Physics 408 – Exam 2 **Name** Solution

You are graded on your work (with partial credit where it is deserved) so please do not just write down answers with no explanation (or skip important steps)!

Please give clear, well-organized, understandable solutions.

$h = 6.63\times10^{-34}$ J s [Planck's constant]
$c = 3.00\times10^{8}$ m/s [speed of light]
$\sigma_B = 5.67\times10^{-8}\ \frac{\mathrm{W}}{\mathrm{m}^2\cdot K^4}$ [Stefan-Boltzmann constant]
$1\ \mathrm{eV} = 1.60\times10^{-19}$ J and $\frac{1\ \mathrm{eV}}{k} = 11{,}600$ K
average distance of Earth from Sun $= 1.50\times10^{11}$ m

$k = 1.38\times10^{-23}$ J/K [Boltzmann constant]
$m_e = 9.11\times10^{-31}$ kg [mass of electron]
$G = 6.67\times10^{-11}\ \frac{\mathrm{N\,m}^2}{\mathrm{kg}^2}$ [gravitational constant]
$R_\odot = 6.96\times10^{8}$ m [radius of Sun]
$R_\oplus = 6.38\times10^{6}$ m [radius of Earth]

See back page for small integral table.

The variables have their usual meanings: E = energy, S = entropy, V = volume, N = number of particles, T = temperature, P = pressure, μ = chemical potential, B = applied magnetic field, C_V = heat capacity at constant volume, k = Boltzmann constant. Also, $\langle\cdots\rangle$ represents an average.

You should know this, but:

$$\langle n(\varepsilon_i)\rangle = \frac{1}{e^{(\varepsilon_i-\mu)/kT}\pm1} \quad ; \quad F = -kT\ln Z \quad , \quad dF = -SdT - PdV + \mu dN \quad ; \quad \text{flux} = e\sigma_B T^4$$

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1. Let us see when we get Bose-Einstein condensation, and when we do not, by considering a set of N spinless bosons with a density of states

$$D(\varepsilon) = A\varepsilon^n$$

where A and n are constants. Choose the zero of energy by setting

$$\varepsilon_1 = 0$$

where ε_1 is the energy of the single-particle ground state.

(a) (5) For $T \to 0$, by assuming that $\langle n_1 \rangle \to N$ continuously (i.e., that there is Bose-Einstein condensation at a nonzero temperature), show that

$$\langle n_1 \rangle \to -\frac{kT}{\mu} \qquad \text{and thus that} \qquad \mu \to -\frac{kT}{N}$$

where $\langle n_1 \rangle$ is the number of bosons in the ground state.

State each essential step in your reasoning clearly.

$$\frac{1}{e^{(\varepsilon_1-\mu)/kT} - 1} \longrightarrow \text{very large } N \Rightarrow e^{-\mu/kT} \approx 1 \quad \text{(nearly equal to 1)}$$

$$\Rightarrow -\frac{\mu}{kT} \ll 1 \quad \text{(nearly equal to 0)}$$

$$\Rightarrow e^{-\mu/kT} \approx 1 - \frac{\mu}{kT}$$

Then $\langle n_1 \rangle = \dfrac{1}{e^{-\mu/kT} - 1} \approx -\dfrac{kT}{\mu}$

and $\mu = -\dfrac{kT}{N}$ to a very good approximation.

(b) (5) Using the approximation $\mu = 0$ (suggested by the result of part (a)), represent all the bosons that are not in the ground state at finite T by an integral involving the density of states. Show that

$$N - \langle n_1 \rangle = f(T) \int_0^\infty \frac{x^{s-1}}{e^x - 1} dx$$

while at the same time obtaining the function $f(T)$ and the constant s. **Again, be clear in each step.**

$$\boxed{N - \langle n_1 \rangle =} \int_0^\infty \langle n(\varepsilon) \rangle D(\varepsilon)\, d\varepsilon$$

$$= \int_0^\infty \frac{1}{e^{(\varepsilon - \mu)/kT} - 1} A \varepsilon^n d\varepsilon$$

$$= \boxed{A (kT)^{n+1} \int_0^\infty \frac{x^n}{e^x - 1} dx} \quad , \quad x \equiv \frac{\varepsilon - \mu}{kT} = \frac{\varepsilon}{kT}$$

with $\mu \approx 0$

(c) (5) Now, the Riemann zeta function can be defined by

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x - 1} dx$$ and term-by-term integration of the series expansion gives $$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}$$

which converges only for $s > 1$ and not e.g. $s = 1$. ($\Gamma(s)$ is the gamma function, which is finite for $s > 0$. We are concerned only with real numbers here.)

The criterion for having Bose-Einstein condensation is that the above treatment gives a finite result for $N - \langle n_1 \rangle$ which goes to zero as $T \to 0$. (If the excited states can hold only a finite number of particles, the rest of the particles are forced to condense into the ground state. This is the basic reason that Bose-Einstein condensation occurs.) Determine the values of n (in $D(\varepsilon) = A\varepsilon^n$) for which there will be Bose-Einstein condensation. **Again, be clear.**

$n = s - 1$, or $s = n + 1$, and $s > 1$ $\Rightarrow$ $\boxed{n > 0}$

(c) (i) (3) Will there be Bose-Einstein condensation for **nonrelativistic** bosons in **3** dimensions? Explain, based on your result in part (b). Recall that

$$D(\varepsilon)d\varepsilon = \frac{4\pi p^2 dp}{h^3 / V}$$

in 3 dimensions, where $\varepsilon = p^2 / 2m$ is the energy corresponding to the magnitude p of the momentum.

$$D(\varepsilon)\,d\varepsilon = 4\pi \frac{V}{h^3} (2m\varepsilon)^{1/2}\, m\, d\varepsilon$$

since $d\varepsilon = \frac{2p\,dp}{2m} \Rightarrow p\,dp = m\,d\varepsilon$

and also $p = (2m\varepsilon)^{1/2}$

$\Rightarrow D(\varepsilon) \propto \varepsilon^{1/2}$ so $\boxed{n = \frac{1}{2} > 0 \Rightarrow \text{yes}}$

(ii) (3) Will there be Bose-Einstein condensation for **nonrelativistic** bosons in **2** dimensions (e.g. for a thin film or adsorbed atoms)? Explain, based on your result in part (b). Recall that

$$D(\varepsilon)d\varepsilon = \frac{2\pi p dp}{h^2 / A}$$

in 2 dimensions, where again $\varepsilon = p^2 / 2m$.

$$D(\varepsilon)\,d\varepsilon = 2\pi \frac{A}{h^2} \cdot p\,dp = 2\pi \frac{A}{h^2} m\,d\varepsilon \quad \text{from above}$$

$\Rightarrow D(\varepsilon) = \text{constant}$ so $\boxed{n = 0\text{, and } n \text{ not } > 0 \Rightarrow \text{no}}$

(iii) (2) Will there be Bose-Einstein condensation for **ultrarelativistic** bosons in **3** dimensions? Explain, based on your result in part (b). Ultrarelativistic means $\varepsilon = cp$.

$$D(\varepsilon)\,d\varepsilon = 4\pi \frac{V}{h^3} \left(\frac{\varepsilon}{c}\right)^2 d\left(\frac{\varepsilon}{c}\right) \Rightarrow D(\varepsilon) \propto \varepsilon^2$$

so $\boxed{n = 2 > 0 \Rightarrow \text{yes}}$

(iv) (2) Will there be Bose-Einstein condensation for **ultrarelativistic** bosons in **2** dimensions? Explain, based on your result in part (b). Again, ultrarelativistic means $\varepsilon = cp$.

$$D(\varepsilon)\,d\varepsilon = 2\pi \frac{A}{h^2} \frac{\varepsilon}{c} \frac{d\varepsilon}{c} \Rightarrow D(\varepsilon) \propto \varepsilon$$

so $\boxed{n = 1 > 0 \Rightarrow \text{yes}}$

2. Let us determine the properties of a degenerate (i.e. highly quantum-mechanical) ideal gas of N fermions with the density of states

$$D(\varepsilon) = aN\varepsilon^n.$$

(a) (6) At $T = 0$, calculate the Fermi energy ε_F in terms of N, a, and n.

$$N = \int_0^{\varepsilon_F} aN\varepsilon^n \, d\varepsilon = aN \left[\frac{\varepsilon^{n+1}}{n+1} \right]_0^{\varepsilon_F} = \frac{aN}{n+1} \varepsilon_F^{n+1}$$

$$\Rightarrow \varepsilon_F^{n+1} = \frac{n+1}{a}$$

and $\boxed{\varepsilon_F = \left(\frac{n+1}{a} \right)^{\frac{1}{n+1}}}$

(b) (6) Again at $T = 0$, calculate the average energy per fermion, $\frac{\langle E \rangle}{N}$, in terms of ε_F and n. I.e., use the result of part (a) to get $\frac{\langle E \rangle}{N} =$ (number which is determined by only n) $\times\, \varepsilon_F$.

[Hint: your result should reduce to $\frac{\langle E \rangle}{N} = \frac{3}{5}\varepsilon_F$ for nonrelativistic fermions in 3 dimensions.]

$$\langle E \rangle = \int_0^{\varepsilon_F} \varepsilon \, D(\varepsilon) \, d\varepsilon$$

$$= \int_0^{\varepsilon_F} aN\varepsilon^{n+1} \, d\varepsilon$$

$$= aN \left[\frac{\varepsilon^{n+2}}{n+2} \right]_0^{\varepsilon_F}$$

$$\Rightarrow \boxed{\frac{\langle E \rangle}{N}} = \frac{\varepsilon_F}{n+2} \, a \, \varepsilon_F^{n+1} = \frac{\varepsilon_F}{n+2} \underset{\text{from part (a)}}{(n+1)} = \boxed{\frac{n+1}{n+2} \varepsilon_F}$$

$$\left[n = \tfrac{1}{2} \Rightarrow \frac{\langle E \rangle}{N} = \frac{\frac{1}{2}+1}{\frac{1}{2}+2} \varepsilon_F = \frac{3}{5} \varepsilon_F \right]$$

(c) (6) Write down the expressions for $\langle E \rangle$ and $\varepsilon_F N$ at a nonzero temperature T, using the Fermi function $f(\varepsilon,T) = \frac{1}{e^{(\varepsilon-\mu)/kT}+1}$. These are integrals involving the density of states.

$$\langle E \rangle = \int_0^{\infty} \varepsilon \, f(\varepsilon,T) \, D(\varepsilon) \, d\varepsilon$$

$$\varepsilon_F N = \int_0^{\infty} \varepsilon_F \, f(\varepsilon,T) \, D(\varepsilon) \, d\varepsilon$$

(d) (7) Using the semi-rigorous approach in which the chemical potential μ is approximated by the Fermi energy ε_F in $\frac{\partial f(\varepsilon,T)}{\partial T}$ and $D(\varepsilon)$ is approximated by $D(\varepsilon_F)$ within the integral, show that the heat capacity at constant volume is given by

$$C_V = \text{constant} \times D(\varepsilon_F) T^S$$

where you will obtain the constant and the exponent S. **(See the last page for integrals.)**

$$C_V = \frac{\partial}{\partial T}\langle E\rangle = \frac{\partial}{\partial T}\left(\langle E\rangle - \varepsilon_F \langle N\rangle\right)$$

constant

$$= \int_0^\infty (\varepsilon - \varepsilon_F)\,\frac{\partial f(\varepsilon,T)}{\partial T}\, D(\varepsilon)\, d\varepsilon$$

$$\frac{\partial f(\varepsilon,T)}{\partial T} = -\frac{1}{\left(e^{(\varepsilon-\varepsilon_F)/kT}+1\right)^2}\, e^{(\varepsilon-\varepsilon_F)/kT}\left(\frac{\varepsilon-\varepsilon_F}{k}\right)\left(-\frac{1}{T^2}\right)$$

$$= +\frac{1}{(e^x+1)^2}\, e^x\, \frac{x}{T}$$

with $\mu \approx \varepsilon_F$ in this approach (which is semi-rigorous)

For $\frac{\varepsilon_F}{kT} >> 1$ this is sharply peaked around $\varepsilon = \varepsilon_F$, or $x = 0$, so

$$\boxed{C} \approx D(\varepsilon_F)\frac{(kT)^2}{T}\int_{-\varepsilon_F/kT}^{\infty} dx\, x^2\, \frac{e^x}{(e^x+1)^2} \quad \text{with } \frac{\varepsilon_F}{kT} >> 1$$

$$\approx D(\varepsilon_F)\, k^2 T \int_{-\infty}^{\infty} dx\, x^2\, \frac{e^x}{(e^x+1)^2}$$

$$= \boxed{\frac{1}{3}\pi^2 D(\varepsilon_F) k^2 T}$$

from integral table

3. The Sun can be approximately treated as a blackbody with a surface temperature of 6000 K.
(See the front page for constants.)

(a) (9) Calculate the flux of solar radiant energy at Earth's orbital distance (in W/m^2).

$$\frac{\text{energy emitted}}{\text{time}} = \sigma_B T_{Sun}^4 \cdot 4\pi R_\odot^2$$

$$= 4.5 \times 10^{26} \text{ W} \quad \text{after numbers used}$$

$$\boxed{\text{flux} =} \frac{4.5 \times 10^{26} \text{ W}}{4\pi (1.50 \times 10^{11} \text{ m})^2}$$

$$= \boxed{1580 \frac{W}{m^2}}$$

(b) (9) Take the earth to be in a thermal steady state, radiating as much energy (averaged over the day) as it receives from the Sun. Estimate the average surface temperature of the Earth. [Note that the Earth receives energy over a circular cross-section and radiates over a spherical surface.]

$$(\sigma_B T_{Earth}^4)(4\pi R_\oplus^2) = (\pi R_\oplus^2)(1580 \tfrac{W}{m^2})$$

$$\Rightarrow \boxed{T_{Earth} = 290 \text{ K}}$$

(c) In maximizing the Planck distribution of radiation as a function of wavelength, one obtains

$$x = 5\left(1 - e^{-x}\right) \quad , \quad x \equiv \frac{hc}{\lambda kT}.$$

(i) (3) Calculate the value of x by iteration, listing your first 2 successive values here, below.

Put $x = 5$ on right for quick convergence,
to about $\boxed{x = 4.965}$

(ii) (4) Use your result to estimate the average wavelength of the radiation emitted by the Earth, preferably given in $\mu m = 10^{-6}$ m.

$$\lambda = \frac{hc}{5kT} \approx \boxed{10 \mu m}$$

or 4.965

4. *Variation of density (and thus pressure) with height in isothermal atmosphere*

See the front page for some relevant equations involving the Helmholtz free energy F and its relation to the chemical potential μ, which can then be obtained from F after F is obtained from the partition function Z.

The partition function for molecules at a height z in the atmosphere is approximately given by

$$\ln Z = N \ln z_{mol} - (N \ln N - N) \quad , \quad z_{mol}(z) = e^{-mgz/kT}\, z_{mol}(0) \quad , \quad z_{mol}(0) = \frac{V}{\lambda_{th}^3} \quad , \quad \lambda_{th} = \frac{h}{(2\pi mkT)^{1/2}} \quad .$$

(The partition function for a single molecule is here called z_{mol} to avoid confusion with the height z. Recall that the last term in $\ln Z$ comes from $\ln N!$ in Stirling's approximation.)

(a) (12) Obtain $\mu(z)$, the chemical potential as a function of height, in terms of z, the molecular mass m, the acceleration of gravity g, the thermal de Broglie wavelength λ_{th}, the temperature T, and the number density $n(z) = \dfrac{N}{V}$.

$$F = -kT \ln Z$$

$$= -kT\,(N \ln z_{mol}) + kT\,(N \ln N - N)$$

$$\boxed{\mu =} \left(\frac{\partial F}{\partial N}\right)_{V,T} \qquad \text{since} \quad dF = -S\,dT - P\,dV + \mu\, dN$$

$$= -kT \ln z_{mol} + kT\,(\ln N + \underbrace{N \cdot \tfrac{1}{N} - 1}_{=0})$$

$$= -kT \ln \left(\frac{e^{-mgz/kT}}{\lambda_{th}^3} \cdot \frac{V}{N} \right)$$

$$= \boxed{-kT \ln \left(\frac{e^{-mgz/kT}}{\lambda_{th}^3} \cdot \frac{1}{n(z)} \right)}$$

(b) (13) Assuming thermal equilibrium and a constant temperature T, determine $n(z)$ as a function of z and T:

$$n(z) = n(0) \times \text{function of } z \text{ and } T \ .$$

$\mu(z) = \mu(0)$ for thermal equilibrium,
with T assumed constant here

$$\Rightarrow \quad -kT \ln\left(\frac{e^{-mgz/kT}}{\lambda_{th}^3}\frac{1}{n(z)}\right) = -kT \ln\left(\frac{1}{\lambda_{th}^3}\frac{1}{n(0)}\right)$$

$z = 0$

$$\Rightarrow \quad \frac{e^{-mgz/kT}}{n(z)} = \frac{1}{n(0)}$$

$$\Rightarrow \quad \boxed{n(z) = n(0)\, e^{-mgz/kT}}$$

$$\int_{-\infty}^{\infty} dx\, x^2 \frac{e^x}{\left(e^x+1\right)^2} = \frac{1}{3}\pi^2$$

$$\int_0^\infty e^{-ax^2}\, dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}$$ for $a > 0$ (the Gaussian integral)

$$\int_0^\infty x^2 e^{-ax^2}\, dx = \frac{1}{4}\sqrt{\frac{\pi}{a^3}}$$ for $a > 0$

$$\int_0^\infty x^{2n} e^{-ax^2}\, dx = \frac{2n-1}{2a}\int_0^\infty x^{2(n-1)} e^{-ax^2}\, dx$$

$$\int_0^\infty x^3 e^{-ax^2}\, dx = \frac{1}{2a^2}$$ when $a > 0$