Nonnormal Modes and Quantum Noise

Anthony E. Siegman
McMurtry Professor of Engineering Emeritus
Stanford University

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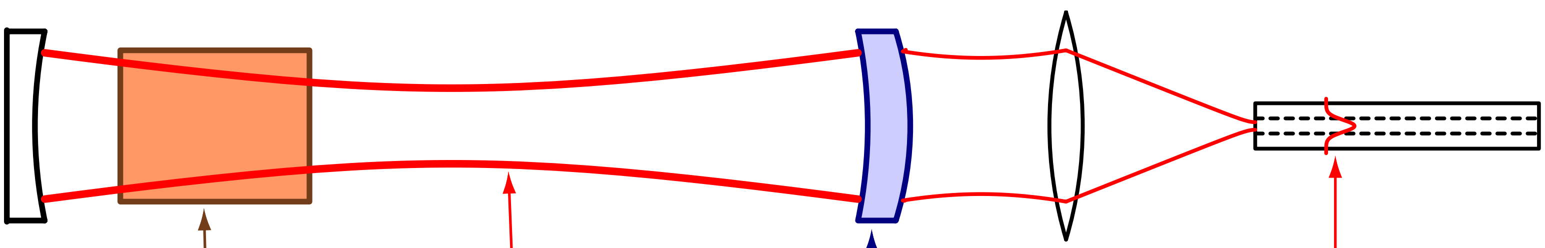
Idealized stable-cavity laser

Assumed to have a lossless optical cavity, finite output coupling, an ideal laser medium, and orthogonal laser cavity modes:



Resulting quantum-noise-limited Schawlow-Townes linewidth:

$$\Delta f_{\mathrm{osc}} = \frac{N_2}{N_2 - N_1} \times \frac{\pi\, h f\, \Delta f_{\mathrm{cav}}^2}{P_{\mathrm{osc}}}$$

Normal modes

Linear and lossless physical systems generally have a set of orthogonal, or “normal”, eigenmodes which are

- Solutions to a Hermitian operator
- Guaranteed to be orthogonal
- Guaranteed to form a complete basis set

The orthogonality of these normal modes provides the basis for many fundamental physical concepts

Examples

Examples of common optical systems with normal or orthogonal eigenmodes:

- Closed metal waveguides and microwave cavities
- Dielectric resonators
- Index-guided optical fibers and waveguides
- Stable optical resonators and lensguides
  (at least in the ideal limit)

The orthogonality of these normal modes provides the foundation for many fundamental physical concepts

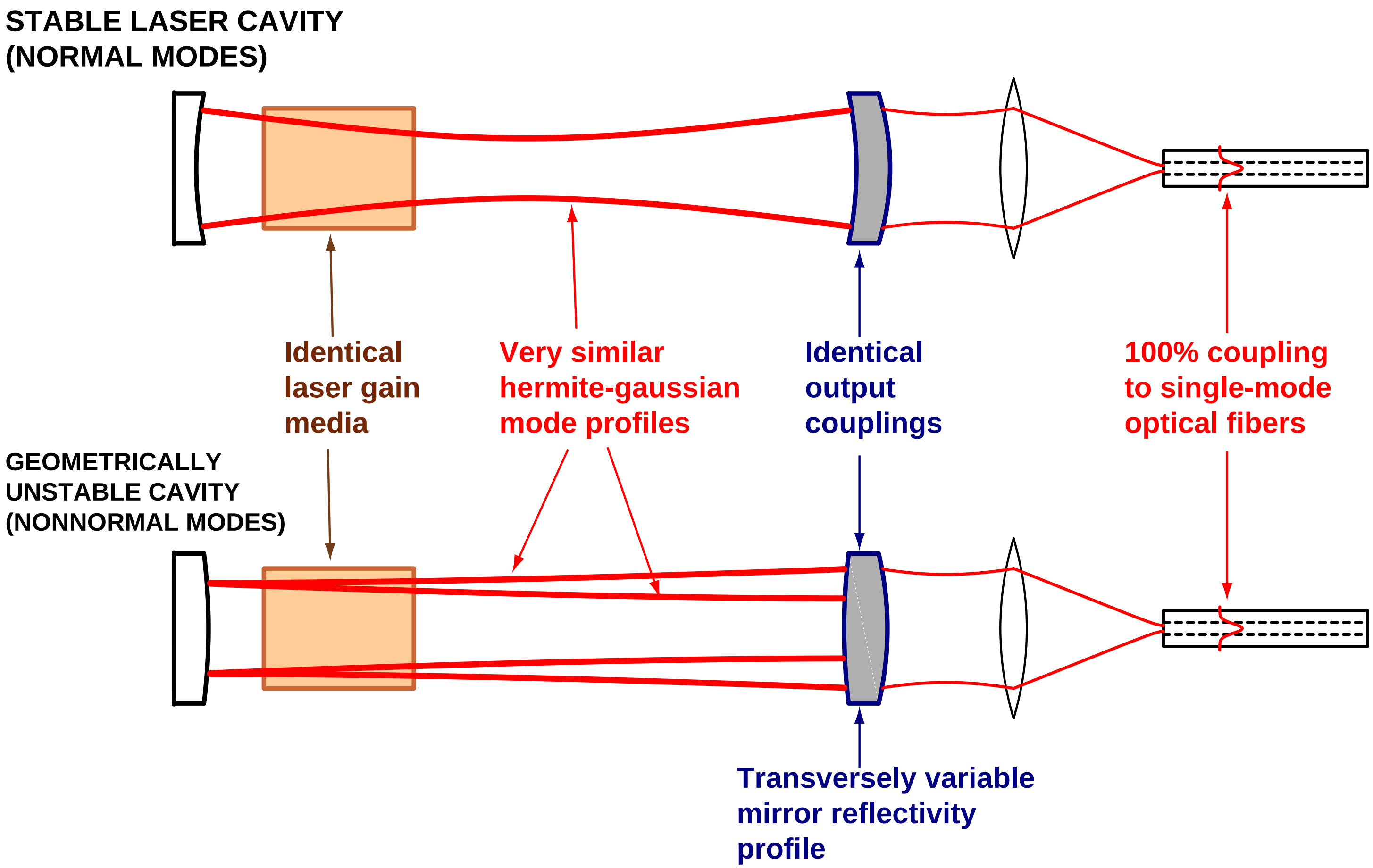
Universal properties of normal mode systems

Systems with normal eigenmodes have universal properties:

1. Total system power or energy is sum of powers or energies in individual modes.
2. Concept of second quantization leads to basic concept of "photons"
3. Mode matching couples input signal entirely into one mode
4. Standard eigenmode expansion procedures apply
5. Standard quantum noise value of one noise photon per mode leads to Schawlow-Townes linewidth for laser oscillators

Normal-mode and nonnormal-mode lasers

Two lasers with exactly the same laser parameters:



But very different Schawlow-Townes linewidths

Not so normal modes . . .

Some common optical systems, however, have distinctly nonorthogonal or nonnormal eigenmodes.

- These are still linear systems (e.g., passive optical cavities or waveguides)
- They still have "modes" (eigenmodes)
- But these modes are not orthogonal

and this loss of orthogonality leads to major changes in the mathematical, physical, and quantum properties of these "nonnormal systems"

Nonnormal optical systems

- Nonnormal optical systems are governed by equations that are still linear, but are nonhermitian
- As a result, these systems have nonorthogonal eigenmodes
- And this leads to major changes in all of the fundamental mode properties of these systems

Examples

Examples of common optical systems with nonnormal (that is, nonorthogonal) eigenmodes:

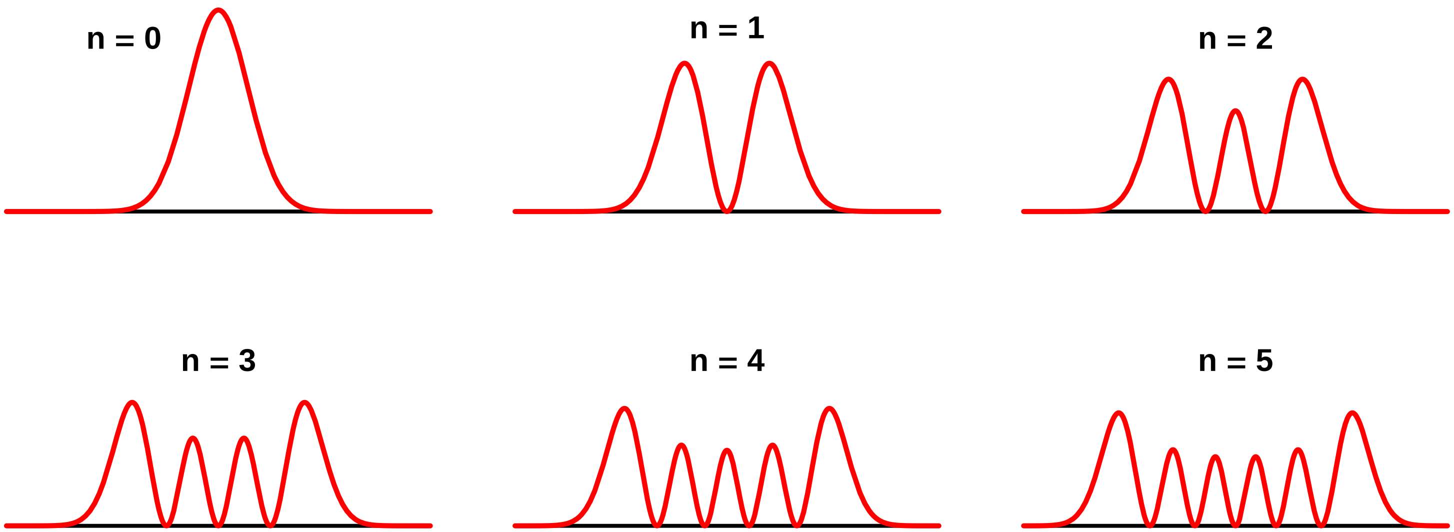
- Gain-guided semiconductor lasers
- Loss-guided or gain-guided ducts
- Unstable optical resonators
- Finite-diameter stable resonators
- Birefringent systems having optical "twist"

Normal mode example: stable resonator modes

The transverse modes of a stable laser cavity are real-valued Hermite-gaussian functions:

$$u_n(x) = H_n(\sqrt{2}x/w)\exp[-x^2/w^2] = H_n(ax)\,e^{-x^2/2a^2}$$

with a purely real spot size or scale factor $a \equiv \sqrt{2}/w$

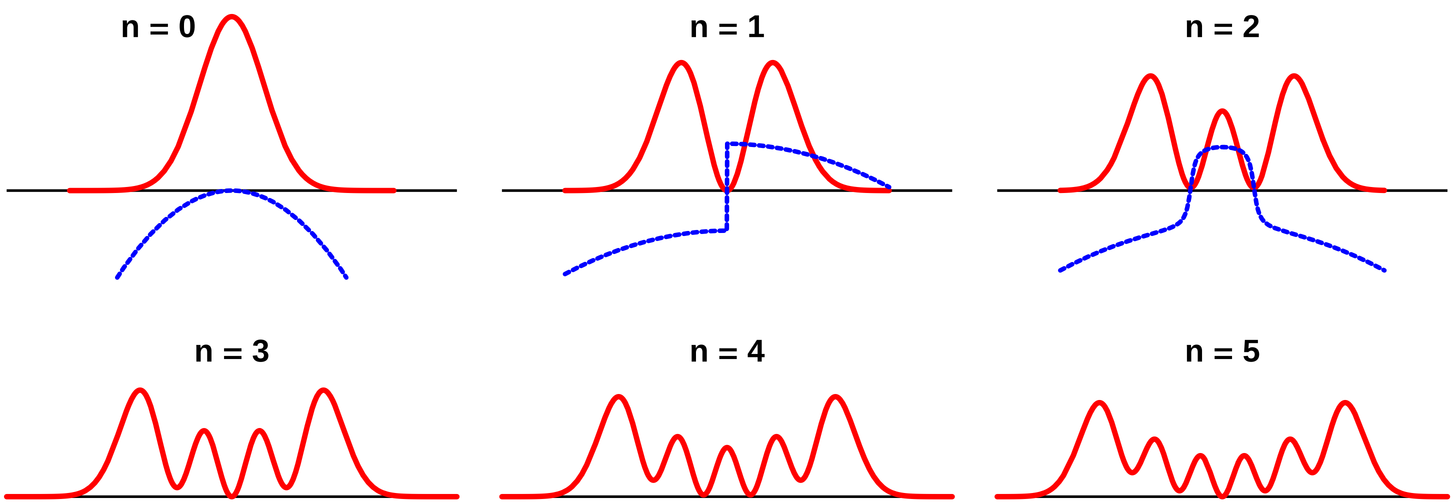


Nonnormal example: gain-guided laser modes

Gain-guided or VRMlaser cavities can have complex-valued (and hence nonorthogonal) Hermite-gaussian cavity modes:

$$u_n(x) = H_n(\tilde{a}\,x)\,e^{-x^2/2\tilde{a}^2}$$

with a complex-valued scale factor $\tilde{a} = e^{j\theta} \times \sqrt{2}/w$



Eigenmode equations

The eigenmodes of optical waveguides and resonators (whether normal or not) are the solutions of some appropriate linear equation, e.g.:

1) The wave equation for propagating modes in optical waveguides

$$\left[\nabla_x^2 + k^2(x)\right] \tilde{u}_n(x) = \beta_n^2 \, \tilde{u}_n(x)$$

2) An integral equation ("Fox and Li equation") for resonant modes in optical cavities

$$\int K(x, x') \, \tilde{u}_n(x') \, dx' = \tilde{\gamma}_n \, \tilde{u}_n(x)$$

Operator formulation

These equations can be rewritten in a generalized operator formalism:

$$L\,\tilde{u}_n(x) = \tilde{\gamma}_n\,\tilde{u}_n(x)$$

and the operators for many physical systems will be <u>hermitian</u>, meaning that

$$L \equiv L^{\dagger} \equiv (L^T)^*$$

where

$$L^* \equiv \text{ordinary complex conjugation}$$
$$L^T \equiv \text{transposition of variables}$$
$$L^{\dagger} \equiv \text{hermitian conjugate, or adjoint}$$

Hermitian operators

Hermitian operators will always have a complete set of eigenfunctions or “normal modes” which will satisfy both the operator equation and the boundary conditions

$$L\,\tilde{u}_n(x) = \tilde{\gamma}_n\,\tilde{u}_n(x)$$

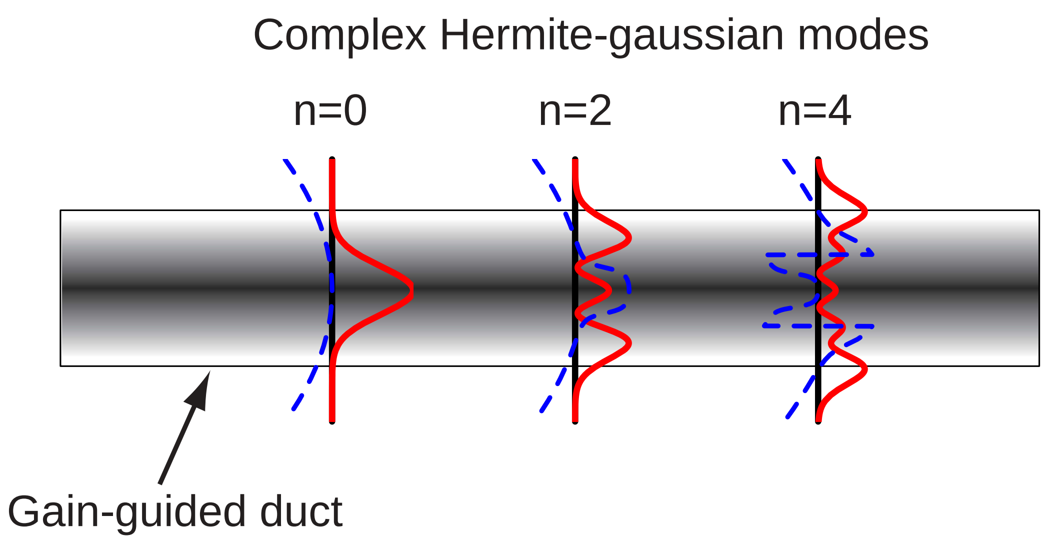
These normal modes will always be orthogonal

$$\int \tilde{u}_n^*(x)\tilde{u}_m(x)\,dx = \delta_{nm}$$

and will form a complete basis set, such that any state $\tilde{u}(x)$ of the system can be written as

$$\tilde{u}(x) = \sum_n \tilde{c}_n \tilde{u}_n(x)$$

Example: parabolic gain-guided waveguide



The eigenmodes of an optical fiber or duct with tapered gain guiding, as well as index guiding

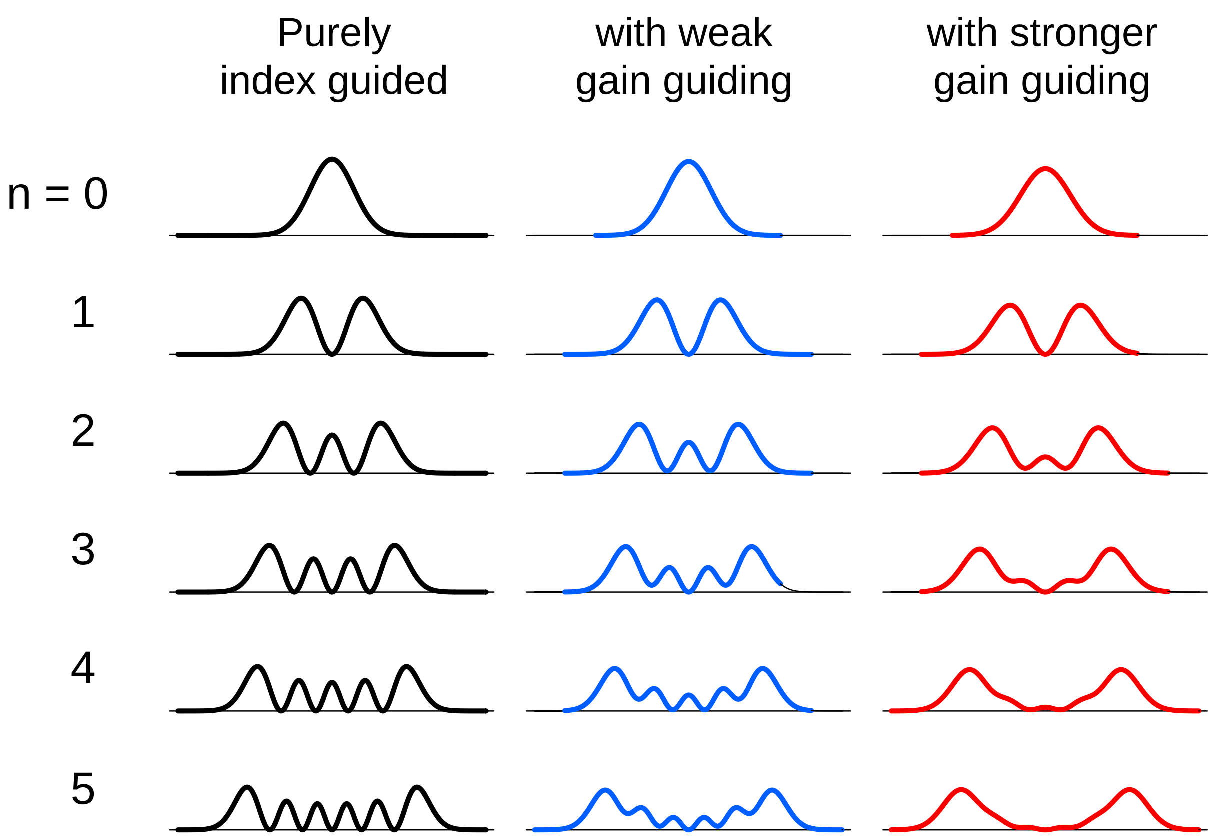
$$n(x) = n_0 - \frac{n_2 x^2}{2}, \qquad g(x) = g_0 - \frac{g_2 x^2}{2}$$

are complex-valued Hermite-Gaussian functions

$$\tilde{u}_n(x) = \tilde{u}_{n0}\, H_n(\tilde{a}x)\, \exp[-\tilde{a}^2 x^2/2]$$

Effects of gain guiding

Amplitude profiles of higher-order complex HG modes change significantly with increased gain guiding



Phase profiles are also distorted and become nonspherical

Complex-valued Hermite-gaussians

These Hermite-Gaussian eigenmodes

$$\tilde{u}_n(x) = \tilde{u}_{n0}\, H_n(\tilde{a}x)\, \exp[-\tilde{a}^2 x^2/2]$$

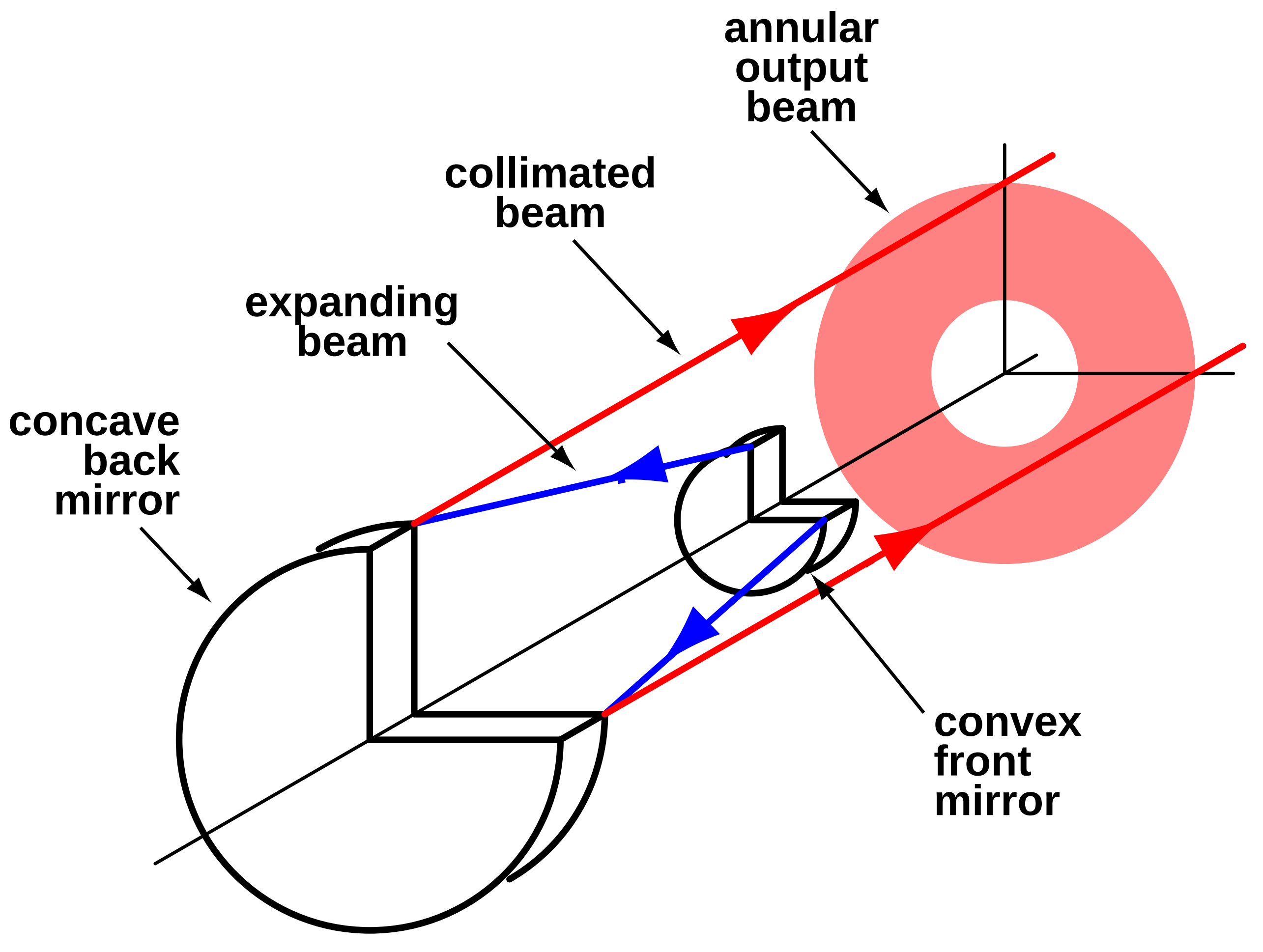
have a complex-valued scale factor (equivalent to a "complex-valued spot size")

$$\tilde{a} = \left(\frac{2\pi}{\lambda_0}\right)^{1/2} \left(n_0 n_2 + j\, \frac{\lambda_0}{2\pi} g_2\right)^{1/4} \equiv |\tilde{a}|\; e^{j\theta}$$

As a result, these modes are distinctly nonorthogonal or nonnormal for the gain-guided case with $g_2 > 0$

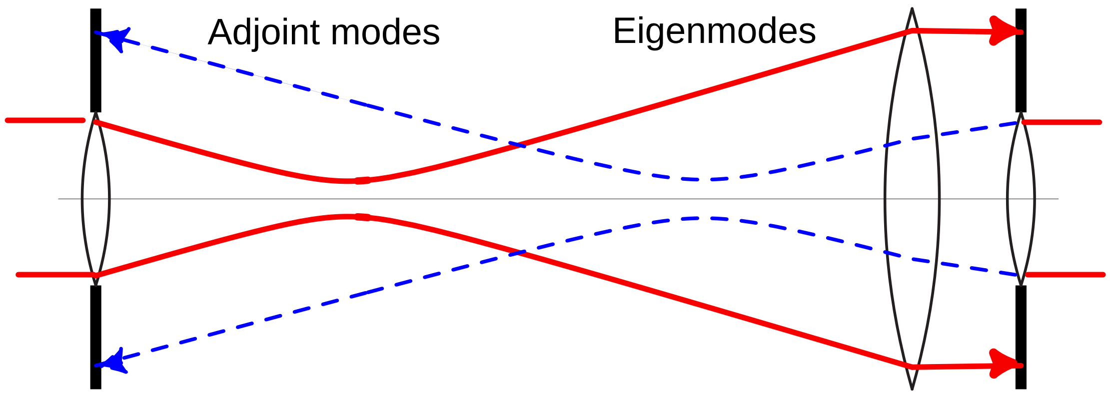
Another example: unstable optical resonators

Unstable optical resonators have clear-cut resonant modes
— but the modes are not orthogonal



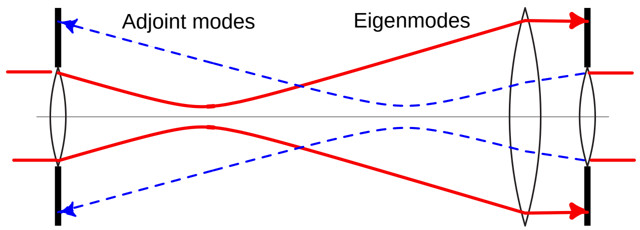
Unstable resonator eigenmodes

Consider equivalent unstable lensguide

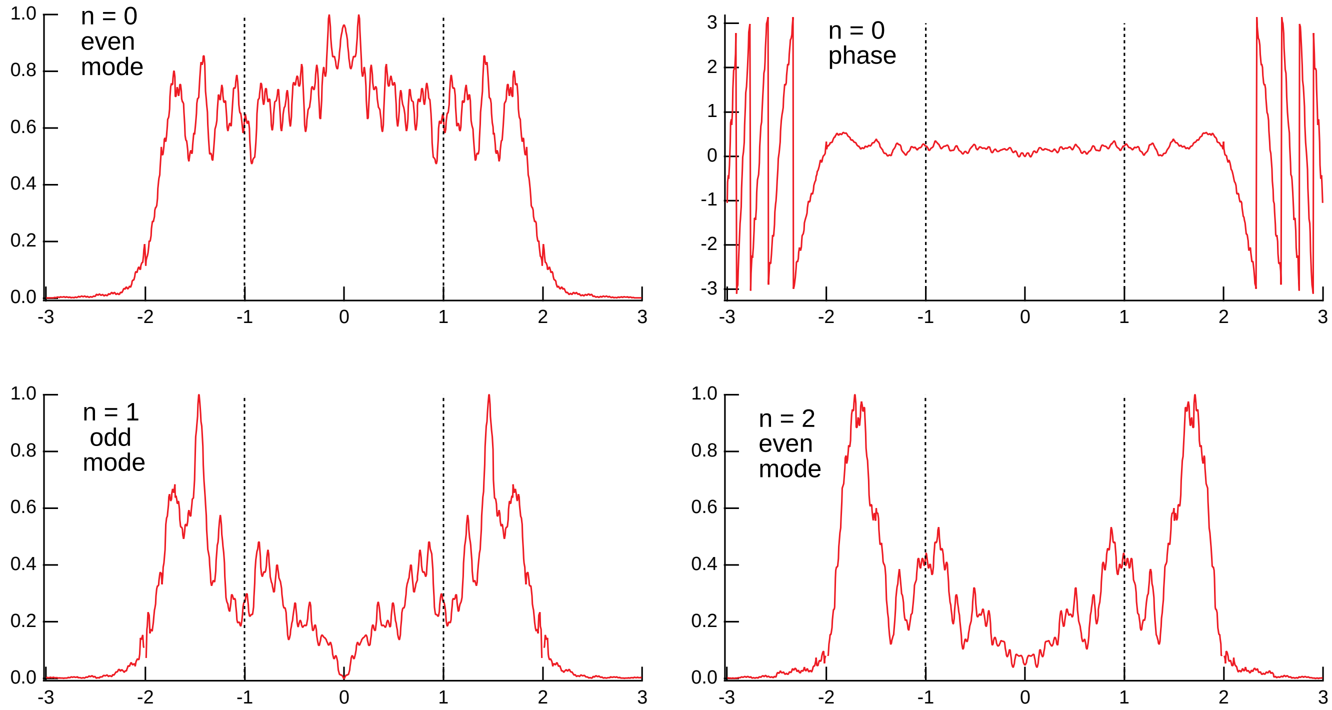


- Solid lines show right-going eigenmodes for an unstable lensguide (or a ring unstable resonator)
- Dashed lines show left-going modes for the same lensguide (or other way around the ring resonator)
- Right- and left-going modes have identical eigenvalues but mode patterns have an "adjoint" relationship

Examples of unstable-resonator mode profiles



Typical mode profiles:



Why are these systems nonnormal?

1) In gain-guided systems:

$$\left[\nabla_x^2 + \tilde{k}^2(x)\right] \tilde{u}_n(x) = \beta_n^2 \, \tilde{u}_n(x)$$

- Gain guiding makes wave vector $\tilde{k}$ complex-valued
- Wave equation operator is no longer hermitian

2) In unstable optical resonators:

$$\int K(x, x') \, \tilde{u}_n(x') \, dx' = \tilde{\gamma}_n \, \tilde{u}_n(x)$$

- Wave equation is fully hermitian
- But boundary conditions at ∞ are not hermitian
- Huygens integral operator then becomes nonhermitian

Mathematical properties of nonnormal operators

Nonhermitian operators are mathematically unfriendly:

- Not guaranteed to even have eigensolutions

$$L\,\tilde{u}_n(x) \stackrel{?}{=} \tilde{\gamma}_n\,\tilde{u}_n(x)$$

- Eigenfunctions, if they exist, are not orthogonal

$$\int \tilde{u}_n^*(x)\tilde{u}_m(x)\,dx \neq 0$$

- And they may or may not form a complete set

$$\tilde{u}(x) \stackrel{?}{=} \sum_n \tilde{c}_n\tilde{u}_n(x)$$

Eigenmodes and adjoint functions

Suppose a nonhermitian operator L has a set of eigenmodes $\tilde{u}_n$ satisfying

$$L\,\tilde{u}_n(x) = \tilde{\gamma}_n\,\tilde{u}_n(x)$$

Then its adjoint operator $L^\dagger$ will also have a set of adjoint functions $\tilde{v}_n$ satisfying

$$L^\dagger\,\tilde{v}_n(x) = \tilde{\gamma}_n^*\,\tilde{v}_n(x)$$

These adjoint functions are not physical modes of the nonnormal system (call them "adjoint functions", not "adjoint modes") — though these adjoint functions $\tilde{v}_n$ will have the same eigenvalues $\tilde{\gamma}_n$ as the eigenmodes $\tilde{u}_n$

Nonorthogonality

Eigenmodes $\tilde{u}_n$ of a nonnormal system, if they do exist,
can be <u>normalized</u>

$$M_{nn} \equiv \int_{-\infty}^{\infty} \tilde{u}_n^*(x)\, \tilde{u}_n(x)\, dx = 1$$

but they are <u>not orthogonal</u> to each other

$$M_{nm} \equiv \int_{-\infty}^{\infty} \tilde{u}_n^*(x)\, \tilde{u}_n(x)\, dx \neq 0 \qquad (n \neq m)$$

Biorthogonality

The physical eigenmodes $\tilde{u}_n$ are instead biorthogonal to the adjoint functions $\tilde{v}_n$

$$\int_{-\infty}^{\infty} \tilde{v}_n^*(x)\, \tilde{u}_m(x)\, dx = \delta_{nm} = \begin{cases} 1, & n = m \\ 0, & n \neq m \end{cases}$$

The adjoint functions, like the eigenmodes, are also nonorthogonal, and have a normalization greater than unity

$$K_{nm} \equiv \int_{-\infty}^{\infty} v_n^*(x)\, \tilde{v}_m(x)\, dx = \begin{cases} K_{nn} > 1, & n = m \\ K_{nm} \neq 0, & n \neq m \end{cases}$$

These K_{nn} and K_{nm} values have physical significance as adjoint coupling coefficients and excess quantum noise coefficients

Unusual properties of nonnormal systems

1. Total power or energy no longer given by sum of powers or energies in individual modes.

2. Second quantization lost; basic concept of "photons" seriously muddied.

3. Mode matching replaced by adjoint coupling: more power into one mode than total power in whole system .

4. Major changes required in eigenmode expansion procedures

5. Laser modes experience <u>excess quantum noise</u>, leading to large increase in Schawlow-Townes linewidth

1) Total energy $\neq$ sum of energies per mode

Expand fields of nonnormal system in terms of nonnormal eigenmodes and evaluate total power or energy:

$$\mathcal{E}(x) = \sum_{n=0}^{N} \tilde{c}_n \, \tilde{u}_n(x)$$

$$\begin{aligned}
\text{Energy} &= \int_{-\infty}^{\infty} |\mathcal{E}(x)|^2 \, dx \\
&= \sum_{n=0}^{N} |\tilde{c}_n|^2 \; + \; \sum_{n \neq m} \tilde{c}_n^* c_m M_{nm} \\
&= \sum_{n} \text{Energies per mode} + \sum_{n \neq m} \text{“cross-mode terms”}
\end{aligned}$$

Energy in individual modes greater than total energy in system, because cross-mode terms can be negative.

2) "Photons" in normal mode systems

Classical energy in fields of a normal laser cavity:

$$\int_{-\infty}^{\infty} |\mathcal{E}(x)|^2 \, dx = \sum_{n=0}^{N} |\tilde{c}_n|^2 = \sum_{n=0}^{N} \tilde{c}_n^* \, \tilde{c}_n$$

Converting coefficients $\tilde{c}_n$ and $\tilde{c}_n^*$ into quantum operators $\mathbf{a}_n$ and $\mathbf{a}_n^\dagger$ transforms this into a quantum Hamiltonian:

$$\mathcal{H} = \sum_{n=0}^{N} \mathbf{a}_n^\dagger \mathbf{a}_n \, \hbar\omega_{qn} = \sum_n \text{[SHO Hamiltonians]}$$

Each mode becomes quantized simple harmonic oscillator; one photon = one quantum of any one of these oscillators

Procedure is called <u>second quantization</u>

Nonnormal systems no longer have photons?

Classical energy for a nonnormal system however becomes

$$\int_{-\infty}^{\infty} |\mathcal{E}(x)|^2 \, dx = \sum_{n=0}^{N} |\tilde{c}_n|^2 \; + \; \sum_{n \neq m} \tilde{c}_n^* c_m M_{nm}$$

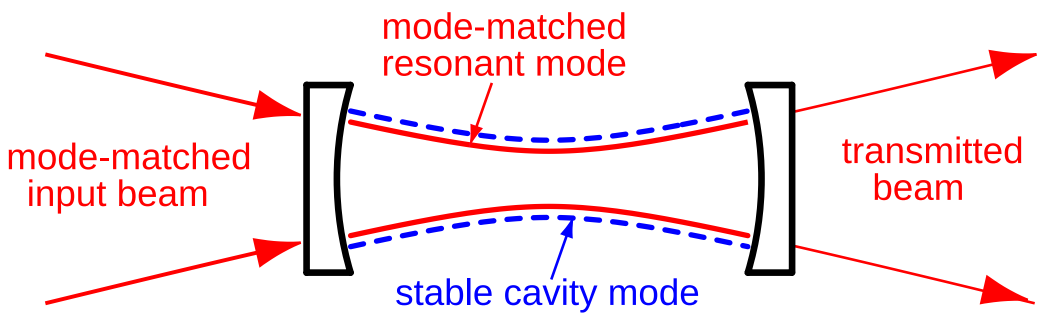
Cross-terms no longer vanish; quantum Hamiltonian becomes

$$\mathcal{H} = \sum_{n=0}^{N} \mathbf{a}_n^\dagger \mathbf{a}_n \, \hbar\omega_{qn} + \sum_{n \neq m} \mathbf{a}_n^\dagger \mathbf{a}_m M_{nm} \, \hbar \sqrt{\omega_{qn}\omega_{qm}} \; .$$

Cavity modes no separate into individual harmonic oscillators

Process of second quantization thus eliminated, or at least seriously muddied

3) Mode matching vs. adjoint coupling

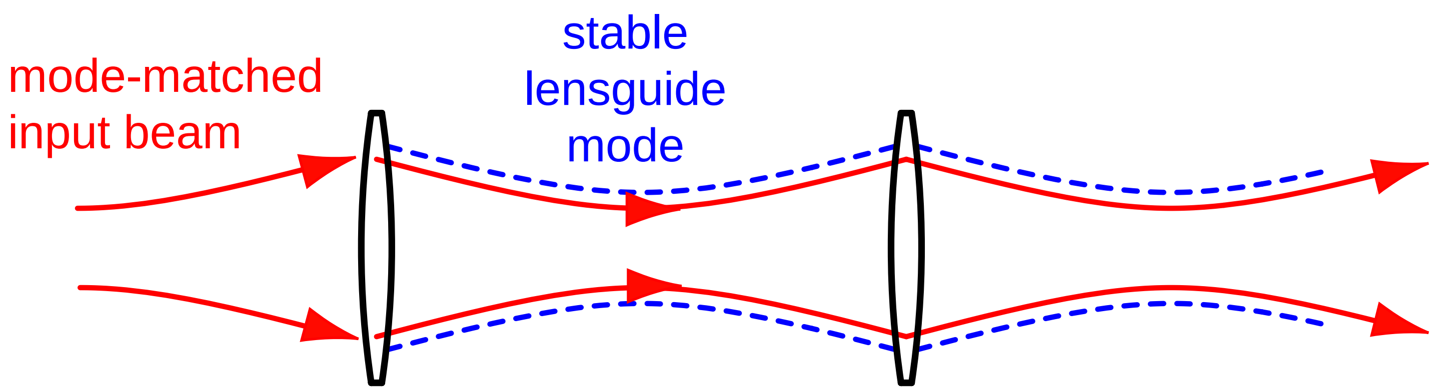


Mode matching is usual way of injecting an input signal into an optical lensguide or cavity

- Input wavefront matched to one selected eigenmode of lensguide or cavity (often lowest-order gaussian mode)
- Delivers entire energy into that one selected mode

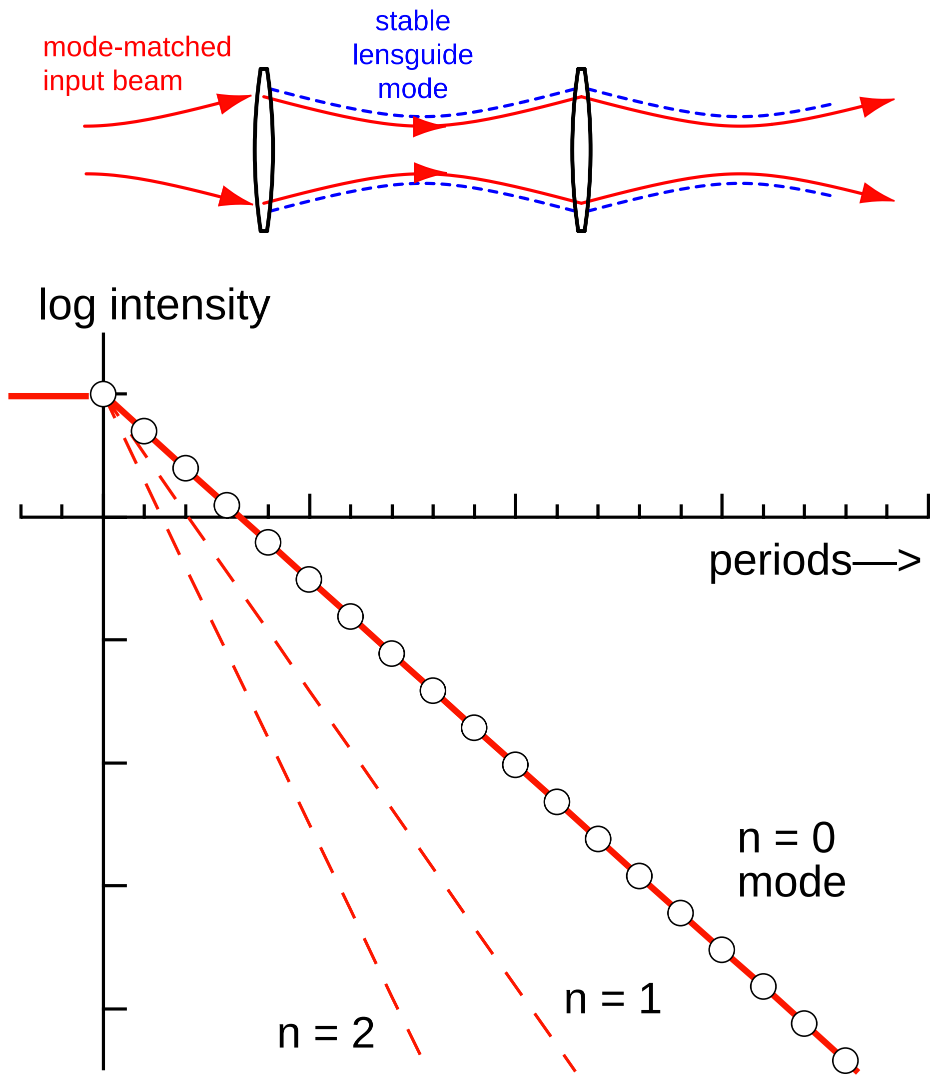
Mode matching into normal-mode lensguide

Example of mode matching into stable lensguide



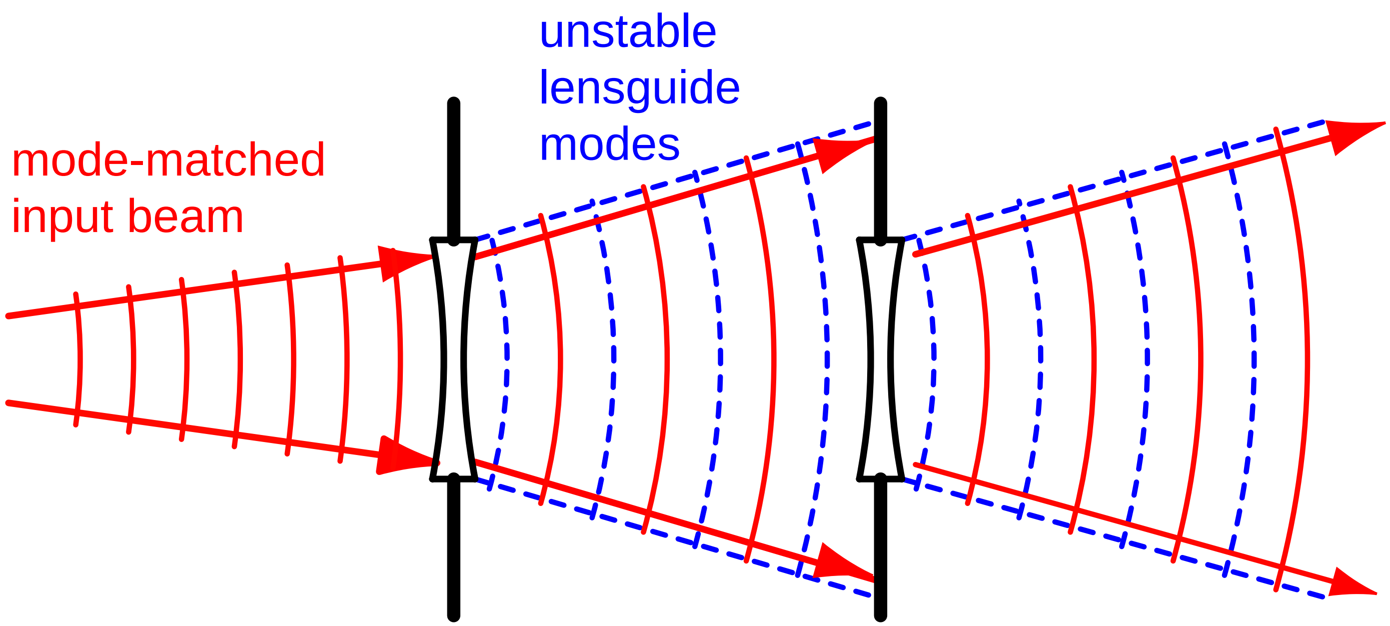
Entire input energy coupled into selected mode
(most often lowest-order mode)

Graphic interpretation of mode matching



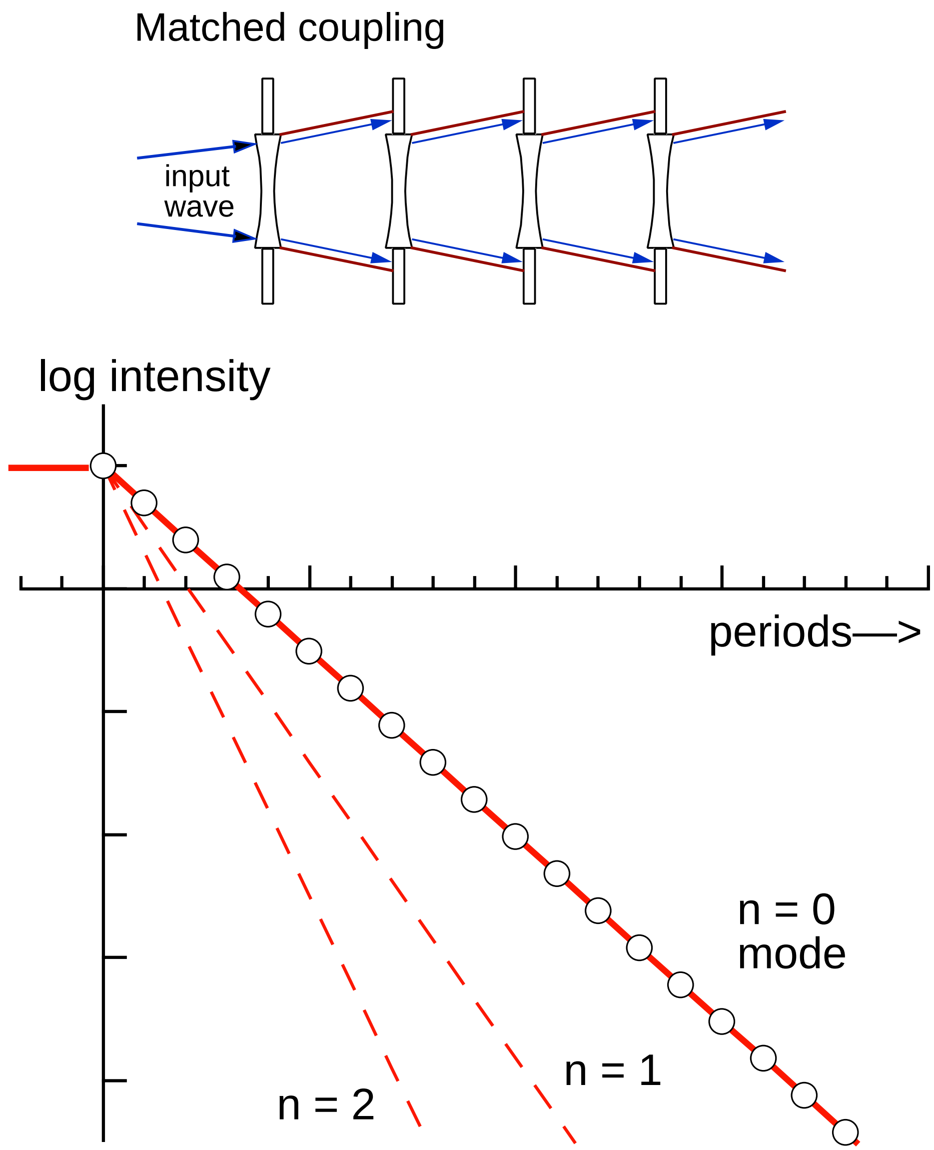
Mode matching into a nonnormal lensguide

Can also mode match into a nonnormal system:



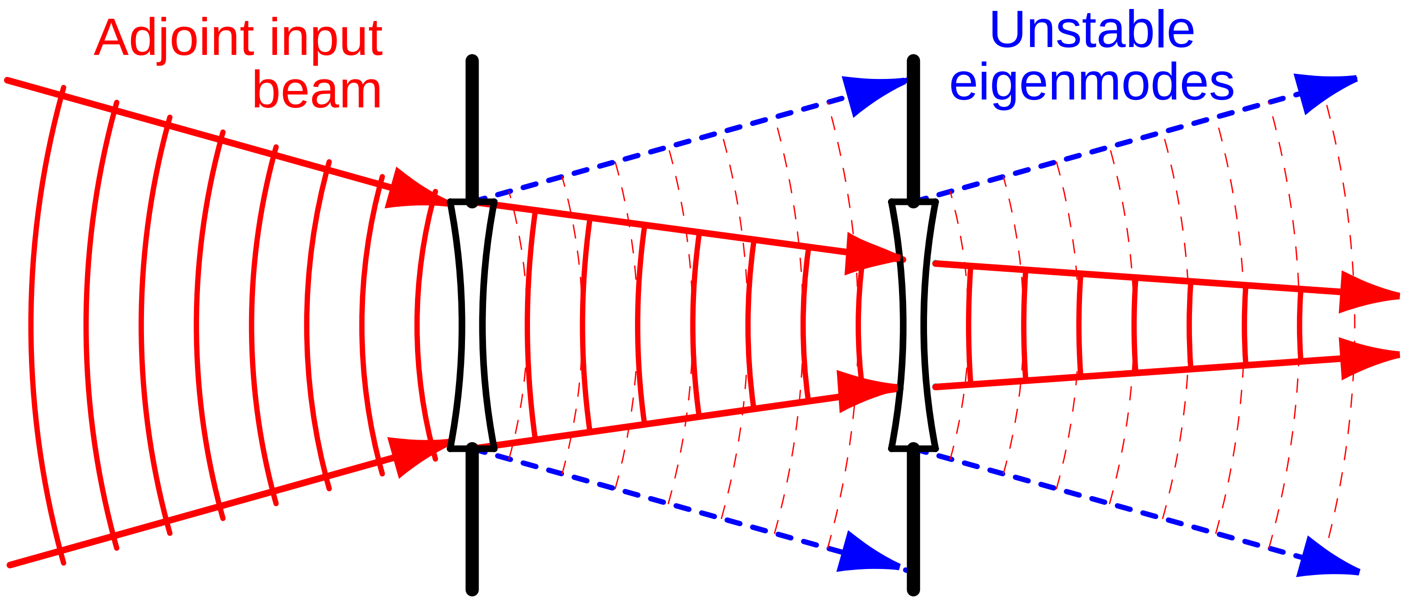
Input energy again goes into single selected nonnormal mode

Nonnormal mode matching



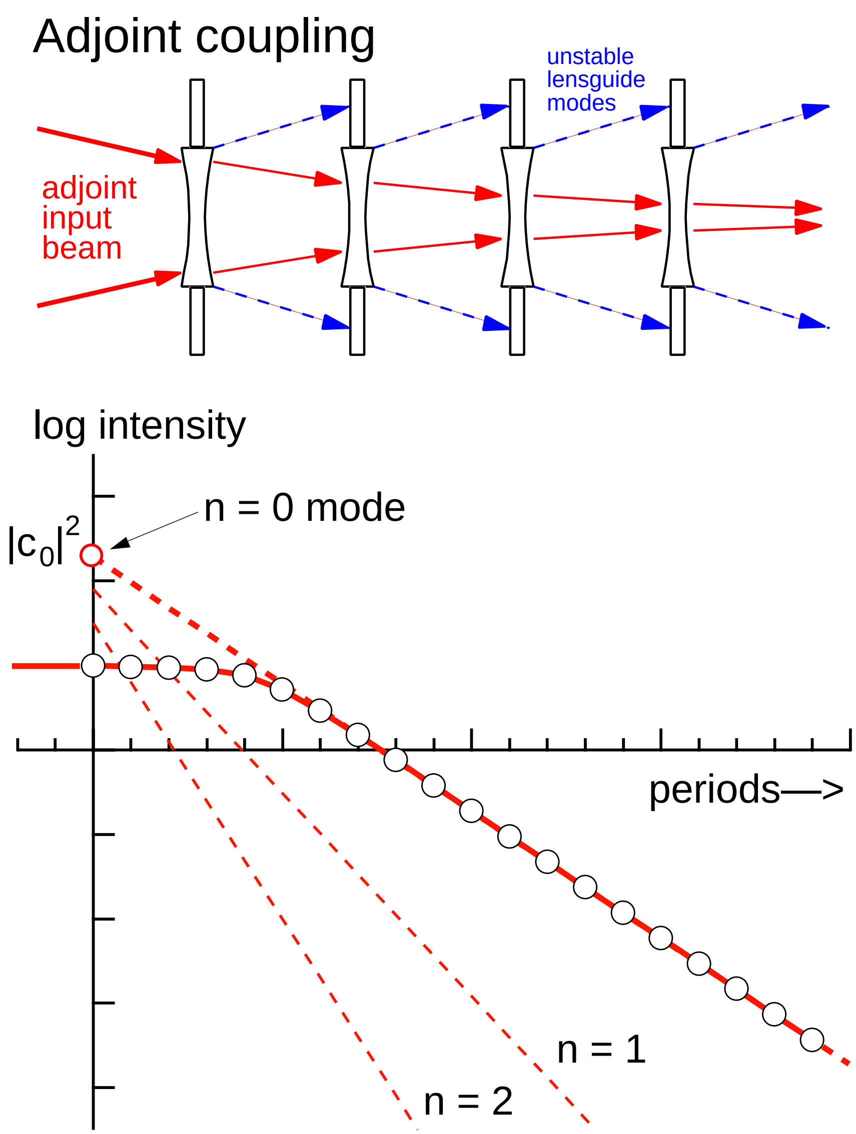
Adjoint coupling into nonnormal system

Adjoint coupling to nonnormal system is quite different:



- Input energy excites multiple modes of the system
- With greater than unity coupling per mode

Graphic interpretation of adjoint coupling



General properties of adjoint coupling

Adjoint coupling into nonnormal system means:

- Injected wavefront matched not to selected mode, but to adjoint function for selected mode
- Selected eigenmode excited with greater than unity input coupling
- Unavoidably also excites other eigenmodes
- Large (but negative) cross-power terms conserve energy
- Excess coupling factor for mode n equals "Petermann factor" $K_{nn} > 1$ for that adjoint function
- All this is possible only with nonnormal modes

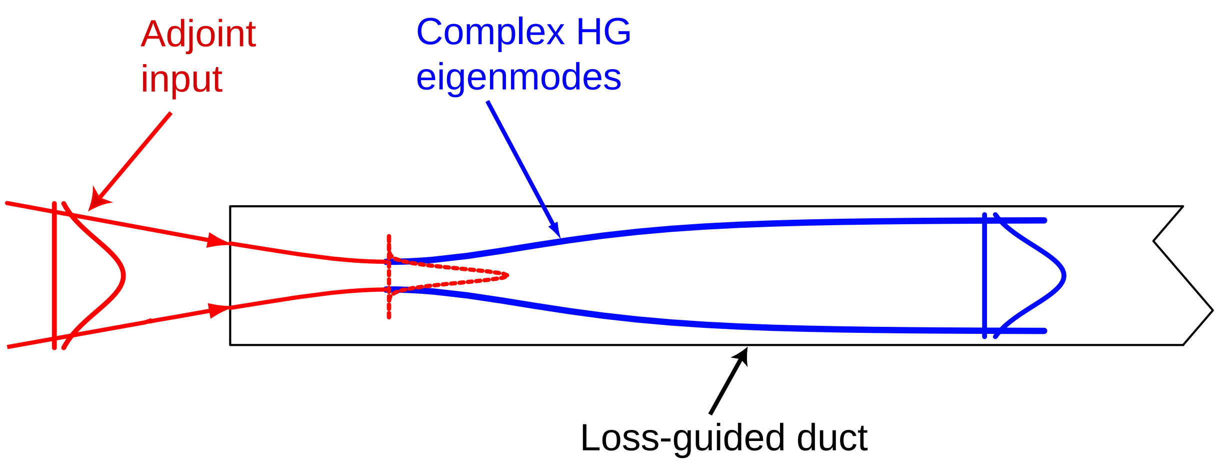
4) Expansions in nonnormal eigenmodes

Can fields in nonnormal optical system be expanded as a superposition of nonnormal eigenmodes of the system?

$$\tilde{f}(x) = \sum_{n=0}^{\infty} \tilde{c}_n \, \tilde{u}_n(x)$$

Answer is "yes"—but not in usual overlap integral fashion

Example: eigenmode expansion of adjoint coupling into complex HG modes of a loss–guided duct



First try "quadrature expansion"

To find expansion coefficients $\tilde{c}_n$

$$\tilde{f}(x) = \sum_{n=0}^{\infty} \tilde{c}_n \, \tilde{u}_n(x)$$

try usual quadrature method: multiply both sides by $\tilde{v}_n^*(x)$, and use <u>bi</u>orthogonality relation

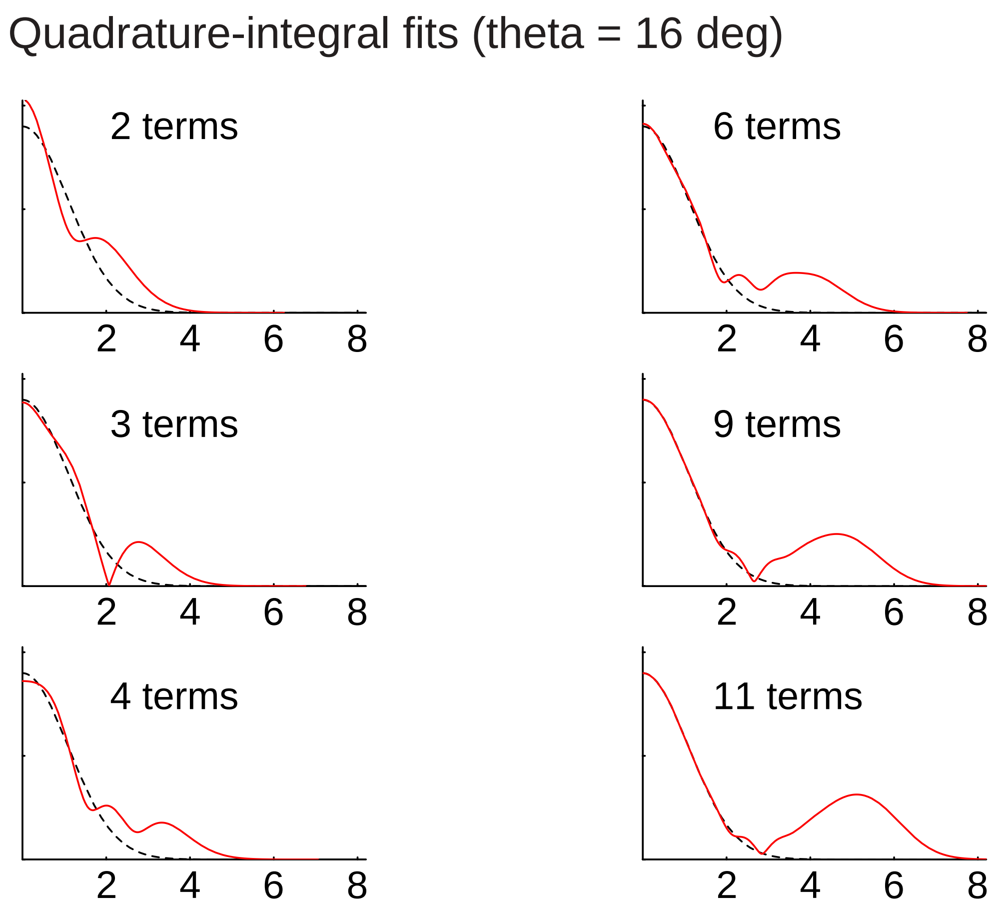
$$\int \tilde{v}_n^*(x)\tilde{u}_m(x)\, dx = 0$$

thereby obtaining "quadrature coefficients"

$$\tilde{c}_n = \int_{-\infty}^{\infty} \tilde{v}_n^*(x)\, \tilde{f}(x)\, dx = \text{"quadrature coefficients"}$$

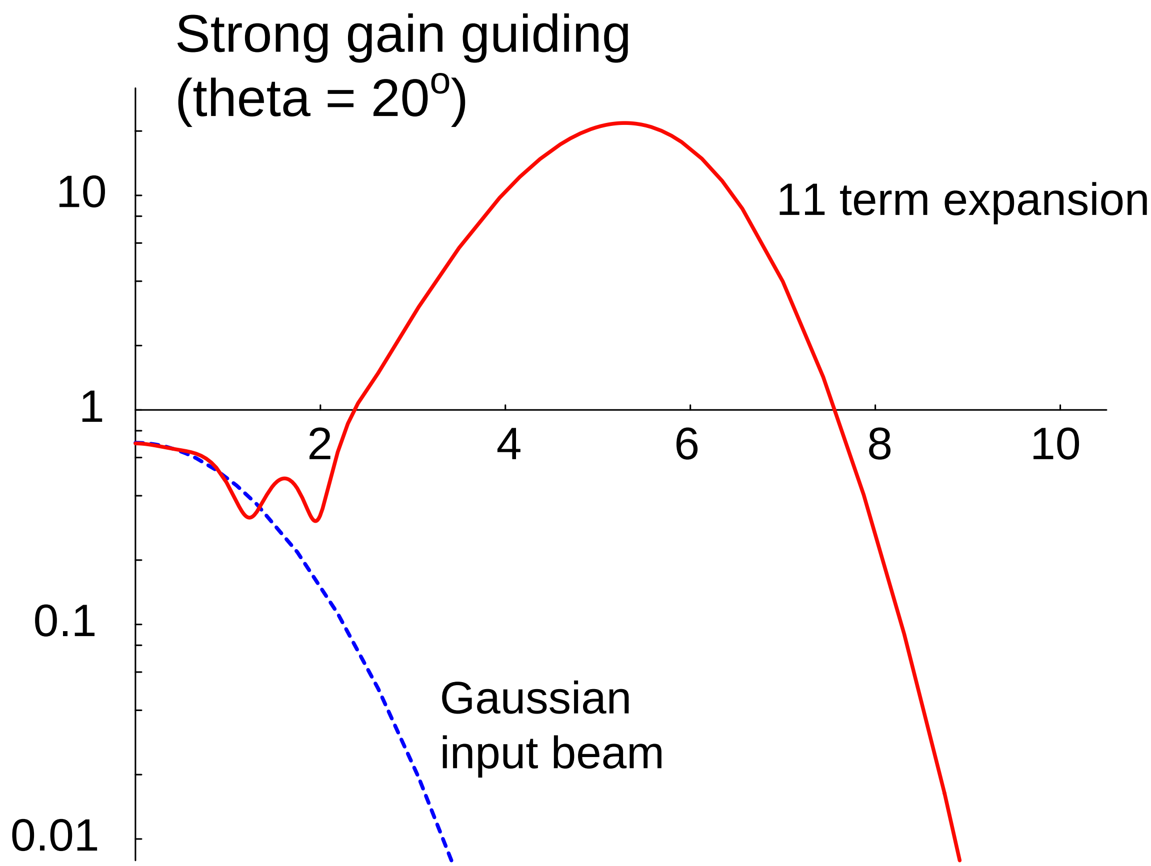
Quadrature expansion may not converge

Expansions using quadrature coefficients converge slowly if at all — often diverge for strong enough gain guiding



Quadrature fit for stronger gain guiding

Quadrature expansions diverge wildly for still stronger gain guiding



Minimum error expansion procedure

Is there a better way? By writing mean-square error for finite N-term eigenmode expansion as

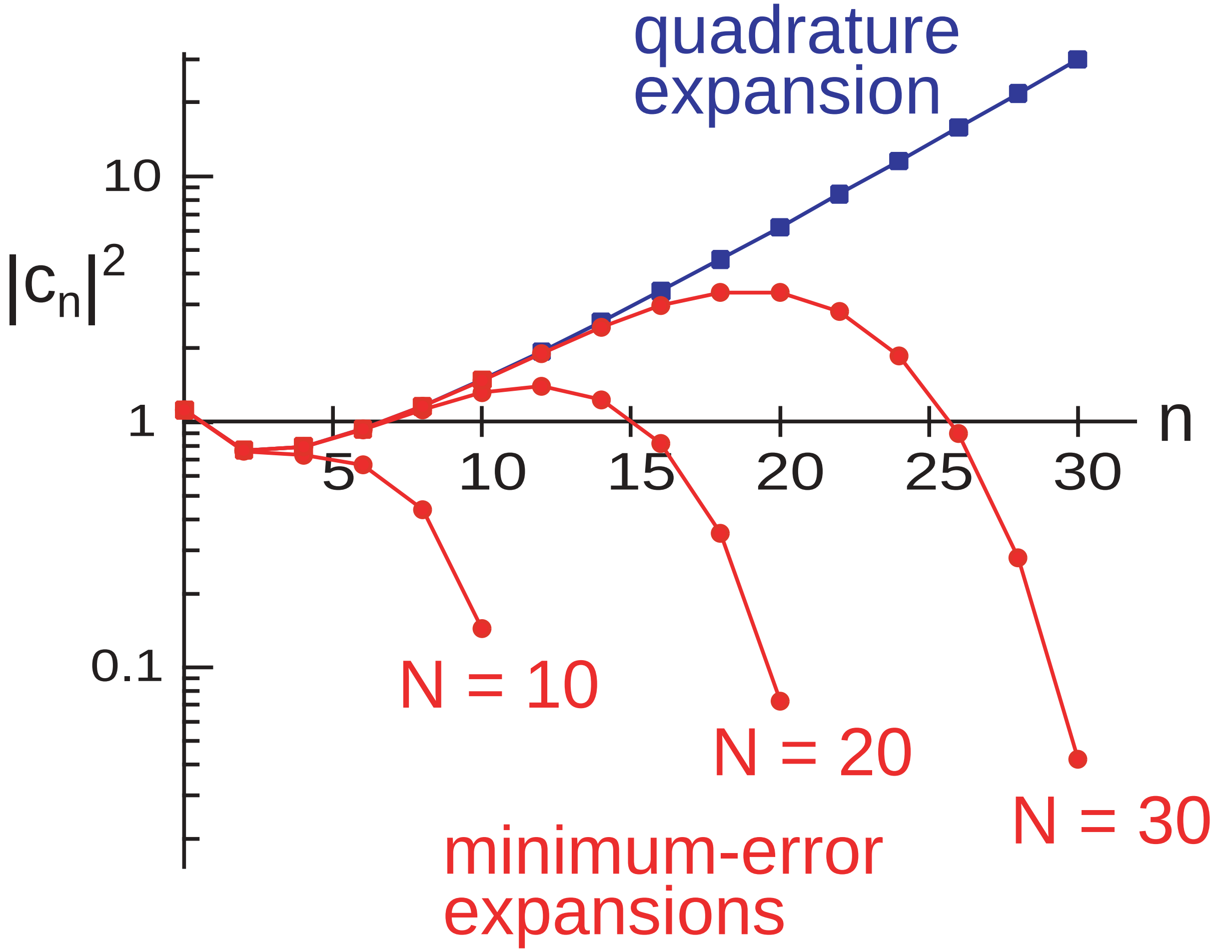
$$\begin{aligned}\epsilon_N &= \int_{-\infty}^{\infty} \left| \tilde{f}(x) - \sum_n \tilde{c}_n \tilde{u}_n(x) \right|^2 dx \\ &= 1 - \sum_n \tilde{c}_n^* f_n - \sum_n \tilde{c}_n f_n^* + \sum_n \sum_m \tilde{c}_n^* \tilde{c}_m M_{nm}\end{aligned}$$

one can derive a matrix inversion procedure to find "minimum error coefficients"

- Produces expansions which do converge well with increasing numbers of terms.
- But coefficients themselves change as number of terms is increased

Minimum-error vs. quadrature coefficients

Typical example:



5) Quantum noise in laser oscillators

In normal-mode laser cavities, spontaneous emission from atoms produces quantum noise equivalent to "one noise photon per mode"

$$\frac{dn}{dt} = \kappa\,(n+1)\,N_2 \,-\, \kappa\,n\,N_1$$

This leads to quantum-limited Schawlow-Townes linewidth for laser oscillators

$$\Delta f_L = \frac{N_2}{N_2 - N_1} \,\times\, \frac{\pi\,hf\,\Delta f_c^2}{P_{osc}}$$

Also leads to standard quantum-limited noise figure for laser amplifiers

Excess noise factor for nonnormal modes

Spontaneous emission rate in nonnormal lasers increases to K_p noise photons per mode

$$\frac{dn}{dt} = \kappa\,(n + K_p)\,N_2 \;-\; \kappa\,n\,N_1$$

$$K_p = \int \tilde{v}_0^*(x)\,\tilde{v}_0(x)\,dx$$

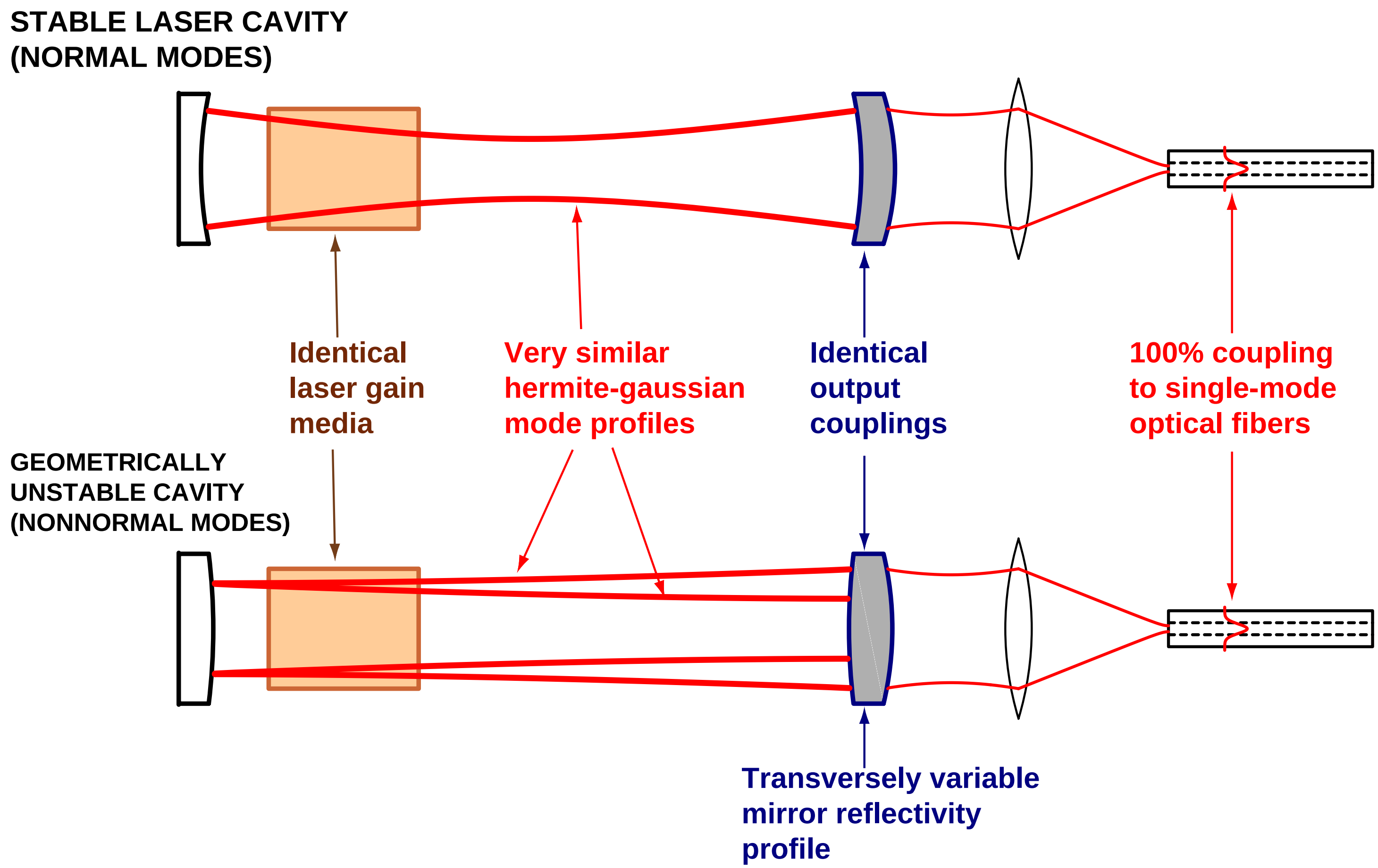
$= \text{Petermann excess noise factor } (>1)$

This leads to measurable increase in the quantum-limited linewidth for lasers having nonnormal cavity modes

$$\Delta f_L = K_p \times \frac{N_2}{N_2 - N_1} \;\times\; \frac{\pi\,hf\,\Delta f_c^2}{P}$$

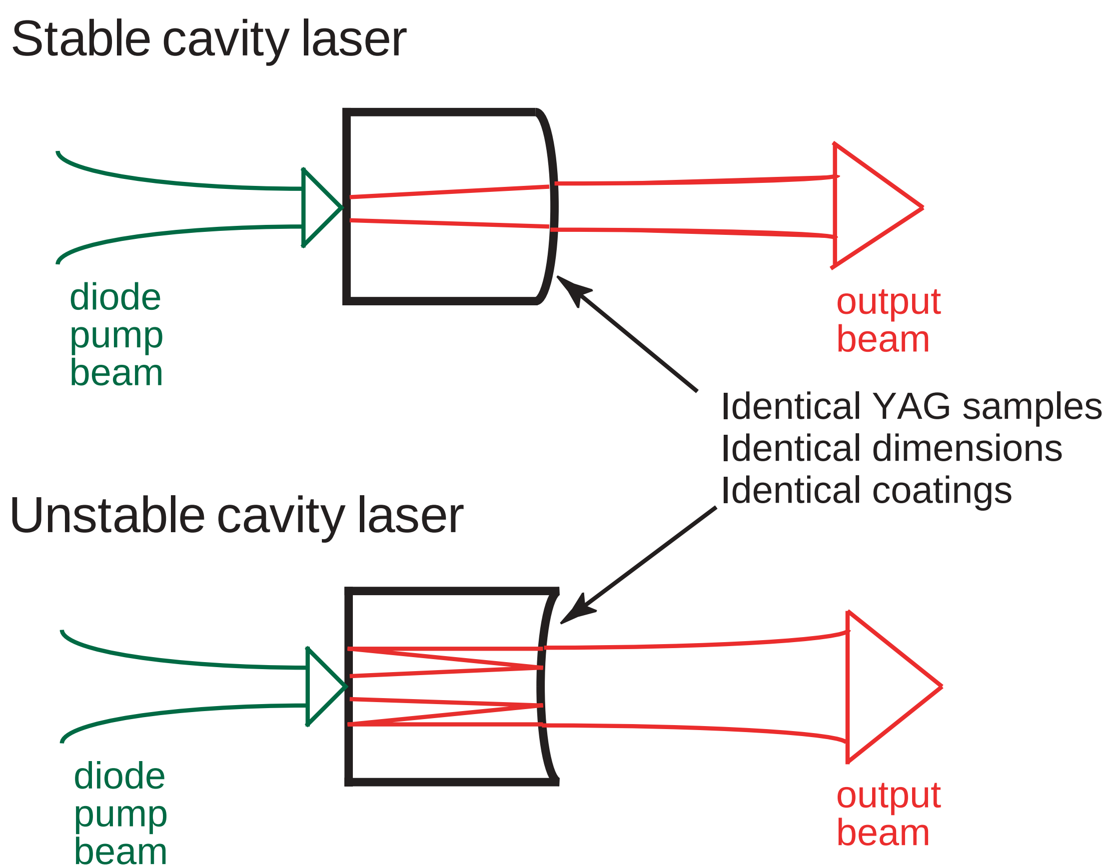
Identical laser parameters

but very different Schawlow-Townes linewidths



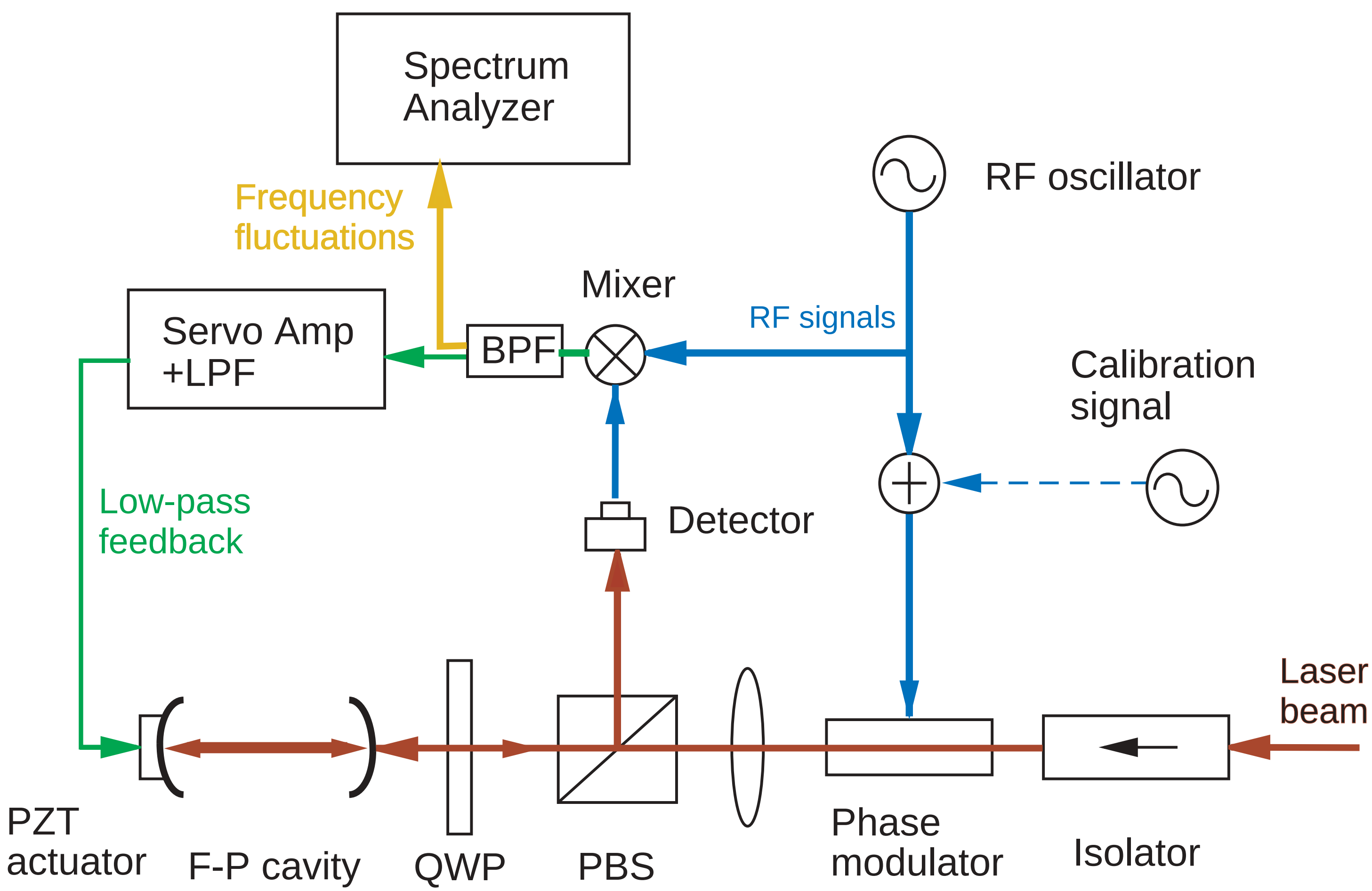
Experiment: stable & unstable mini-YAG lasers

Compare quantum linewidths of identical miniature monolithic stable and unstable resonators



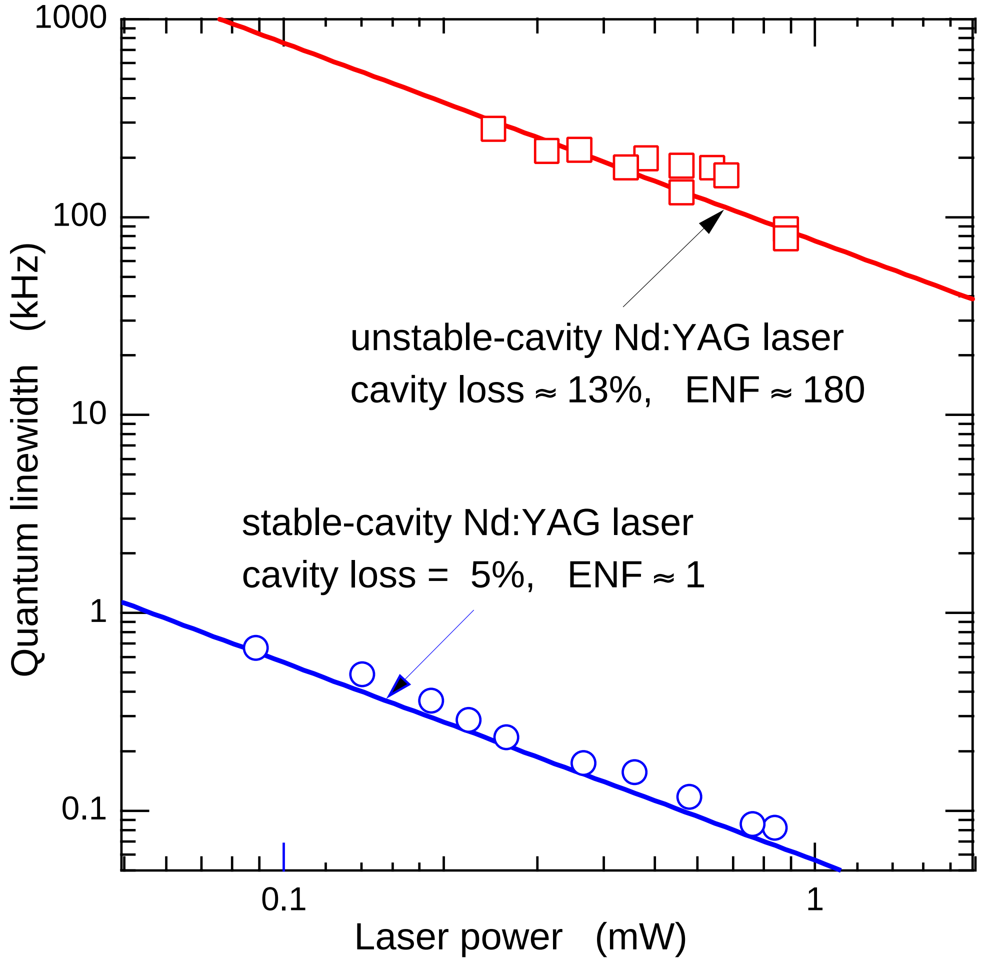
Pound-Drever spectrum measurement system

Measure quantum noise sidebands using modified
Pound-Drever stabilization system



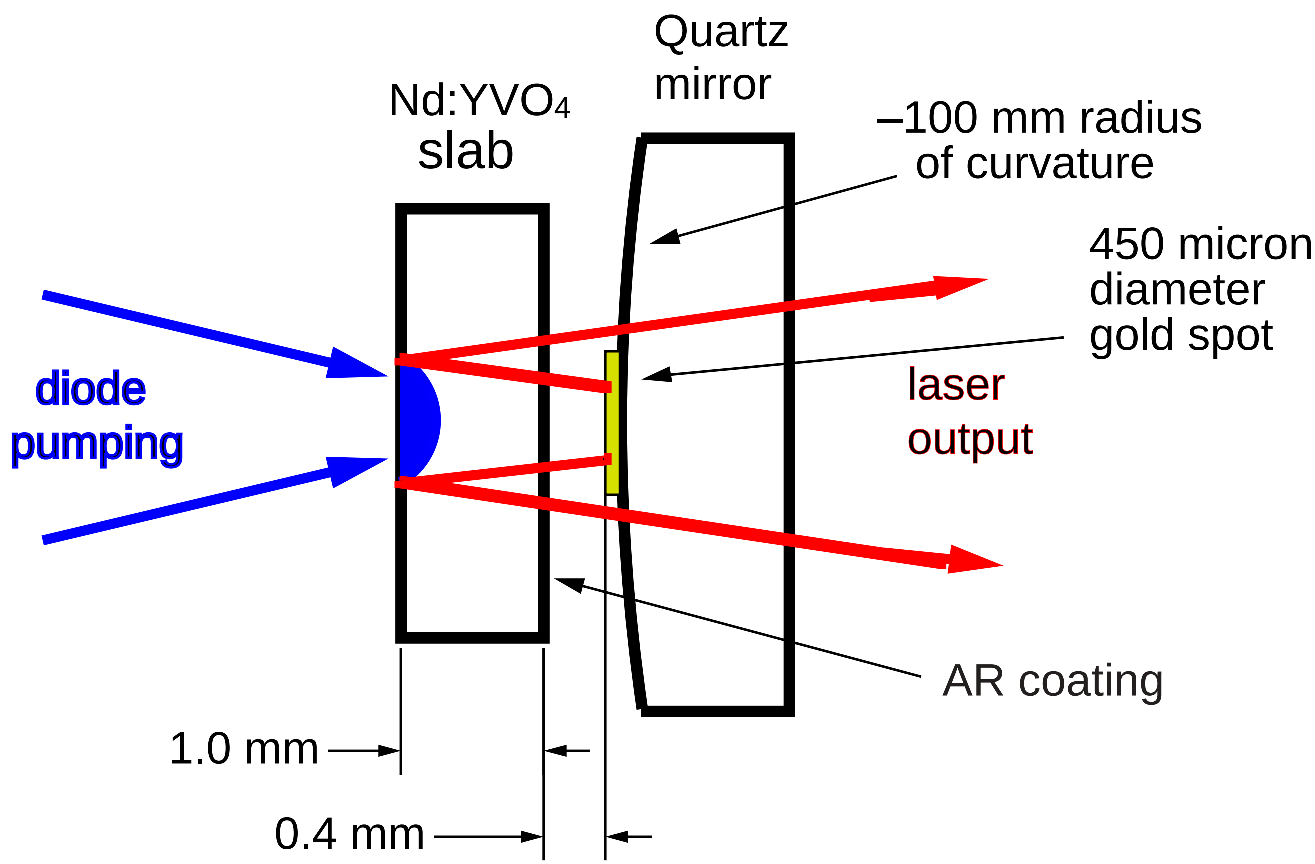
Measured excess noise factors

Experimental result: large excess quantum noise for unstable-resonator (nonnormal-mode) laser



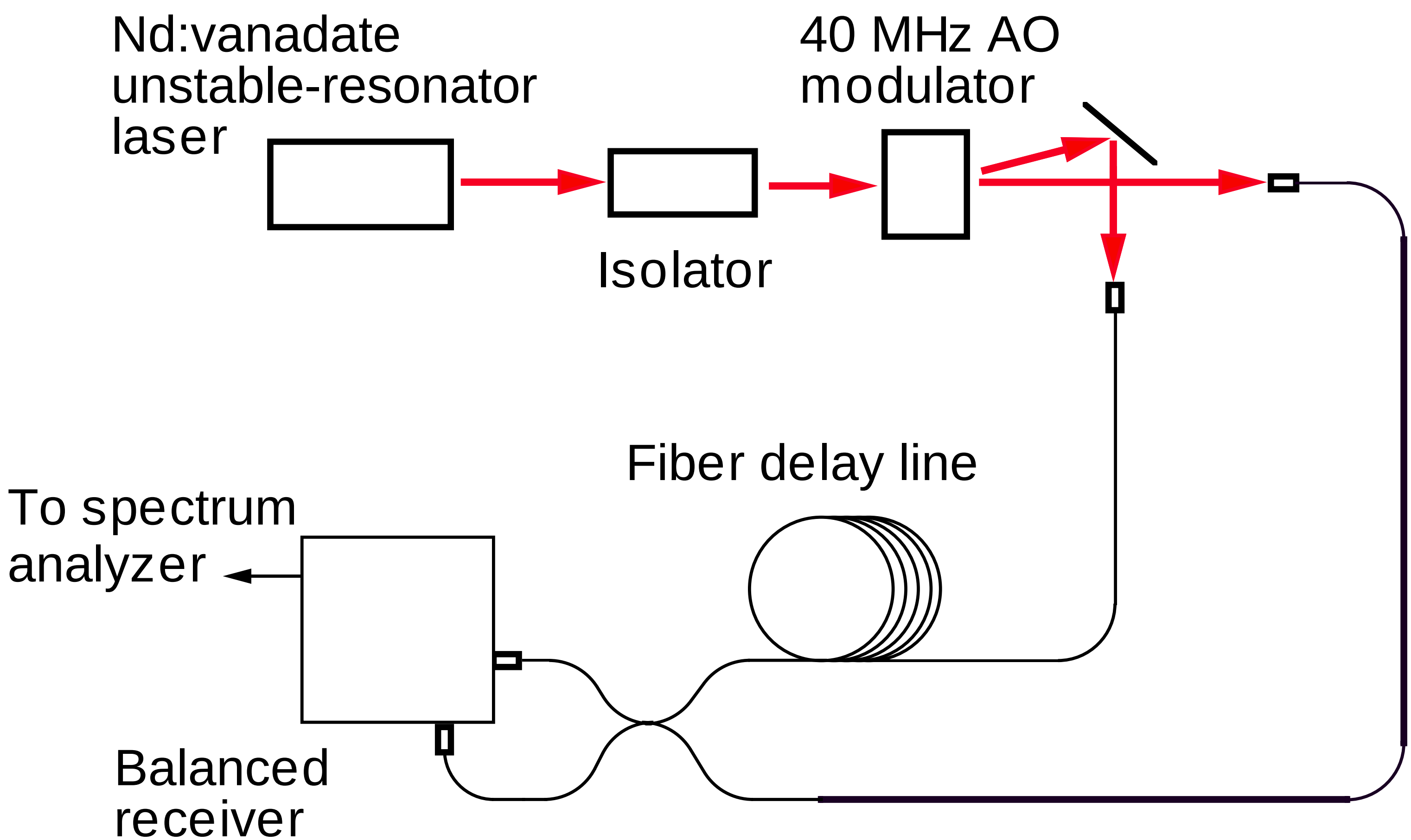
Vanadate unstable resonator laser

More definitive experiment using miniature quasi monolithic Nd:vanadate unstable-resonator laser



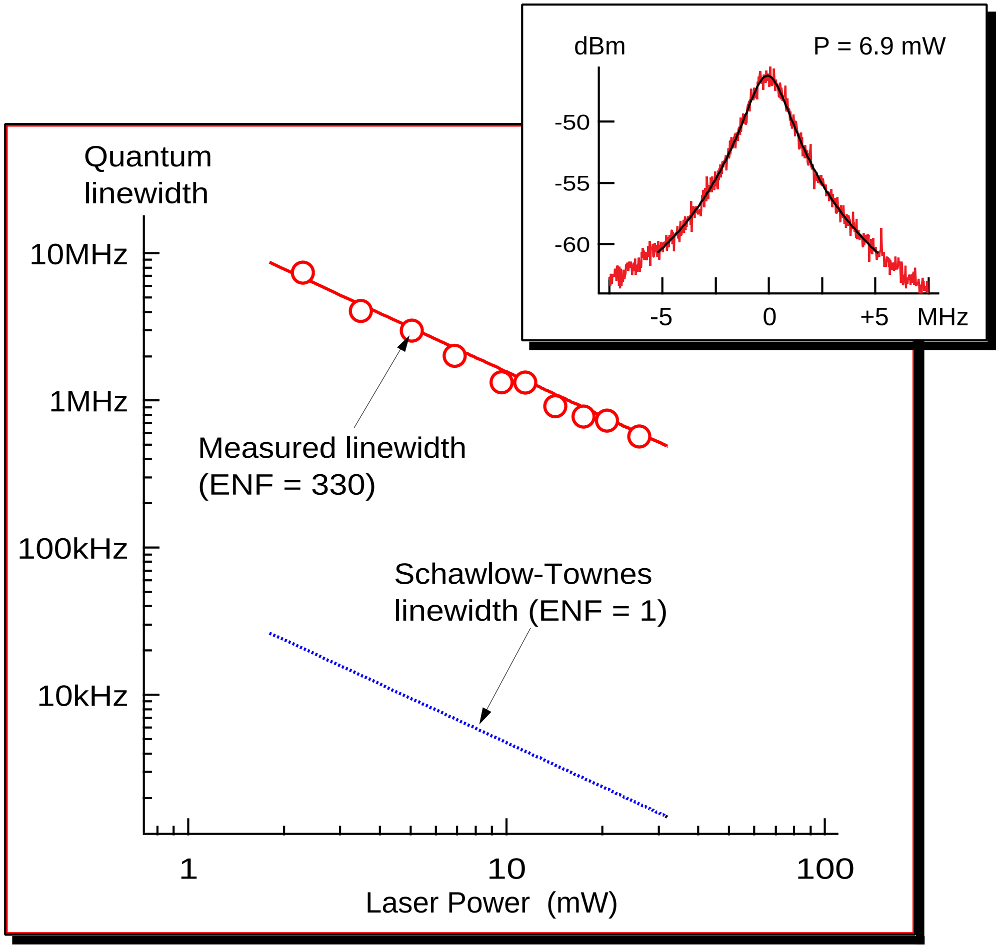
Fiber delay line spectral measurement system

Measure quantum linewidth using self-heterodyne apparatus with optical fiber delay line



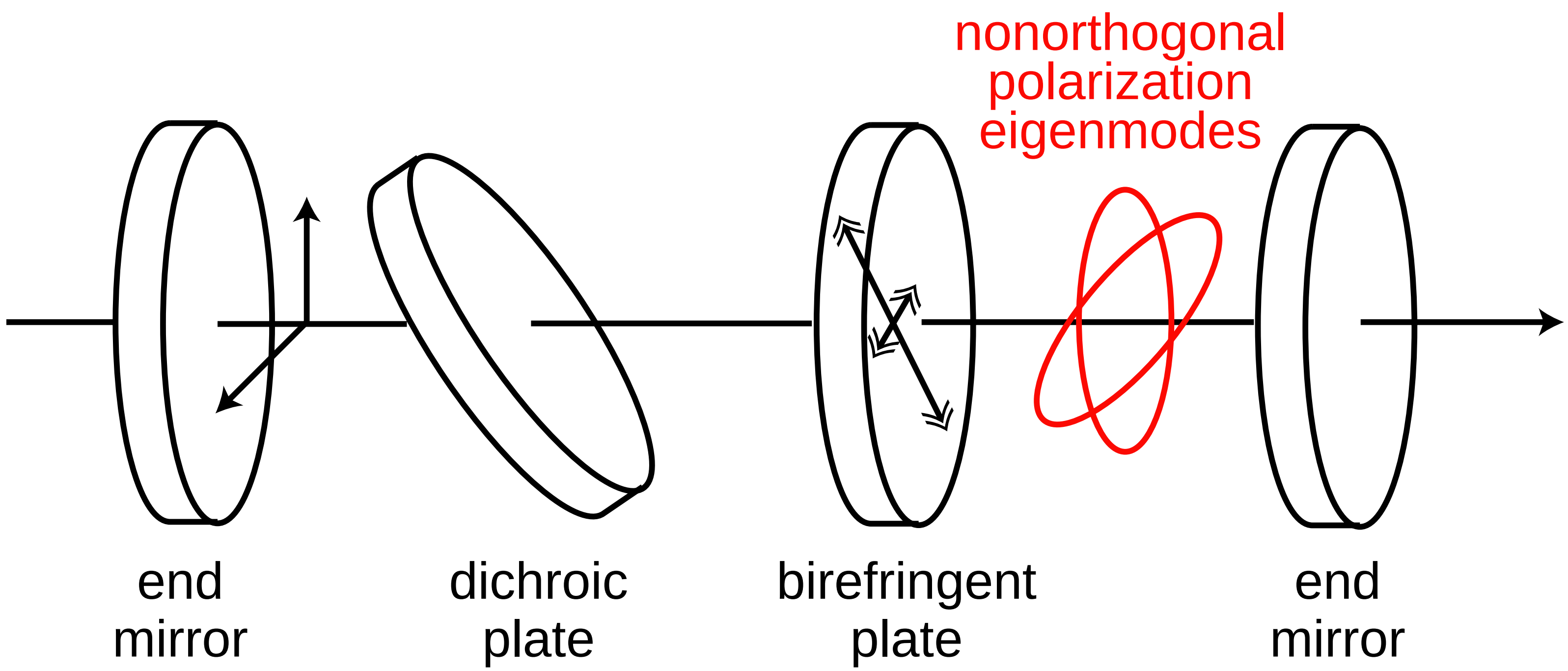
Vanadate unstable resonator measurements

Definitive confirmation of excess quantum linewidth with ENF $\approx$ 330 in hard-edged unstable-resonator laser



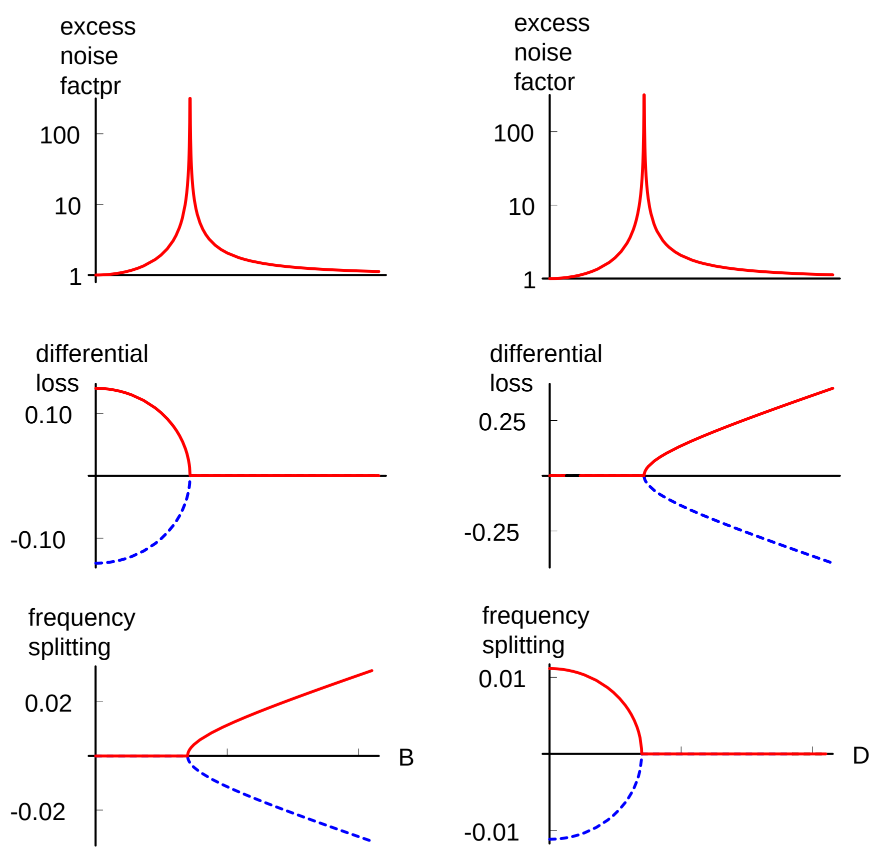
Nonnormal polarization eigenmodes

"Twisted" optical resonator with nonnormal polarization eigenmodes



Excess noise with nonnormal polarization modes

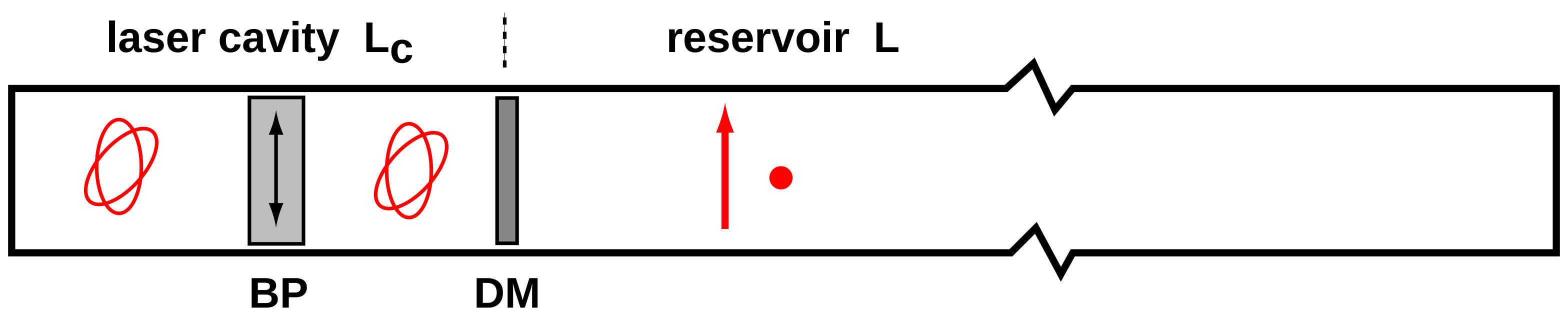
Quantum noise properties of an optical resonator with nonnormal "twisted-polarization" eigenmodes



B = birefringence; D = dichroism

2 X 1D reservoir model for twisted polarization

Analytical model for nonnormal polarization laser looking into one-dimensional dual-polarization transmission-line reservoir



LC = polarization cavity

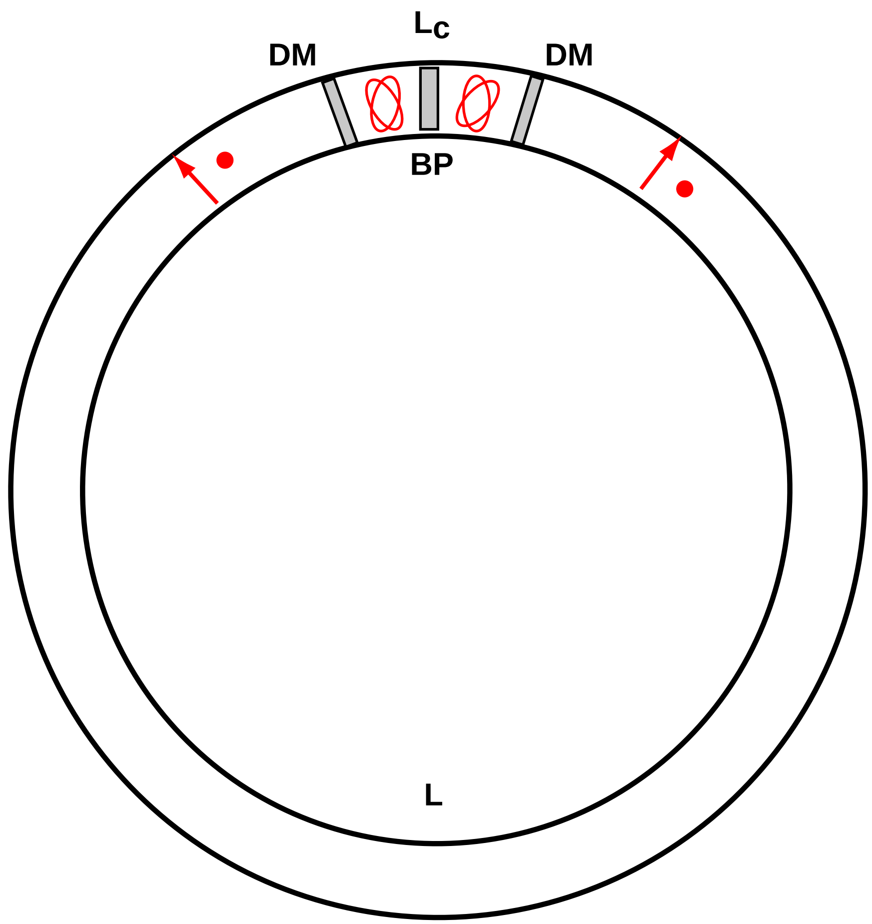
RBP = rotatable birefringent plate

DM = dichroic delectric mirror

L = long waveguide reservoir

Ring reservoir model for twisted polarization

Ring-resonator "2 × 1D" model for twisted polarization laser



Summary

- Some real physical systems are not described by hermitian operators, and therefore do not have a complete set of normal modes
- This leads to significant changes in the physical, mathematical and quantum properties of these nonnormal systems
- Loss of orthogonality is the key driver for all of these unusual effects
- All nonnormal systems are also in one way or another lossy systems (due to internal losses or output coupling) — but not all lossy systems are nonnormal systems
- Fully quantum treatments are being developed

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