Physics 624, Quantum II -- Final Exam

Please show all your work on the separate sheets provided (and be sure to include your name).

You are graded on your work on those pages, with partial credit where it is deserved.

You receive credit for clear, well-organized, complete work.

An unjustified final answer receives zero credit.

All problems except the last one are nonrelativistic.

For the harmonic oscillator:

$$a = \sqrt{\frac{m\omega}{2\hbar}}\, x + i\frac{1}{\sqrt{2m\hbar\omega}}\, p \quad , \quad a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\, x - i\frac{1}{\sqrt{2m\hbar\omega}}\, p$$

For the Pauli matrices:

$$(\mathbf{a}\cdot\sigma)(\mathbf{b}\cdot\sigma) = \mathbf{a}\cdot\mathbf{b} + i\sigma\cdot(\mathbf{a}\times\mathbf{b})$$

1. Consider a unitary symmetry operator S that commutes with the Hamiltonian:

$$[S,H]=0 \quad .$$

(S is time-independent in the Schrödinger picture.)

(a) (3) Show that the matrix elements of H are then invariant under S:

$$\langle\alpha'|H|\beta'\rangle=\langle\alpha|H|\beta\rangle \qquad \text{where} \qquad |\alpha'\rangle=S|\alpha\rangle \quad , \quad |\beta'\rangle=S|\beta\rangle .$$

(b) (10) Suppose that S represents a *continuous* symmetry, with

$$S=e^{-i\varepsilon G/\hbar}$$

where ε is a real parameter. Using the infinitesimal version of this transformation, show that G is conserved, and that any expectation value $\langle\alpha|G|\alpha\rangle$ is constant with respect to time.

(c) (3) Give 3 examples of continuous symmetries and their corresponding conservation laws.

(d) (3) Now suppose that S represents a *discrete* symmetry. Show that S is conserved, and that any expectation value $\langle\alpha|S|\alpha\rangle$ is constant with respect to time.

(e) (6) Suppose that S is replaced by the time-reversal operator T .

(i) What is different about T (as compared to the general operator S defined in the top line above)?

(ii) Is $[T,H]=0$ always true? (Explain, of course, with examples.)

(iii) Is $[T,H]=0$ ever true? (Explain, of course, with examples.)

2. (25) Using the properties of the Pauli matrices, obtain a simple form for

$$e^{i\vec{\sigma}\cdot\vec{\alpha}}$$

in terms of $\cos\alpha$ and $\sin\alpha$, justifying each step explicitly and clearly.

Here the real parameters in $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ specify a rotation, and $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ consists of the Pauli matrices.

3. Recall that Fermi's golden rule is

$$\Gamma = \frac{2\pi}{\hbar}\left[\left|\langle f|\hat{V}|i\rangle\right|^2 \rho\left(\varepsilon_f\right)\right]_{\varepsilon_f=\varepsilon_i}$$

where you should know the meaning of the various quantities in this famous equation. Here we consider a particle with mass m and charge q which is restricted to move in 1 dimension, in the x direction.

It is confined in a 1-dimensional harmonic oscillator potential with angular frequency ω, so that the Hamiltonian is

$$\hat{H} = \frac{1}{2m}\left(\hat{p}_x - \frac{e}{c}\hat{A}_x\right)^2 + m\omega^2 \hat{x}^2 \quad .$$

The particle is also subjected to a quantized electromagnetic field. But it is in a cavity for which there is only a single set of relevant electromagnetic modes, having the same polarization, and in a very narrow range of angular frequencies peaked at ω. The relevant x component for the central mode is

$$\hat{A}_x\left(\vec{r},t\right) = C\left(\hat{a}(\omega)e^{-i\omega t} + \hat{a}^\dagger(\omega)e^{i\omega t}\right) \;,\; \text{with } \hat{A}_y\left(\vec{r},t\right) = \hat{A}_z\left(\vec{r},t\right) = 0 \,.$$

Here $\hat{a}(\omega)$ and $\hat{a}^\dagger(\omega)$ are the usual destruction and creation operators for photons, and C is a real constant.

(a) (20) Starting with the above equations (being clear in each step), calculate the quantity

$$\left|\langle f|\hat{V}|i\rangle\right|^2$$

which is needed to obtain the rate of spontaneous emission of one photon by the particle, if it is initially in the harmonic oscillator state n and there are initially no photons in the cavity. Here $\hat{V}$ is the relevant perturbation in the Hamiltonian resulting from the interaction of the particle with the mode in the electromagnetic field which corresponds to conservation of energy.

Give your answer in terms of n and the various constants.

(b) (5) (i) The classical electromagnetic field $A\left(\vec{r},t\right)$ is real. What does this property require of the corresponding field operator $\hat{A}\left(\vec{r},t\right)$?

(ii) What would be the difference in the form of a quantized field operator for a classical scalar field $\phi\left(\hat{r},t\right)$ that is not real? Write down the basic form of such a field, $\hat{\phi}\left(\hat{r},t\right)$, which corresponds to the form of $\hat{A}_x\left(\vec{r},t\right)$ above.

4. The Dirac equation with an electromagnetic field is

$$i\hbar\frac{\partial\psi}{\partial t}=\left(c\alpha\cdot\left(-i\hbar\nabla-\frac{q}{c}\mathbf{A}\right)+\beta mc^2\right)\psi+qV\psi \quad , \quad \alpha=\begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix} \quad , \quad \beta=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad , \quad \psi=\begin{pmatrix} \phi \\ \chi \end{pmatrix}$$

where α, β are the 3-dimensional vectors whose components are the 4×4 matrices α_i, β_i; σ is the 3-dimensional vector whose components are the 2×2 Pauli matrices σ_i ; ∇ is the usual del operator; $\mathbf{A}$ and V are the 3-vector and "scalar" potentials of the electromagnetic field; and ψ is the 4-component wavefunction composed of the "large" and "small" 2-component parts ϕ and χ.

(a) (5) Write down the two coupled equations for ϕ and χ.

(b) (3) Show that the second equation can be rewritten as

$$\chi=\frac{1}{2mc}\left(-i\hbar\nabla-\frac{q}{c}\mathbf{A}\right)\cdot\sigma\phi-\frac{1}{2mc^2}\left(i\hbar\frac{\partial}{\partial t}-mc^2-qV\right)\chi \quad .$$

(c) (15) Showing each major step explicitly and clearly, use the lowest-order result for χ to obtain the equation for ϕ alone which involves the spin operator $\mathbf{S}$ and the magnetic field $\mathbf{B}$, plus the various other quantities.

(d) (2) What is the principal difference between the term involving $\mathbf{S}$ and $\mathbf{B}$ here and the corresponding expression for the magnetic dipole moment of a classical spinning charged object interacting with a magnetic field?

5. (5 points extra credit, based on talks)

(a) What is the Fulling-Davies-Unruh effect?

(b) If you are asked, are neutrinos Dirac fermions or Majorana fermions, what is the correct response?

(c) What is the big difference between exchanging two identical particles in 2 and in 3 dimensions?

(d) What is the main method of separation of uranium isotopes that is classified for security reasons?

(e) In what promising new material do electrons behave like Dirac fermions?

Have a pleasant summer!