these operators, the quantum numbers n_+ and n_- for the oscillators are related to the usual angular momentum quantum numbers j and m through

$$\left|n_+ , n_-\right\rangle \rightarrow \left|j , m\right\rangle$$

with the appropriate correspondence between these two sets of quantum numbers.

We can also define the operators $K_+ \equiv a_+^\dagger a_-^\dagger$ and $K_- \equiv a_+ a_-$.

All your answers below are first to be calculated in terms of n_+ , n_- , and new states $\left|n_+' , n_-'\right\rangle$, but finally given in terms of j, m, and new states with the form $\left|j', m'\right\rangle$.

(a) (12) Calculate $J_+\left|n_+ , n_-\right\rangle \rightarrow J_+\left|j , m\right\rangle$ and $J_-\left|n_+ , n_-\right\rangle \rightarrow J_-\left|j , m\right\rangle$.

(b) (13) Calculate $K_+\left|j , m\right\rangle$ and $K_-\left|j , m\right\rangle$ in the same way, starting with the oscillator picture..

2. Perform the calculations below using the raising and lowering operators for orbital angular momentum,

$$-i\hbar e^{\pm i\phi}\left(\pm i\frac{\partial}{\partial\theta}-\cot\theta\frac{\partial}{\partial\phi}\right)$$

where

$$J_{\pm}|j,m\rangle=\sqrt{(j\mp m)(j\pm m+1)}\,\hbar|j,m\pm 1\rangle$$

together with

$$-i\hbar\frac{\partial}{\partial\phi}Y_{\ell,m}(\theta,\phi)=m\hbar Y_{\ell,m}(\theta,\phi)$$

and the normalization condition $\int d\Omega\, Y^{*}_{\ell,m}(\theta,\phi)\,Y_{\ell',m'}(\theta,\phi)=\delta_{\ell,\ell'}\delta_{m,m'}$, $d\Omega=d\theta\sin\theta\, d\phi$.

(a) (9) Calculate $Y_{1,1}(\theta,\phi)$ (i.e. the spherical harmonic for $\ell=1$ and $m=1$), leaving the normalization constant c to be determined in part (b).

(b) (8) Calculate the normalization constant c . (There is an arbitrary phase factor, of course, which you may choose.)

(c) (8) Calculate $Y_{1,0}(\theta,\phi)$.

3. (25) Consider the rotation of a 2-component spinor. We expect that the sequence of Euler rotations

$$D^{(1/2)}(\alpha,\beta,\gamma) = e^{-i\sigma_3\alpha/2}\, e^{-i\sigma_2\beta/2}\, e^{-i\sigma_3\gamma/2}$$
$$= \begin{pmatrix} e^{-i(\alpha+\gamma)/2}\cos\frac{\beta}{2} & -e^{-i(\alpha-\gamma)/2}\sin\frac{\beta}{2} \\ e^{i(\alpha-\gamma)/2}\sin\frac{\beta}{2} & e^{i(\alpha+\gamma)/2}\cos\frac{\beta}{2} \end{pmatrix}$$

is equivalent to a single rotation

$$D^{(1/2)}(\phi) = e^{-i\vec{\sigma}\cdot\hat{n}\,\phi/2}$$
$$= \overleftrightarrow{1}\cos\frac{\phi}{2} - i\vec{\sigma}\cdot\hat{n}\sin\frac{\phi}{2}$$

where $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ are the Pauli matrices.

Find ϕ (the angle of rotation) and the components of $\hat{n} = (n_x, n_y, n_z)$ (the axis of rotation) in terms of the Euler angles α, β, γ .

4. The electric quadrupole moment Q is defined by

$$Q = e\langle \alpha, j, m = j | (3z^2 - r^2) | \alpha, j, m = j \rangle \quad \text{with} \quad Y_2^0 = \left(\frac{5}{16\pi}\right)^{1/2} \frac{3z^2 - r^2}{r^2} \ .$$

For example, $j = \frac{1}{2}$ for the proton (as in a recent colloquium) and $j = 1$ for the deuteron (bound state of proton and neutron). Q reflects the nature of the forces between neutrons and protons in a nucleus and also the shape of the nucleus.

(a) (6) Are there any systems in nature for which $Q = 0$ is required by the selection rules for spherical tensor operators $T_q^{(k)}$? What can you say about these systems in general?

(b) (6) What do the selection rules say about the *dipole matrix elements*

$$\langle \alpha' j' m' | x | \alpha \, j \, m \rangle \quad \text{with} \quad Y_1^{\pm 1} = \mp \left(\frac{3}{4\pi}\right)^{1/2} \frac{x \pm iy}{\sqrt{2}\, r} \quad ?$$

(You should get the same ones that we encountered in treating absorption of radiation by an atom.)

(c) (13) In general, a spherical tensor operator is required to satisfy the commutation relations

$$\left[J_\pm, T_q^{(k)}\right] = \hbar\sqrt{(k \mp q)(k \pm q + 1)}\, T_{q\pm 1}^{(k)} \quad , \quad \left[J_z, T_q^{(k)}\right] = \hbar q\, T_q^{(k)}$$

where

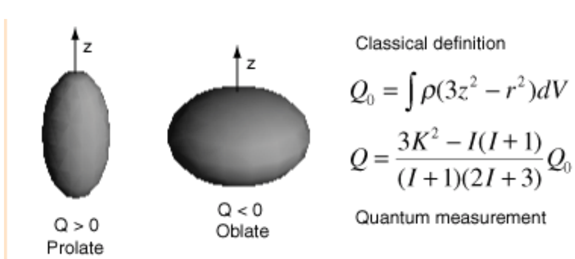
$$J_\pm = J_x \pm iJ_y$$

Let $\vec{J}^{\,2}\{A\} = \left[J_x, [J_x, A]\right] + \left[J_y, [J_y, A]\right] + \left[J_z, [J_z, A]\right]$.

Show that

$$\vec{J}^{\,2}\{T_q^{(k)}\} = c(k) T_q^{(k)}$$

and determine the coefficient $c(k)$.



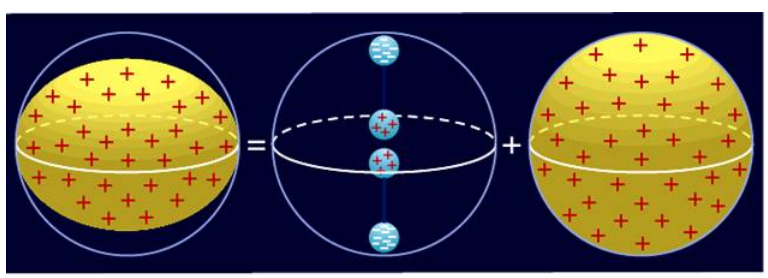


Figure 2. Nuclear quadrupole resonance requires that the nuclei under scrutiny display electric quadrupole moments. Such quadrupole moments arise when the distribution of positive electric charge in the nucleus is not perfectly spherical. For example, a slightly oblate (pumpkin-like) distribution of positive charge *(left)* can be thought of as the sum of a quadrupolar distribution *(center)* and a spherical distribution *(right)*.

Credits for figures: http://hyperphysics.phy-astr.gsu.edu/hbase/nuclear/elequad.html , http://www.kyklotron.com/NQR.html , http://www.allacronyms.com