Physics 624 Final Exam Solution

1. (a) $\langle\alpha'|H|\beta'\rangle = \langle\alpha|S^{\dagger}HS|\beta\rangle = \langle\alpha|S^{-1}SH|\beta\rangle = \langle\alpha|H|\beta\rangle$

(b) To lowest order in ε, $S = 1 - i\varepsilon G/\hbar$ and

$$\left(1 - i\varepsilon \frac{G}{\hbar}\right)H - H\left(1 - i\varepsilon\frac{G}{\hbar}\right) = 0 \Rightarrow GH - HG = 0 \quad \text{or } [G,H]=0$$

so $\frac{d}{dt}G = \frac{1}{i\hbar}[G,H] = 0$ by Heisenberg equation of motion

meaning G is conserved, and $\frac{d}{dt}\langle\alpha|G|\alpha\rangle = 0$ in the Heisenberg picture, because $|\alpha\rangle$ is constant in the Heisenberg picture.

(c) translational invariance in space ⇒ momentum conservation
" " " time ⇒ energy "
rotational invariance ⇒ angular momentum conservation
[gauge invariance ⇒ charge conservation]

(d) $\frac{dS}{dt} = \frac{1}{i\hbar}[S,H] = 0 \Rightarrow S$ conserved & $\frac{d}{dt}\langle\alpha|S|\alpha\rangle = 0$

[e.g., the parity of a state is conserved]

(e) (i) <u>anti</u> - unitary

(ii) $T\vec{p} = -\vec{p}\,T$ [not +]

$T\vec{J} = -\vec{J}\,T$ [not +]

(iii) $T\vec{x} = +\vec{x}\,T$

$TH = +HT$ if $H = \frac{p^2}{2m} + V(\vec{r})$

2. $e^{i\vec{\sigma}\cdot\vec{\alpha}} = 1 + i\vec{\sigma}\cdot\vec{\alpha} - \frac{1}{2!}(\vec{\sigma}\cdot\vec{\alpha})^2 - \frac{1}{3!}i\vec{\sigma}\cdot\vec{\alpha}\,(\vec{\sigma}\cdot\vec{\alpha})^2 + \dots$

(implied sums)

$$(\vec{\sigma}\cdot\vec{\alpha})^2 = (\sigma_i\alpha_i)(\sigma_j\alpha_j) = \frac{1}{2}(\sigma_i\alpha_i\sigma_j\alpha_j + \sigma_j\alpha_j\sigma_i\alpha_i)$$

$$= \frac{1}{2}\underbrace{(\sigma_i\sigma_j + \sigma_j\sigma_i)}_{=2\delta_{ij}}\alpha_i\alpha_j = \alpha_i\alpha_i = \alpha^2$$

$$\therefore e^{i\vec{\sigma}\cdot\vec{\alpha}} = \left(1 - \frac{1}{2!}\alpha^2 + \dots\right) + i\vec{\sigma}\cdot\hat{\alpha}\left(\alpha - \frac{1}{3!}\alpha^3 + \dots\right), \quad \hat{\alpha} = \frac{\vec{\alpha}}{\alpha}$$

$$= \cos\alpha + i\vec{\sigma}\cdot\hat{\alpha}\sin\alpha$$

3. (a) In the interaction picture, the operators for both the harmonic oscillator and the quantized electromagnetic field are Heisenberg operators, with e.g. $\hat{a}(t) = \hat{a}(0)e^{-i\omega t}$ for the harmonic oscillator and $\hat{a}_{\omega}(t) = \hat{a}_{\omega}(0)e^{-i\omega t}$ for the field mode with angular frequency ω.

The $\hat{A}^2$ part of the interaction is irrelevant because it does not correspond to one photon emission. In the Coulomb gauge, $\vec{\nabla}\cdot\vec{A}=0$, the relevant interaction is $-2\cdot\frac{1}{2m}\cdot\frac{e}{c}\hat{A}_x\hat{p}_x$. The relevant part of $\hat{A}_x$ for emission is $C\,\hat{a}^{\dagger}_{\omega}(0)\,e^{+i\omega t}$. The relevant part of $\hat{p}_x$ with energy conserved is $-i\sqrt{\frac{m\hbar\omega}{2}}\,a$,

since $\hat{p}_x = -i(\hat{a}-\hat{a}^{\dagger})\sqrt{\frac{m\hbar\omega}{2}}$ from a & $a^{\dagger}$ on first page of exam.

Then the full relevant interaction is

$$\hat{V} = -2\cdot\frac{1}{2m}\cdot\frac{e}{c}\cdot C\,\hat{a}^{\dagger}_{\omega}(0)\,e^{+i\omega t}\,(-i)\sqrt{\frac{m\hbar\omega}{2}}\,\hat{a}(0)\,e^{-i\omega t}$$

$$= i\,\frac{e}{c}\sqrt{\frac{\hbar\omega}{2m}}\;C\,\hat{a}^{\dagger}_{\omega}(0)\,\hat{a}(0).$$

Also, $|i\rangle = |n_{field}=0,\, n\rangle$ & $\langle f| = \langle n_{field}=1,\, n-1|$,

so $\langle f|\hat{V}|i\rangle = i\,\frac{e}{c}\sqrt{\frac{\hbar\omega}{2m}}\;C\,\langle n_{field}=1,\, n-1|\hat{a}^{\dagger}_{\omega}(0)\,\hat{a}(0)|n_{field}=0,\, n\rangle$

$$= i\,\frac{e}{c}\sqrt{\frac{\hbar\omega}{2m}}\;C\cdot\sqrt{1}\cdot\sqrt{n}$$

and $|\langle f|\hat{V}|i\rangle|^2 = \frac{e^2}{c^2}\cdot\frac{\hbar\omega}{2m}\,C^2 n$.

(b) (i) $A^* = A \Rightarrow \hat{A}^{\dagger} = A$, Hermitian

(ii) $\hat{\phi}(\vec{r},t) = C\sum_{\vec{k}}\left(a_{\vec{k}}\,e^{i\vec{k}\cdot\vec{r}-i\omega t} + b^{\dagger}_{\vec{k}}\,e^{-i\vec{k}\cdot\vec{r}}\,e^{+i\omega t}\right)$

[not Hermitian if ϕ not real]

4. (a) $i\hbar\frac{\partial\phi}{\partial t} = c\,\vec{\sigma}\cdot\vec{P}\,\chi + (mc^2+qV)\phi$, $\vec{P}\equiv\vec{p}-\frac{q}{c}\vec{A}$, $\vec{p}\equiv -i\hbar\vec{\nabla}$

$i\hbar\frac{\partial\chi}{\partial t} = c\,\vec{\sigma}\cdot\vec{P}\,\phi + (-mc^2+qV)\chi$

(b) $2mc^2\chi = c\,\vec{\sigma}\cdot\vec{P}\,\phi - i\hbar\frac{\partial\chi}{\partial t} + (mc^2+qV)\chi$

$\Rightarrow \chi = \frac{1}{2mc}\vec{P}\cdot\vec{\sigma}\,\phi - \frac{1}{2mc^2}\left(i\hbar\frac{\partial}{\partial t} - mc^2 - qV\right)\chi$

(c) $i\hbar\frac{\partial\phi}{\partial t} = c\,(\vec{P}\cdot\vec{\sigma})\frac{1}{2mc}(\vec{P}\cdot\vec{\sigma})\phi + (mc^2+qV)\phi$

$(\vec{P}\cdot\vec{\sigma})^2 = \vec{P}\cdot\vec{P} + i\,\vec{\sigma}\cdot(\vec{P}\times\vec{P})$ from equation on first page of exam

$= (-i\hbar\vec{\nabla}-\frac{q}{c}\vec{A})^2 + i\,\vec{\sigma}\cdot\left[(-i\hbar\vec{\nabla}-\frac{q}{c}\vec{A})\times(-i\hbar\vec{\nabla}-\frac{q}{c}\vec{A})\right]$

Now $(\vec{a}\times\vec{a})_k$ really means $\varepsilon_{ijk}a_i a_j$, so $\vec{\nabla}\times\vec{\nabla}=0$ & $\vec{A}\times\vec{A}=0$.

Also, $(\vec{\nabla}\times\vec{A}+\vec{A}\times\vec{\nabla})\phi_\alpha = \vec{\nabla}\phi_\alpha\times\vec{A} + \phi_\alpha\vec{\nabla}\times\vec{A} - \vec{\nabla}\phi_\alpha\times\vec{A} = \vec{B}\phi_\alpha$, so

$i\hbar\frac{\partial\phi}{\partial t} = \frac{1}{2m}(-i\hbar\vec{\nabla}-\frac{q}{c}\vec{A})^2\phi - 2\cdot\frac{q}{2mc}\vec{S}\cdot\vec{B}\,\phi + (qV+mc^2)\phi$

$\vec{S} = \frac{1}{2}\hbar\vec{\sigma}$

(d) $g=2$ rather than 1