

You are graded on your work, with partial credit where it is deserved.

Please give clear, well-organized solutions, and please show enough work to be convincing.

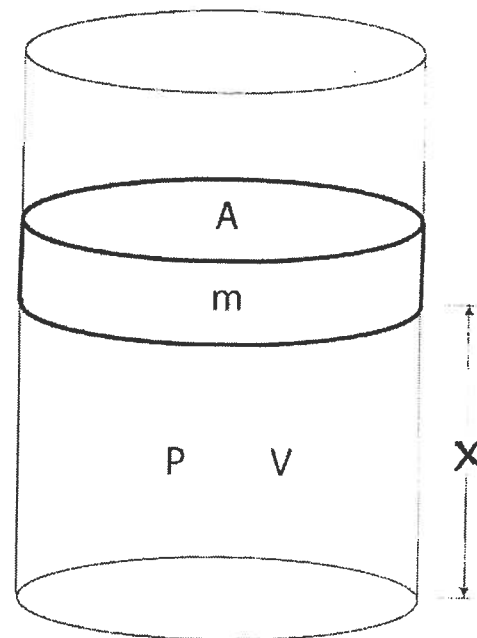
1. We will consider the vibrations of a cylinder with cross-sectional area A and mass m , floating above an ideal gas with heat capacity C_V at constant volume and C_P at constant pressure, and with $\gamma \equiv C_P / C_V$. No energy leaves or enters the gas, so the vibrations are adiabatic.

You may assume the ideal gas equation of state $PV = NkT$, and the fact that $E = E(T)$ for an ideal gas.

But please do not assume other properties, such as the relation between V and P or T during an adiabatic process, since these are equivalent to some of what is asked for below.

(a) (4) Write down the first law of thermodynamics, relating the change in internal energy dE to the heat dQ added to the gas and the work dW done by the gas*.

$$\boxed{dE = dQ - dW}$$



(b) (4) Using the first law, obtain the relation between C_V , dE , and dT .

$$C_V \equiv \frac{dQ}{dT} \text{ with } V \text{ fixed, so that } dW = P dV = 0$$

$$= \frac{dE}{dT} \Rightarrow \boxed{dE = C_V dT}$$

$$[\text{or } C_V = \left(\frac{\partial E}{\partial T}\right)_V]$$

(c) (4) Using the first law, plus the ideal gas equation of state, show that $C_P - C_V = \text{constant}$, and obtain the constant in terms of the number of particles N and the Boltzmann constant k .

$$C_P \equiv \frac{dQ}{dT} \text{ with } P \text{ fixed, so that } dW = P dV = d(PV)$$

$$= \frac{d(E + PV)}{dT}$$

$$= C_V + \frac{\partial(NkT)}{\partial T}$$

$$= C_V + Nk$$

$$\Rightarrow \boxed{C_P - C_V = Nk}$$

since $PV = NkT$,
and $E = E(T) \Rightarrow dE = C_V dT$
for any process in case of
ideal gas

* This is a slight change of notation, and with no explicit indication that Q and W are not proper variables.

(d) (4) For an adiabatic process, show that

$$C_v dT = -PdV.$$

adiabatic means $dQ = 0$, so $dE = dQ - dW = -dW$
 or $C_v dT = -PdV$

(e) (4) Using the above equation, and the ideal gas equation of state, show that (for an adiabatic process)

$$\frac{C_v}{C_p - C_v} (PdV + VdP) = -PdV.$$

$$PV = NkT \Rightarrow dT = \frac{d(PV)}{Nk} \Rightarrow C_v dT = \frac{C_v}{Nk} d(PV)$$

from (c) $\Rightarrow \frac{C_v}{C_p - C_v} (PdV + VdP) = -PdV$ from (d)

(f) (4) Now use the above result to obtain a simple relation between $\frac{dP}{dV}$ and $\frac{P}{V}$.

$$\frac{C_v}{C_p - C_v} = \frac{1}{\gamma - 1} \Rightarrow PdV + VdP = -(\gamma - 1)PdV$$

$$\Rightarrow VdP = -\gamma PdV$$

$$\Rightarrow \frac{dP}{dV} = -\gamma \frac{P}{V}$$

(g) (4) If F is the upward force on the cylinder due to the gas, and $X = X_0 + x$ is the length of the cylindrical volume of gas (as shown on the first page), show that for small amplitude oscillations

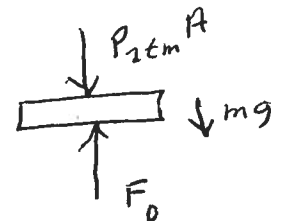
$$\frac{dF}{dX} \approx -\gamma \frac{F_0}{X_0}, \text{ where } F_0 = mg + P_{atm} A$$

and X_0 is the value of X in equilibrium.

$$F = PA \text{ and } V = XA, \text{ so from above}$$

$$\frac{dF}{dX} = -\gamma \frac{F}{X} \approx -\gamma \frac{F_0}{X_0}$$

where in equilibrium $F_0 = mg + P_{atm} A$



(h) (4) Now show that the adiabatic vibrations of the cylinder are described by

$$\frac{d^2x}{dt^2} = -Kx$$

while also obtaining K in terms of m , F_0 , and the other constants in Part (g).

[Hint: first note that the net force ΔF on the cylinder is given by $\Delta F/x \approx dF/dX$.]

$$\frac{\Delta F}{x} \approx \frac{dF}{dX} \approx -\gamma \frac{F_0}{X_0} \quad \text{and} \quad \Delta F = m \frac{d^2x}{dt^2}$$

$$\therefore m \frac{d^2x}{dt^2} = -\gamma \frac{F_0}{X_0} x \Rightarrow \boxed{\frac{d^2x}{dt^2} = -Kx}, \quad \boxed{K \equiv \frac{\gamma F_0}{m X_0}}$$

(i) (4) Finally, obtain $\gamma \equiv C_p/C_v$ in terms of the angular frequency $\omega = 2\pi/T$ with which the cylinder vibrates. [Here T is the period of oscillation.]

$$x = x_0 \cos(\omega t + \phi) \quad \Rightarrow \quad \frac{d^2x}{dt^2} = -\omega^2 x$$

↑ phase

$$\Rightarrow \omega^2 = K = \frac{\gamma F_0}{m X_0}$$

$$\Rightarrow \boxed{\gamma = \frac{m X_0}{F_0} \omega^2}$$

(j) (4) How would the equation of Part (f) above, and the final result for the vibrational angular frequency ω , change if the gas were held at constant temperature T , so that the vibrations were isothermal rather than adiabatic?

$$PV = NkT = \text{constant}$$

$$\Rightarrow P dV + V dP = 0$$

$$\Rightarrow \boxed{\frac{dP}{dV} = -\frac{P}{V}}$$

and $\boxed{\omega^2 = \frac{F_0}{m X_0}}$ [with γ replaced by 1]

2. For a (classical, monatomic, nonrelativistic) ideal gas we had $Z = \frac{1}{N!} \left(\frac{V}{\lambda_{th}^3} \right)^N$.

(a) (5) By replacing a sum by an integral, obtain Stirling's approximation $\ln N! \approx N \ln N - N$.

$$\boxed{\ln N!} = \ln \prod_{n=1}^N n = \sum_{n=1}^N \ln n \underbrace{\Delta n}_1 \approx \int_1^N \ln n \, dn = \left[n \ln n - n \right]_1^N \approx \boxed{N \ln N - N}$$

(b) (5) Then obtain a simple expression for $\ln Z$ in terms of N , $\lambda_{th} = \frac{h}{(2\pi mkT)^{1/2}}$, and

$$v = \frac{V}{N}$$

$$\begin{aligned} \boxed{\ln Z} &\approx N \ln \left(\frac{V}{\lambda_{th}^3} \right) - (N \ln N - N) \\ &= N \ln \left(\frac{V/N}{\lambda_{th}^3} \right) + N \\ &= \boxed{N \ln \left(\frac{v}{\lambda_{th}^3} \right) + N} \end{aligned}$$

(c) (5) Using the fact that $\langle E \rangle = \sum_j p_j E_j$, where $p_j = \frac{e^{-E_j/kT}}{Z}$, obtain $\langle E \rangle$ in terms of $\frac{\partial \ln Z}{\partial T}$.

$$\frac{\partial \ln Z}{\partial T} = \frac{1}{Z} \frac{\partial}{\partial T} \sum_j e^{-E_j/kT} = \frac{1}{Z} \sum_j e^{-E_j/kT} \left(-\frac{E_j}{k}\right) \left(-\frac{1}{T^2}\right)$$

$$= \frac{1}{kT^2} \sum_j p_j E_j$$

$$= \langle E \rangle$$

$$\Rightarrow \boxed{\langle E \rangle = kT^2 \frac{\partial \ln Z}{\partial T}}$$

(d) (5) Use your result above to calculate $\langle E \rangle$ for this ideal gas.

$$\boxed{\langle E \rangle} = kT^2 \frac{\partial}{\partial T} \left[N \ln T^{3/2} + \text{function of } N \text{ and } V \right]$$

$$= kT^2 \cdot N \cdot \frac{3}{2} \frac{1}{T}$$

$$= \boxed{\frac{3}{2} NkT}$$

3. Recall that the Carnot cycle for a Carnot engine consists of two isothermal processes, at temperatures T_{hot} and T_{cold} (with heat Q_{hot} received from one reservoir, and heat Q_{cold} given to the other reservoir) and two adiabatic processes.

(a) (5) For any heat engine doing work W , what is the definition of the efficiency?

$$e \equiv \frac{W}{Q_{hot}}$$

(b) (5) For a Carnot engine, relate Q_{cold}/Q_{hot} to T_{cold}/T_{hot} . [Hint: Consider ΔS for one cycle.]

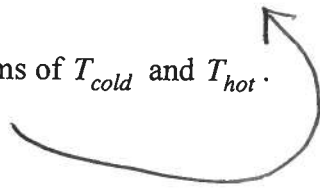
$$\Delta S = 0 \quad \text{for one full cycle} \quad [\text{for any engine}]$$

$$\Delta S = + \frac{Q_{hot}}{T_{hot}} - \frac{Q_{cold}}{T_{cold}} \quad \text{for Carnot cycle}$$

$$\therefore \frac{Q_{cold}}{Q_{hot}} = \frac{T_{cold}}{T_{hot}} \quad \text{for Carnot}$$

Then $e = \frac{W}{Q_{hot}} = \frac{Q_{hot} - Q_{cold}}{Q_{hot}} = 1 - \frac{Q_{cold}}{Q_{hot}} = 1 - \frac{T_{cold}}{T_{hot}}$

(c) (5) Obtain the efficiency for a Carnot engine in terms of T_{cold} and T_{hot} .



(d) (5) If the operating temperatures are 900 K and 300 K, what is the Carnot efficiency?

$$e = 1 - \frac{300 \text{ K}}{900 \text{ K}} = 1 - \frac{1}{3} \approx 67\%$$

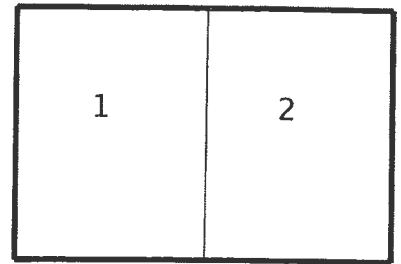
4. (20) Two systems, 1 and 2, are separated by a partition that permits energy and particles to flow through, and which is also movable. Then we have

$$E = E_1 + E_2, \quad V = V_1 + V_2, \quad N = N_1 + N_2, \quad S = S_1 + S_2$$

where the total energy E , total volume V , and total number of particles N are constant, but E_1 , V_1 , and N_1 can vary. Recall that

$$dS_i = \frac{1}{T_i} dE_i - \frac{P_i}{T_i} dV_i + \frac{\mu_i}{T_i} dN_i \quad (i=1,2)$$

where T is the temperature, P the pressure, and μ the chemical potential.



$$\begin{aligned} E_2 &= E - E_1 \\ V_2 &= V - V_1 \\ N_2 &= N - N_1 \end{aligned}$$

By maximizing the total entropy, determine the 3 conditions for the two systems to be in thermal, mechanical, and chemical equilibrium.

$$0 = \frac{\partial S}{\partial E_1} = \underbrace{\frac{\partial S_1}{\partial E_1}}_{\frac{1}{T_1}} + \frac{\partial S_2}{\partial E_2} \left(\frac{\partial E_2}{\partial E_1} \right) = \frac{1}{T_1} - \frac{1}{T_2} \Rightarrow \boxed{T_2 = T_1}$$

$$0 = \frac{\partial S}{\partial V_1} = \underbrace{\frac{\partial S_1}{\partial V_1}}_{-\frac{P_1}{T_1}} + \frac{\partial S_2}{\partial V_2} \left(\frac{\partial V_2}{\partial V_1} \right) = -\frac{P_1}{T_1} + \frac{P_2}{T_2} \Rightarrow \boxed{P_2 = P_1}$$

$$0 = \frac{\partial S}{\partial N_1} = \underbrace{\frac{\partial S_1}{\partial N_1}}_{\mu_1} + \frac{\partial S_2}{\partial N_2} \left(\frac{\partial N_2}{\partial N_1} \right) = \frac{\mu_1}{T_1} - \frac{\mu_2}{T_2} \Rightarrow \boxed{\mu_2 = \mu_1}$$