

Physics 633 Exam 1

Please show all significant steps clearly in all problems.

1. In class, we applied Noether's theorem to relativistic bosons and fermions. Let us now apply it to nonrelativistic spin 1/2 fermions, for which the field operator $\Psi(x)$ is a 2-component spinor and the Lagrangian density is

$$\mathcal{L} = \Psi^\dagger(x) \left[i\hbar \frac{\partial}{\partial t} - h(\vec{x}) \right] \Psi(x) \quad (1)$$

with

$$h(\vec{x}) = -\frac{\hbar^2}{2m} \nabla^2 + V(\vec{x}). \quad (2)$$

(a) (6) By extremalizing the action S , determine the equations of motion for the quantized fields $\Psi(x)$ and $\Psi^\dagger(x)$.

(b) (6) Calculate the conjugate momentum fields π and π^\dagger .

[Note: The answers below, to (c), (d), and (e), should have a simple, physically recognizable form.]

(c) (6) In treating Noether's theorem, we found that the conserved four-momentum is given by

$$cP^\mu = \int d^3x \left(c\pi \frac{\partial \Psi}{\partial x_\mu} - \mathcal{L} g^{\mu\nu} \right). \quad (3)$$

From this expression, calculate the Hamiltonian H .

(d) (6) Using Eq. (3), calculate the 3-momentum \vec{P} .

(e) (6) Finally, calculate the spin angular momentum \vec{M} by using the result we obtained in conjunction with Noether's theorem,

$$cM^{ij} = \int d^3x \, c\pi S^{ij} \Psi. \quad (4)$$

You can determine the 2×2 matrix S^{ij} by comparing the definition of S^{ij} ,

$$\delta \Psi = \frac{1}{2} \varepsilon_{ij} S^{ij} \Psi \quad (5)$$

for a rotation given by

$$\delta x_i = \varepsilon_{ij} x^j, \quad (6)$$

with the requirement that a 2-component spinor transforms as

$$\delta \Psi = -\frac{i}{4} \varepsilon_{ij} \sigma^{ij} \Psi \quad (7)$$

where the 2×2 matrix σ^{ij} is given by

$$\sigma^{ij} = \sigma_k \quad \text{i,j,k=1,2,3 or cyclic permutation} \quad (8)$$

and the σ_k are the three Pauli matrices.

2. In this problem, let us choose units such that $\hbar = c = 1$, in order to avoid complicating factors of $\hbar c$.

Consider a one-component complex bosonic or fermionic field ψ , with the equation of motion

$$\left[i \frac{\partial}{\partial t} - H(\vec{x}) \right] \psi(x) = 0, \quad (9)$$

where $H(\vec{x})$ is an operator which involves only the spatial coordinates. Define $\omega_{\vec{k}}$ by

$$\hbar \omega_{\vec{k}} = e^{-i \vec{k} \cdot \vec{x}} H(\vec{x}) e^{i \vec{k} \cdot \vec{x}}. \quad (10)$$

Also let

$$G(x) = \int \frac{d^4 k}{(2\pi)^4} e^{-ikx} G(k) \quad (11)$$

with

$$G(k) = \frac{1}{k_0 - \omega_{\vec{k}} + i\varepsilon \operatorname{sgn}(\omega_{\vec{k}})} \quad , \quad \varepsilon \rightarrow 0+ \quad , \quad \operatorname{sgn}(u) = u/|u| \quad (12)$$

and

$$kx = k_0 x^0 - \vec{k} \cdot \vec{x} = \omega t - \vec{k} \cdot \vec{x}. \quad (13)$$

(a) (10) For the case $t > 0$, show that $G(x)$ can be written as

$$G(x) = a \int \frac{d^3 k}{(2\pi)^3} e^{-i\omega_{\vec{k}} t} e^{i \vec{k} \cdot \vec{x}} \quad (14)$$

and, at the same time, evaluate a . [Do a contour integration, drawing a picture and explaining each step. You will have a check on your answer when you do part (b).]

(b) (10) The Green's function is defined in terms of the vacuum expectation value of a time-ordered product, with

$$iG(x - x') = \langle 0 | \psi(x) \psi^\dagger(x') | 0 \rangle \quad (15)$$

for the case $t > t'$. Here we can represent the field operator as

$$\psi(x) = \frac{1}{\sqrt{V}} \sum_{\vec{k}} \left[a(\vec{k}) e^{-i\omega_{\vec{k}} t} e^{i \vec{k} \cdot \vec{x}} + b^\dagger(\vec{k}) e^{i\omega_{\vec{k}} t} e^{-i \vec{k} \cdot \vec{x}} \right]. \quad (16)$$

Show that $G(x)$ is again given by the same expression you obtained for (14), in the case $t > t'$, so that the representation (11)-(12) is justified. (The case $t < t'$ clearly goes through the same way.)

(c) (10) Show that $G(x - x')$ really is a Green's function:

$$\left[i \frac{\partial}{\partial t} - H(\vec{x}) \right] G(x - x') = \delta(x - x') \quad (17)$$

3. In obtaining the nonrelativistic limit of the Dirac equation, we had

$$H_{eff} = \frac{(\vec{p} - e\vec{A})^2}{2m} + e\phi - \frac{p^4}{8m^3} - \frac{e}{2m}\vec{\sigma} \cdot \vec{B} + \frac{e[\nabla^2\phi]}{8m^2} + \frac{e}{4m^2}\vec{\sigma} \cdot ([\vec{\nabla}\phi] \times \vec{p}). \quad (18)$$

(a) (5) For a central potential, $\phi = \phi(r)$ where r is the radial coordinate, rewrite the last term as a product involving $d\phi/dr$.

(b) (5) For a Coulomb potential, $\phi(r) = Ze/(4\pi r)$ (in rationalized Gaussian units), simplify the next-to-last term.

(c) (10) Give a clear physical interpretation of each of the six terms in the above expression, after the last two terms have been replaced by the results of parts (a) and (b).

4. (20) The Kramers-Heisenberg formula is

$$\frac{d\sigma}{d\Omega} = r_0^2 \left(\frac{\omega'}{\omega} \right) \left| \delta_{AB} \vec{\epsilon}^{(\alpha)} \cdot \vec{\epsilon}^{(\alpha')} - \frac{1}{m} \sum_I \left[\frac{(\vec{p} \cdot \vec{\epsilon}^{(\alpha')})_{BI} (\vec{p} \cdot \vec{\epsilon}^{(\alpha)})_{IA}}{E_I - E_A - \hbar\omega} + \frac{(\vec{p} \cdot \vec{\epsilon}^{(\alpha)})_{BI} (\vec{p} \cdot \vec{\epsilon}^{(\alpha')})_{IA}}{E_I - E_A + \hbar\omega'} \right] \right|^2 \quad (19)$$

where the notation is taken from Sakurai.

(a) (15) Using a picture, and clear but simple arguments, show that the following is true for Thomson scattering:

If light is scattered through 90° , it will be 100% polarized.

You do not need to do any serious mathematics in this problem, but you will need to give a complete and rigorous argument.

(b) (5) Describe a simple experiment to observe this effect.