

You are graded on your work, with partial credit where it is deserved.

Please give clear, well-organized solutions.

$$G = 6.67 \times 10^{-11} \frac{\text{N m}^2}{\text{kg}^2} \quad [\text{gravitational constant}] \quad M_{\odot} = 1.99 \times 10^{30} \text{ kg} \quad [\text{solar mass, i.e. mass of Sun}]$$

$$h = 6.63 \times 10^{-34} \text{ J s} \quad [\text{Planck's constant}] \quad k = 1.38 \times 10^{-23} \text{ J/K} \quad [\text{Boltzmann constant}]$$

$$\sigma_B = 5.67 \times 10^{-8} \frac{\text{W}}{\text{m}^2 \cdot \text{K}^4} \quad [\text{Stefan-Boltzmann constant}] \quad R_{\oplus} = 6378 \text{ km} \quad [\text{radius of Earth}]$$

$$m_e = 9.11 \times 10^{-31} \text{ kg} \quad [\text{mass of electron}] \quad m_n = 1.67 \times 10^{-27} \text{ kg} \quad [\text{mass of neutron}]$$

1. (a) (4) Using a kinetic argument (with particles rebounding off an enclosing wall) show that, for any ideal gas,

$$P = \frac{1}{3} \langle \vec{p} \cdot \vec{v} \rangle \frac{N}{V} .$$

Here P is the pressure, N is the number of particles, V is the volume, and \vec{p} and \vec{v} are a particle's momentum and velocity. (Recall that \vec{p} gives the force due to a single particle, and \vec{v} gives the number of particles arriving in a time Δt .)

(b) (4) For a nonrelativistic ideal gas, use the result of part (a) to show that $P = a \frac{\langle E \rangle}{V}$, while at the same time determining the constant a . Here $\langle E \rangle$ is the energy of the system.

(c) (4) For an ultrarelativistic ideal gas, use the result of part (a) to show that $P = b \frac{\langle E \rangle}{V}$, while at the same time determining the constant b .

(d) (4) As usual, let us assume periodic boundary conditions, so that $L = n_x \lambda_x$, etc., where λ_x is a de Broglie wavelength. Show from this that the volume in momentum space for a single allowed momentum \vec{p} is h^3 / V , where $V = L^3$.

(e) (4) Consider a sphere in momentum space of radius p_F . For spin $1/2$ fermions (e.g. electrons or neutrons or ${}^3\text{He}$ atoms) show that the number of states inside this sphere is $\frac{8\pi}{3} \frac{V}{h^3} p_F^3$.

(f) (4) Now specialize to nonrelativistic spin $1/2$ fermions, and rewrite the result of part (e) in terms of h , the particle mass m , V , and the Fermi energy ϵ_F .

(g) (4) Show that the Fermi energy is given by $\varepsilon_F = \frac{\hbar^2}{2m} (3\pi^2 n)^{2/3}$, where $\hbar = h / 2\pi$ and $n = N / V$ is the number density of the particles.

(h) (4) Given that the Fermi temperature $T_F = \varepsilon_F / k$ of a typical metal is about 30,000 K, and that the number density is very roughly the same for a metal and for ${}^3\text{He}$ atoms, but that a ${}^3\text{He}$ atom has about 6000 times the mass of an electron, estimate the Fermi temperature in ${}^3\text{He}$.

(i) (4) Given that the number density of electrons in a white dwarf is about 10^6 times greater than in the metal of part (h), roughly estimate the Fermi temperature T_F in a white dwarf.

(j) (4) Given that the number density of neutrons in a neutron star is about 10^{15} times greater than that of electrons in the metal of part (h) (since the radius of a nucleon is about 10^{-5} times the radius of an atom), but that a neutron has about 2000 times the mass of an electron, estimate the Fermi temperature T_F in a neutron star.

(k) (4) Combine the relevant results earlier in this problem to obtain an expression for the degeneracy pressure P for a white dwarf, with the electrons treated nonrelativistically.

(l) (4) It is reasonable to estimate that the pressure due to gravity at the center of a white dwarf or neutron star is roughly $P = G\rho^2 R^2$, where R is the radius and ρ is the average mass density of the object (white dwarf or neutron star).

By equating the degeneracy pressure of part (k) to this gravitational pressure, determine how the radius R depends on the mass M . I.e., show that $R \propto M^X$, while at the same time determining X .

(m) (4) Use the equation of part (k), retaining the particle mass m , to crudely estimate R_{wd} / R_{ns} , the ratio of the radius of a white dwarf to that of a neutron star, if they both have the same mass (e.g., one solar mass M_\odot).

2. (a) (5) Using a sketch of the Fermi-Dirac distribution function, explain why the heat capacity of a metal is only about 1% of what was expected classically, and why it is proportional to the temperature (rather than being a constant, as was again expected classically).

(b) (5) Explain why electrons find it energetically preferable to react with protons to form neutrons in the high density of a neutron star.

3. The chemical potential in an atmosphere is given by

$$\mu(z) = mgz + kT \ln(n(z)\lambda_{th}^3) \quad , \quad \lambda_{th} \equiv \frac{h}{(2\pi mkT)^{1/2}}$$

where z is the height and T is the constant temperature.

(a) (10) Calculate the number density $n(z)$ as a function of height (and the other parameters, such as the particle mass m and the acceleration of gravity g).

(b) (10) The atomic mass is $16 \times 1.66 \times 10^{-27}$ kg for oxygen. Assuming an average temperature of 250 K for a crude estimate, calculate the ratio $n(20 \text{ km}) / n(0)$ for molecular oxygen. ($g = 9.8 \text{ m/s}^2$)

4. (18) The value of the radiant energy flux density at the Earth from the Sun normal to the incident rays is called the solar constant. The observed value integrated over all emission wavelengths is

$$\text{solar constant} = 0.136 \text{ J/(s cm}^2\text{)} .$$

Take the earth to be in a thermal steady state, radiating as much energy (averaged over the day and over the surface area of the Earth) as it receives from the Sun.

Estimate the average surface temperature of the Earth.

Have a good weekend!