Physics 408 -- Exam 2 **Name** Solution

You are graded on your work, with partial credit where it is deserved.

Please give clear, well-organized solutions.

$h = 6.63\times10^{-34}$ J s [Planck's constant] $k = 1.38\times10^{-23}$ J/K [Boltzmann constant]

$\sigma_B = 5.67\times10^{-8}\ \frac{\text{W}}{\text{m}^2\cdot K^4}$ [Stefan-Boltzmann constant] $c = 3.00\times10^{8}$ m/s [speed of light]

$$\langle n_k\rangle = \frac{1}{e^{(\varepsilon_k-\mu)/kT}\pm1}$$

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1. The distance between the Earth and the Sun is $R = 1.50\times10^{11}$ m, and the radius of the Sun is $r = 6.96\times10^{8}$ m. The radiant energy flux (or intensity) of sunlight at the position of the Earth is $S = \text{solar constant} = 1.36\times10^{3}\ \frac{\text{J/s}}{\text{m}^2}$.

(a) (9) Calculate the total rate of energy generation of the Sun in J/s.

$$P = \left(1.36\times10^{3}\ \frac{\text{J/s}}{\text{m}^2}\right)(4\pi)(1.50\times10^{11}\ \text{m})^2$$

$$= \boxed{3.8\times10^{26}\ \frac{\text{J}}{\text{s}}}$$

(b) (9) Recalling the Stefan-Boltzmann law, energy flux from perfect blackbody $= \sigma_B T^4$, calculate the temperature at the surface of the Sun.

$$\text{flux} = \frac{3.8\times10^{26}\ \frac{\text{J}}{\text{s}}}{(4\pi)(6.96\times10^{8}\ \text{m})^2} = 6.24\times10^{7}\ \frac{\text{J/s}}{\text{m}^2}$$

$$T = \left(\frac{6.24\times10^{7}\ \frac{\text{J/s}}{\text{m}^2}}{5.67\times10^{-8}\ \frac{\text{W}}{\text{m}^2\cdot\text{K}^4}}\right)^{1/4} = \boxed{5800\ \text{K}}$$

(c) (9) The solar spectrum peaks at about 518 nm, or 518×10^{-9} m. According to the Wien displacement law, with the Planck distribution written in terms of the wavelength λ,

$$\lambda_{\text{peak}} = \frac{hc}{4.97\,kT}.$$

Calculate the surface temperature of the Sun from this equation. How does it compare with your answer to part (b)?

$$T = \frac{(6.63 \times 10^{-34}\ \text{J}\cdot\text{s})(3.00 \times 10^{8}\ \text{m/s})}{(4.97)(1.38 \times 10^{-23}\ \text{J/K})(518 \times 10^{-9}\ \text{m})}$$

$$= 5600\ \text{K}$$

which is reasonably consistent

(d) (3) If the Sun is yellow, how can we treat it as a perfect blackbody?

black under reflection

but yellow under thermal emission

2. For a quantum ideal gas of ultrarelativistic electrons, the energy ε is related to the momentum p by $\varepsilon = cp$, so the density of states has the form

$$D(\varepsilon) = A\varepsilon^2$$

where A is a constant.

(a) (10) Obtain A in terms of the number of electrons N and the Fermi energy ε_F.

$$N = \int_0^{\varepsilon_F} A\,\varepsilon^2\, d\varepsilon = A\,\frac{\varepsilon_F^3}{3} \Rightarrow \boxed{A = \frac{3N}{\varepsilon_F^3}}$$

(b) (10) Then show that the average energy per electron, $\frac{E}{N}$, is given by

$$\frac{E}{N} = \text{constant} \times \varepsilon_F$$

while at the same time obtaining the constant. (Here E is the usual total energy of the N electrons, which we sometimes write as $\langle E \rangle$.)

$$E = \int_0^{\varepsilon_F} \varepsilon \cdot A\varepsilon^2\, d\varepsilon = A\,\frac{\varepsilon_F^4}{4} = \frac{3N}{\varepsilon_F^3} \cdot \frac{\varepsilon_F^4}{4}$$

$$\Rightarrow \boxed{\frac{E}{N} = \frac{3}{4}\varepsilon_F}$$

3. In this problem we consider adsorbed atoms on a surface, which can be treated as a 2-dimensional classical ideal gas (but with a binding energy ε_0), in thermal equilibrium with the vapor above the surface, which can be treated as a 3-dimensional classical ideal gas. You may start with the Helmholtz free energies

$$F_{ads} = -N_{ads}kT\ln\left(\frac{z_{ads}}{N_{ads}}\right) - N_{ads}kT \quad , \quad z_{ads} = \frac{A}{\lambda_{th}^2}e^{\varepsilon_0/kT}$$

$$F_{gas} = -N_{gas}kT\ln\left(\frac{z_{gas}}{N_{gas}}\right) - N_{gas}kT \quad , \quad z_{gas} = \frac{V}{\lambda_{th}^3} \quad , \quad \lambda_{th} \equiv \frac{h}{(2\pi mkT)^{1/2}}$$

in an obvious notation.

(a) (5) Given that the Helmholtz free energy is defined as $F = \langle E\rangle - TS$, obtain the expression for dF in terms of dT, dV, and dN.

$dF = d\langle E\rangle - d(TS)$

$= (TdS - PdV + \mu dN) - (TdS + SdT)$

$= -SdT - PdV + \mu dN$

(b) (5) Using the relation between dF and dN, obtain the expression for the chemical potential μ as a partial derivative of F.

Clearly $\mu = \left(\frac{\partial F}{\partial N}\right)_{T,V}$ with $V \to A$ in 2 dim.

(c) (5) Using the expression for F_{ads} near the top of the page, obtain μ_{ads} as a function of T, z_{ads}, and N_{ads}.

$\mu_{ads} = \frac{\partial F_{ads}}{\partial N_{ads}}$

$= -kT\ln\left(\frac{Z_{ads}}{N_{ads}}\right) - N_{ads}kT\cdot(-1)\frac{1}{N_{ads}} - kT$ ($\frac{\partial \ln N_{ads}}{\partial N_{ads}}$)

$= -kT\ln\left(\frac{Z_{ads}}{N_{ads}}\right)$

(d) (5) Using the expression for F_{gas} near the top of the page,, obtain μ_{gas} as a function of T, z_{gas}, and N_{gas}.

As in Part (c),

$$\boxed{\mu_{gas} = -kT \ln\left(\frac{z_{gas}}{N_{gas}}\right)}$$

(e) (5) Obtain the relation between $\frac{N_{gas}}{V}$ and $\frac{N_{ads}}{A}\frac{e^{-\varepsilon_0/kT}}{\lambda_{th}}$.

equilibrium $\Rightarrow \mu_{ads} = \mu_{gas}$

$$\Rightarrow \frac{z_{ads}}{N_{ads}} = \frac{z_{gas}}{N_{gas}}$$

$$\Rightarrow \frac{A}{N_{ads}\lambda_{th}^2} e^{\varepsilon_0/kT} = \frac{V}{N_{gas}\lambda_{th}^3}$$

$$\Rightarrow \boxed{\frac{N_{gas}}{V} = \frac{N_{ads}}{A}\frac{e^{-\varepsilon_0/kT}}{\lambda_{th}}}$$

(f) (5) Show that that the coverage of the surface by the adsorbed molecules, N_{ads}/A, is proportional to the pressure. I.e., show that

$$P_{gas} = \frac{N_{ads}}{A}\frac{kT}{\lambda_{th}}e^{-\varepsilon_0/kT} .$$

$P_{gas} V = N_{gas} kT$

$$\Rightarrow \frac{N_{gas}}{V} = \frac{P_{gas}}{kT}$$

$$\Rightarrow \boxed{P_{gas} = \frac{N_{ads}}{A}\frac{kT}{\lambda_{th}}e^{-\varepsilon_0/kT}}$$

4. In condensed matter physics there are various kinds of waves which can be treated in the same way as vibrational waves, except that the dispersion relation for the particles (bosons) associated with these waves is more generally

$$\varepsilon(p) = ap^n \quad .$$

Here a and n are constants, ε is the energy, and p is the magnitude of the momentum. Just as for phonons, we have $\mu = 0$ (where μ is the chemical potential in the Bose-Einstein distribution function given on the first page). Also, there is a maximum single-particle energy $\varepsilon_{\max}$ (corresponding to a maximum momentum $p_{\max}$).

Let us consider such waves and quanta (or particles) in N dimensions, where the volume in momentum space in dp at p is $Ap^{N-1}dp$. Also, the volume in momentum space per state is $\frac{h^N}{V}$, where V is the N-dimensional volume occupied by these bosons. (We assume spinless particles.)

(a) (5) Calculate the density of states in the form

$$D(\varepsilon) = B\varepsilon^{\alpha}$$

while at the same time obtaining the constants B and α in terms of the original constants given above.

$$D(\varepsilon)\,d\varepsilon = \frac{A p^{N-1} dp}{h^N/V} = \frac{V}{h^N} A p^{N-1} dp$$

$$\varepsilon = a p^n \Rightarrow p = \left(\frac{\varepsilon}{a}\right)^{1/n} \text{ and } dp = \frac{1}{a^{1/n}} \cdot \frac{1}{n} \varepsilon^{\frac{1}{n}-1} d\varepsilon$$

$$\therefore D(\varepsilon)\,d\varepsilon = \frac{V}{h^N} A \left(\frac{\varepsilon}{a}\right)^{\frac{N-1}{n}} \cdot \frac{1}{a^{1/n}} \frac{1}{n} \varepsilon^{\frac{1}{n}-1} d\varepsilon$$

$$= \frac{V}{h^N} \frac{A}{n} \frac{1}{a^{N/n}} \varepsilon^{\frac{N}{n}-1} d\varepsilon$$

$$\Rightarrow D(\varepsilon) = B\varepsilon^{\alpha} \quad \text{with} \quad \boxed{B = \frac{V}{h^N} \frac{A}{n} \frac{1}{a^{N/n}}} \quad \& \quad \boxed{\alpha = \frac{N}{n} - 1}$$

(b) (5) Write down the expression for the energy $\langle E \rangle$ associated with these particles at temperature T, in terms of your density of states $D(\varepsilon)$ and the Bose-Einstein distribution function $\langle n_k \rangle$ given on the first page (with $\mu = 0$).

$$\langle E \rangle = \int_0^{\varepsilon_{max}} \varepsilon \langle n(\varepsilon) \rangle D(\varepsilon)\, d\varepsilon$$

$$\left[\text{with } \langle n(\varepsilon) \rangle = \frac{1}{e^{\varepsilon/kT} - 1} \right]$$

(c) (5) Use the result of part (b) to obtain the heat capacity at constant volume in the limit of **high temperatures** (T very large), in the form

$$C_V \propto T^X$$

while at the same time obtaining the constant X. **But you need not obtain the proportionality constant.**

$$\langle E \rangle = \int_0^{\varepsilon_{max}} \varepsilon \underbrace{\frac{1}{e^{\varepsilon/kT}-1}}_{= \left(1 + \frac{\varepsilon}{kT} + \ldots\right) - 1 \approx \frac{\varepsilon}{kT}} D(\varepsilon)\, d\varepsilon$$

$$= kT \underbrace{\int_0^{\varepsilon_{max}} D(\varepsilon)\, d\varepsilon}_{= N}$$

$$\Rightarrow C_V = \frac{\partial \langle E \rangle}{\partial T} = Nk \qquad [\text{Check: agrees with result for phonons}]$$

or $\boxed{X = 0}$ above [i.e., C_V is constant at high T]

(d) (5) Use the result of Part (b) again, this time to obtain the heat capacity in the limit of **low temperatures** (T very small), in the form

$$C_V \propto T^Y$$

while at the same time obtaining the constant Y. **But you again need not obtain the proportionality constant.**

Now $$\langle E \rangle = \int_0^{\infty} \varepsilon \frac{1}{e^{\varepsilon/kT}-1} \cdot B\varepsilon^{\alpha}\, d\varepsilon$$

$$= B(kT)^{\alpha+2} \int_0^{x_{max}} \frac{x^{\alpha+1}}{e^x - 1}\, dx \quad , \quad x \equiv \frac{\varepsilon}{kT}$$

$$\approx B(kT)^{\alpha+2} \underbrace{\int_0^{\infty} \frac{x^{\alpha+1}}{e^x - 1}\, dx}_{= \text{constant}'} \qquad \text{since } x_{max} \to \infty \text{ as } T \to 0$$

$$\propto T^{\alpha+2}$$

$$C_V = \frac{\partial E}{\partial T} \propto T^{\alpha+2-1} = T^{\alpha+1}$$

So above $\boxed{Y = \alpha - 1} = \left(\frac{N}{n} - 1\right) + 1 = \boxed{\frac{N}{n}}$

[Check: if $N = 3$ and $n = 1$, $Y = 3$ and we regain the Debye T^3 law.]