Physics 408 -- Exam 3 Name Solution

You are graded on your work, with partial credit where it is deserved.

Please give clear, well-organized, understandable solutions.

$h = 6.63\times10^{-34}$ J s	[Planck's constant]	$k = 1.38\times10^{-23}$ J/K	[Boltzmann constant]
$c = 3.00\times10^{8}$ m/s	[speed of light]	$G = 6.67\times10^{-11}\,\dfrac{\text{N m}^2}{\text{kg}^2}$	[gravitational constant]
$m_n = 1.67\times10^{-27}$ kg	[mass of neutron]	$M_\odot = 1.99\times10^{30}\,kg$	[solar mass, i.e. mass of Sun]

The variables have their usual meanings: $E =$ energy, $S =$ entropy, $V =$ volume, $N =$ number of particles, $T =$ temperature, $P =$ pressure, $\mu =$ chemical potential, $B =$ applied magnetic field, $C_V =$ heat capacity at constant volume, $F =$ Helmholtz free energy, $k =$ Boltzmann constant. Also, $\langle\cdots\rangle$ represents an average.

1. The Gibbs free energy G is defined by $G = \langle E\rangle - TS + PV$.

(a) (5) Using the standard expression for $d\langle E\rangle$, obtain dG in terms of dT, dP, and dN.

$$\boxed{dG =} d\langle E\rangle - d(TS) + d(PV)$$

$$= \left(\cancel{TdS} - \cancel{PdV} + \mu dN\right) - \left(\cancel{TdS} + SdT\right) + \left(\cancel{PdV} + VdP\right)$$

$$= \boxed{-SdT + VdP + \mu dN}$$

(b) (5) Then obtain S, V, and μ as partial derivatives of G.

From chain rule for partial derivatives,

$$-S = \left(\frac{\partial G}{\partial T}\right)_{P,N}, \quad V = \left(\frac{\partial G}{\partial P}\right)_{T,N}, \quad \mu = \left(\frac{\partial G}{\partial N}\right)_{T,P}$$

Now specialize to a system, at fixed temperature and pressure, for which

$$G(T,P,N) = -NkT\ln\bar{z} \quad , \quad \bar{z} = \frac{kT}{P\lambda_{th}^3} \quad , \quad \lambda_{th} = \frac{h}{(2\pi mkT)^{1/2}} .$$

The following questions are for this system. You may wish to write $\bar{z} = \dfrac{aT^{5/2}}{P}$.

(c) (5) Calculate the entropy S in terms of terms of the above quantities.

$$G = -NkT\ln\left(\frac{aT^{5/2}}{P}\right)$$

$$\Rightarrow \boxed{S =} -\frac{\partial G}{\partial T}$$

$$= +Nk\ln\left(\frac{aT^{5/2}}{P}\right) + NkT\frac{\partial}{\partial T}\left[\ln T^{5/2} + \ln\left(\frac{a}{P}\right)\right]$$

$$= Nk\ln\left(\frac{aT^{5/2}}{P}\right) + NkT\cdot\frac{5}{2}\frac{1}{T}$$

$$= \boxed{Nk\ln\left(\frac{aT^{5/2}}{P}\right) + \frac{5}{2}Nk}$$

(d) (5) Calculate C_P, the heat capacity at constant pressure.

$$\boxed{C_p =} T\left(\frac{\partial S}{\partial T}\right)_p \qquad [\,đq = T\,dS\,]$$

$$= TNk\frac{\partial}{\partial T}\left[\ln T^{5/2} + \ln\frac{a}{p}\right]$$

$$= NkT \cdot \frac{5}{2}\frac{1}{T}$$

$$= \boxed{\frac{5}{2}Nk}$$

(e) (5) Calculate the connection between V, P, N, and T (i.e., the equation of state).

$$V = \frac{\partial G}{\partial p} = -NkT\frac{\partial}{\partial p}\left[\ln\left(\frac{1}{p}\right) + \ln\left(aT^{5/2}\right)\right]$$

$$= +NkT \cdot \frac{1}{p}$$

$$\Rightarrow \boxed{pV = NkT}$$

(f) (5) Calculate the internal energy $\langle E\rangle$ in terms of k, T, and N.

$$G \equiv \langle E\rangle - TS + pV$$

given

Part (c)

$$\Rightarrow \boxed{\langle E\rangle =} G + TS - pV$$

$$= -NkT\ln\left(\frac{aT^{5/2}}{p}\right) + T\left[Nk\ln\left(\frac{aT^{5/2}}{p}\right) + \frac{5}{2}Nk\right] - NkT$$

Part (e)

$$= \boxed{\frac{3}{2}NkT}$$

(g) (5) Calculate the chemical potential μ in terms of the thermal energy kT and the ratio

$$\frac{\lambda_{th}^3}{v}, \quad v \equiv \frac{V}{N}.$$

from given expression

$$\boxed{\mu =} \frac{\partial G}{\partial N} = -kT\ln\left(\frac{kT}{p\lambda_{th}^3}\right) \qquad \text{with } \frac{p}{kT} = \frac{N}{V}$$

Part (e)

$$= -kT\ln\left(\frac{V/N}{\lambda_{th}^3}\right)$$

$$= \boxed{kT\ln\left(\frac{\lambda_{th}^3}{v}\right)}$$

2. As argued in the textbook, for bosons (in an ideal-gas picture) at low temperatures we have

$$\langle n_1 \rangle + C\frac{\pi^{1/2}}{2}\times 2.612(kT)^{3/2} = N$$

where $\langle n_1 \rangle$ is the number of bosons in the lowest-energy state and C is the constant in the density of states function: $D(\varepsilon) = C\varepsilon^{1/2}$.

(a) (8) Explain why $\langle n_1 \rangle$ must become nonzero (or more precisely macroscopic) at some nonzero transition temperature T_B as $T \to 0$, and obtain T_B in terms of the constants in the equation above.

When constant $\times$ $T^{3/2}$ shrinks below N,

$\langle n_1 \rangle$ must make up the difference,

$$C\frac{\pi^{1/2}}{2} \times 2.612 (k T_B)^{3/2} = N$$

$$\Rightarrow \boxed{T_B = \frac{1}{k}\left(\frac{2N}{C\pi^{1/2} \times 2.612}\right)^{2/3}}$$

[after raising both sides to $\frac{2}{3}$ power]

(b) (8) Obtain $\frac{\langle n_1 \rangle}{N}$ as a function of $\frac{T}{T_B}$.

$$\langle n_1 \rangle + A\,T^{3/2} = N$$

with $A\,T_B^{3/2} = N$

[A = constant]

$$\Rightarrow \langle n_1 \rangle + \frac{N}{T_B^{3/2}}\,T^{3/2} = N$$

$$\Rightarrow \frac{\langle n_1 \rangle}{N} + \left(\frac{T}{T_B}\right)^{3/2} = 1$$

$$\Rightarrow \boxed{\frac{\langle n_1 \rangle}{N} = 1 - \left(\frac{T}{T_B}\right)^{3/2}}$$

(c) (4) Give an example of a system where this basic phenomenon of Bose-Einstein condensation is observed.

^4He and various atomic systems

3. In this problem you are given that $\mu = \varepsilon_F - \dfrac{\pi^2}{6}\dfrac{D'(\varepsilon_F)}{D(\varepsilon_F)}(kT)^2$ for fermions, and that $\dfrac{kT}{\varepsilon_F} \ll 1$. Here $D'(\varepsilon) \equiv \dfrac{dD(\varepsilon)}{d\varepsilon}$. Consider a system of noninteracting fermions for which the density of states is linear in the energy: $D(\varepsilon) = \dfrac{\varepsilon}{\varepsilon_0^2}$ where ε_0 is a constant. Give the answers below in terms of ε_0, k, T, and the number of fermions N.

(a) (5) Calculate the Fermi energy ε_F.

$$N = \int_0^{\varepsilon_F} D(\varepsilon)\,d\varepsilon = \frac{1}{\varepsilon_0^2}\int_0^{\varepsilon_F} \varepsilon\,d\varepsilon = \frac{\varepsilon_F^2}{2\varepsilon_0^2}$$

(b) (5) Calculate the chemical potential μ as a function of T.

$$D'(\varepsilon) = \frac{1}{\varepsilon_0^2}$$

$$\Rightarrow \boxed{\mu =} \varepsilon_F - \frac{\pi^2}{6}\frac{1/\varepsilon_0^2}{\varepsilon_F/\varepsilon_0^2}(kT)^2 = \boxed{\varepsilon_F - \frac{\pi^2}{6}\frac{(kT)^2}{\varepsilon_F}}$$

(c) (5) Calculate the total energy E as a function of T.

$$\boxed{E_0 =} \int_0^{\varepsilon_F} \varepsilon\, D(\varepsilon)\,d\varepsilon = \frac{1}{\varepsilon_0^2}\int_0^{\varepsilon_F}\varepsilon^2\,d\varepsilon = \boxed{\frac{1}{3}\frac{\varepsilon_F^3}{\varepsilon_0^2}}$$

$$\Rightarrow \boxed{E(T) = E_0 + \frac{\pi^2}{6}\frac{\varepsilon_F}{\varepsilon_0^2}(kT)^2}$$

(d) (5) Calculate the heat capacity C_V as a function of T.

$$\boxed{C_V} = \left(\frac{\partial E}{\partial T}\right)_V$$

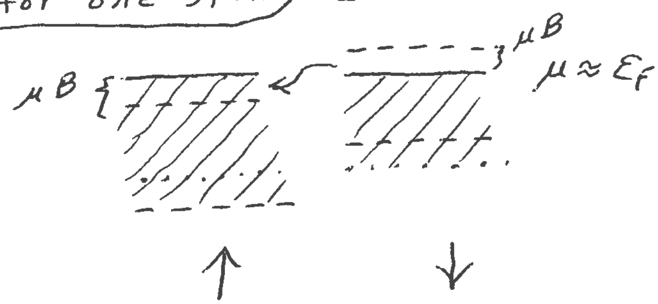
$$= \boxed{\frac{\pi^2}{3}\,\frac{\varepsilon_F}{\varepsilon_0^2}\,k^2\,T}$$

(e) (5) Finally, the Pauli paramagnetism at $T = 0$: Through a simple argument, obtain the magnetic dipole moment M resulting from an applied magnetic field B with $\mu B \ll \varepsilon_F$, where μ is (the z-component of) the magnetic dipole moment of a single fermion. Give M in terms of ε_0 etc.

[spin $\frac{1}{2}$ assumed]

density of states for one spin

number shifted $\approx \frac{1}{2} D(\varepsilon_F)\,\Delta E$, $\Delta E = \mu B$



energy lowered by μB

energy raised by μB

[same μ]

for each shifted, magnetic dipole moment of system changes by 2μ

$$\therefore\ \boxed{M =} \cancel{\tfrac{1}{2}} D(\varepsilon_F)\cdot \mu B \cdot \cancel{2}\mu$$

$$= D(\varepsilon_F)\,\mu^2 B$$

$$= \boxed{\frac{\varepsilon_F}{\varepsilon_0^2}\,\mu^2 B}$$

4. In this problem we treat a neutron star with mass M and radius R. The center feels a gravitational pressure which we estimated in class to be

$$P_{center} = \frac{3}{8\pi}\frac{GM^2}{R^4}.$$

Here we will take the neutrons to be ultrarelativistic, with energy $\varepsilon = cp$, where p is the magnitude of the momentum.

(a) (5) Using the fact that the number of states in a momentum-space shell of thickness dp at p is $2\times\frac{4\pi p^2 dp}{h^3/V}$ (since a neutron has 2 spin states), calculate the neutron number density N/V in terms of the Fermi momentum $p_F = \varepsilon_F / c$. Then write p_F in terms of N/V and Planck's constant h.

$$N = \int_0^{p_F} D(p)\,dp = \frac{V}{h^3}\cdot 8\pi \int_0^{p_F} p^2\,dp = \frac{V}{h^3}\cdot\frac{8\pi}{3}p_F^3 \Rightarrow \boxed{\frac{N}{V} = \frac{8\pi}{3h^3}p_F^3}$$

Then $\boxed{p_F = \left(\frac{3h^3}{8\pi}\frac{N}{V}\right)^{1/3}}$

(b) (5) We showed that very generally $P = \frac{1}{3}\langle\vec{p}\cdot\vec{v}\rangle\frac{N}{V}$. Now let us make the approximation $P \approx \frac{1}{3}p_F c\frac{N}{V}$.

Using the fact that $N = M/m_n$, where m_n is the mass of a neutron, and that $V = \frac{4}{3}\pi R^2$, write P in terms of M, R, h, c, and m_n.

$$P = \frac{1}{3}p_F c\frac{N}{V} = \frac{1}{3}\left(\frac{3h^3}{8\pi}\right)^{1/3} c \left(\frac{N}{V}\right)^{4/3}$$

$$\frac{N}{V} = \frac{M/m_n}{\frac{4}{3}\pi R^3} = \frac{3}{4\pi m_n}\frac{M}{R^3}$$

$$\therefore \boxed{P =} \frac{1}{3}\left(\frac{3}{8\pi}\right)^{1/3} hc \left(\frac{3}{4\pi m_n}\right)^{4/3}\left(\frac{M}{R^3}\right)^{4/3}$$

$$= \boxed{\frac{1}{3}\left(\frac{3}{8\pi}\right)^{5/3}\left(\frac{2M}{m_n}\right)^{4/3}\frac{hc}{R^4}}$$

(c) (5) Equate P to P_{center} to obtain the condition for equilibrium at the center of the neutron star. Can you solve for an equilibrium value for R? How do you interpret this physically?

$$\boxed{\frac{1}{3}\left(\frac{3}{8\pi}\right)^{5/3}\left(\frac{2M}{m_n}\right)^{4/3}\frac{\hbar c}{R^4} = \frac{3}{8\pi}\frac{GM^2}{R^4}}$$

No, R factors cancel.

Becomes more difficult to achieve equilibrium in relativistic regime.

(d) (5) Calculate the value of the mass that corresponds to your equation for equilibrium in Part (c). First get the number in kg, and then in solar masses $M_\odot$. How does this compare with the Chandrasekhar limiting mass for a white dwarf?

$$\frac{1}{3}\left(\frac{3}{8\pi}\right)^{2/3}\left(\frac{2}{m_n}\right)^{4/3}\hbar c = G M^{2/3}$$

$$\Rightarrow \frac{1}{3^{3/2}}\left(\frac{3}{8\pi}\right)\left(\frac{2}{m_n}\right)^2\left(\frac{\hbar c}{G}\right)^{3/2} = M$$

$$\Rightarrow \boxed{M} = \frac{1}{3^{3/2}}\left(\frac{3}{8\pi}\right)\left(\frac{2}{1.67\times10^{-27}\ kg}\right)^2 \times \left(\frac{(6.63\times10^{-34}\ J\,s)(3.00\times10^{8}\ m/s)}{6.67\times10^{-11}\ \frac{N\,m^2}{kg^2}}\right)^{3/2}$$

$$\boxed{= 5.3\times10^{30}\ kg}$$

$$\boxed{= 2.7\ M_\odot}$$

[fortuitously good rough estimate]

Well above $M_{Chandrasekhar} = 1.44\ M_\odot$.

If we were doing a white dwarf, we would have $N \approx \frac{M}{2m_n}$ (since there are 2 nucleons per electron) above, instead of $N = \frac{M}{m_n}$. Then $m_n \to 2m_n$ and we estimate for white dwarf

$$M = \frac{1}{2^2}(2.7\ M_\odot) = 0.68\ M_\odot.$$