Physics 633 Exam 2

Please show all significant steps clearly in all problems.

1. Consider a one-dimensional system of noninteracting fermions having the Lagrangian density

$$\mathcal{L}_0 = i\hbar\, \psi^\dagger (x,t) \frac{\partial \psi (x,t)}{\partial t} - \psi^\dagger (x,t)\, T(x)\, \psi (x,t)\,. \tag{1}$$

Here $T(x)$ operates on functions of x, but not on state vectors in the occupation number representation. When the field $\psi(x,t)$ is quantized, it is replaced by the Heisenberg field operator $\widehat{\psi}(x,t)$, which can be written in terms of the usual Heisenberg destruction operators $\widehat{c}_k(t)$ and a complete orthonormal set of functions $\psi_k(x)$:

$$\widehat{\psi}(x,t) = \sum_k \psi_k(x)\, \widehat{c}_k(t)\,. \tag{2}$$

Since there are both positive energy states labeled by $k+$, with $\varepsilon_{k+} > 0$, and negative-energy states labeled by $k-$, with $\varepsilon_{k-} < 0$, this can be written more explicitly as

$$\widehat{\psi}(x,t) = \sum_{k+} \psi_{k+}(x)\, \widehat{c}_{k+}(t) + \sum_{k-} \psi_{k-}(x,t)\, \widehat{c}_{k-}(t) \tag{3}$$

but you do not need this form until part (e).

(a) (4) Using the usual rules for canonical quantization, obtain the "canonical momentum" π and the second-quantized Hamiltonian density $\widehat{\mathcal{H}}_0$.

(b) (4) Write down the standard anticommutation rules for the operators $\widehat{c}_k$ and $\widehat{c}^\dagger_{k'}$. (You should just assume these.)

(c) (4) From the relations in part (b), derive the corresponding relations for the field operators $\widehat{\psi}(x,t)$ and $\widehat{\psi}^\dagger(x',t)$.

(d) (4) The second-quantized Hamiltonian is given by

$$H_0 = \int d^3x\, \widehat{\mathcal{H}}_0. \tag{4}$$

Let the $\psi_k(x)$ be eigenfunctions of the operator $T(x)$:

$$T(x)\, \psi_k(x) = \varepsilon_k \psi_k(x)\,. \tag{5}$$

From the expression for $\widehat{\mathcal{H}}_0$ in part (a), derive the expression for H_0 in terms of the operators $\widehat{c}_k$ and $\widehat{c}^\dagger_{k'}$.

(e) (4) Perform a canonical transformation from the fermion operators $\widehat{c}_{k-}(t)$ and $\widehat{c}^\dagger_{k-}(t)$ for negative-energy states to operators $\widehat{d}_k(t)$ and $\widehat{d}^\dagger_k(t)$ for positive-energy holes or antiparticles. Then show that H_0 has the form

$$H_0 = \sum_k \left(\hat{c}_k^\dagger \hat{c}_k - \frac{1}{2} \right) \varepsilon_k + \sum_k \left(\hat{d}_k^\dagger \hat{d}_k - \frac{1}{2} \right) \varepsilon_k . \tag{6}$$

In this equation and below, k ranges only over positive-energy states.

(f) (4) Consider the total number of particles and the charge, as given by the operators

$$N = \sum_k \hat{c}_k^\dagger \hat{c}_k + \sum_k \hat{d}_k^\dagger \hat{d}_k . \tag{7}$$

$$Q = \sum_k \hat{c}_k^\dagger \hat{c}_k - \sum_k \hat{d}_k^\dagger \hat{d}_k . \tag{8}$$

Determine whether each of these quantities is conserved. I.e., determine whether $dN/dt = 0$ and $dQ/dt = 0$ hold or not. (Start with the Heisenberg equation of motion.)

(g) (4) Now introduce a model interaction between particles and antiparticles:

$$H = H_0 + \sum_{kk'} V_{kk'} \langle \hat{c}_k^\dagger \hat{d}_k^\dagger \rangle \hat{c}_{k'} \hat{d}_{k'} + \sum_{kk'} V_{k'k}^* \hat{d}_k^\dagger \hat{c}_k^\dagger \langle \hat{d}_{k'} \hat{c}_{k'} \rangle . \tag{9}$$

Determine whether this new Hamiltonian H conserves N and Q, using the same approach as in the preceding part. I.e., calculate dN/dt and dQ/dt from the Heisenberg equation of motion.

(h) (4) In the vacuum state $\mid 0\rangle$, with no particles or antiparticles, determine the values of $\langle 0 \mid \hat{d}_{k'} \hat{c}_{k'} \mid 0 \rangle$ and $\langle 0 \mid \hat{d}_{k'}^\dagger \hat{c}_{k'}^\dagger \mid 0 \rangle$. Then discuss how we should interpret $\langle \hat{d}_{k'} \hat{c}_{k'} \rangle$ and $\langle \hat{c}_k^\dagger \hat{d}_k^\dagger \rangle$ in order for H in part (g) to make any sense. In doing this, make reference to the related ideas of a (i) a coherent state and (ii) the Bogoliubov transformation in our treatment of the BCS theory of superconductivity. Please be as specific as posible.

2. If we have bosons with a zero-point energy (like the zero-point energy of the radiation field), then the noninteracting Hamiltonian in Problem 1 becomes

$$H = \sum_{r,k} \left(\widehat{b}_{rk}^{\dagger} \widehat{b}_{rk} + \frac{1}{2} \right) \hbar\, \omega_k + \sum_{r,k} \left(\widehat{a}_{rk}^{\dagger} \widehat{a}_{rk} - \frac{1}{2} \right) \varepsilon_k . \tag{10}$$

Here $\widehat{a}_{rk}$ and $\widehat{a}_{rk}^{\dagger}$ represent operators for both particles and antiparticles, with $r = 1$ for fermions and $r = 2$ for antifermions.

The bosons are assumed to have two polarizations, and the fermions only one (as is true for massless spin one bosons and massless spin 1/2 fermions in three dimensions). Suppose that we have a kind of "supersymmetry", in the sense that the bosons and fermions have the same energies. It is an interesting fact that the infinite vacuum energies of bosons and fermions cancel in this case.

Now, however, suppose that the $\hbar\, \omega_k = \hbar\, ck$ are different from the ε_k because the fermions and bosons have different boundary conditions. Specifically, the boson states are forced to go to zero at $x = 0$ and $x = d$, in a Casimir problem. (Recall that there are conducting plates at $z = 0$ and $z = d$ in the three-dimensional Casimir problem.) Then the wavenumber is quantized:

$$k = n \frac{\pi}{d} \ , \ n = 1, 2, ... \tag{11}$$

The fermion states, on the other hand, have no such boundary condition, so the fermion energy is independent of the separation d. You can therefore forget the fermions beyond this point.

(a) (25) Calculate the change in the zero-point energy of the boson field due to this boundary condition, as a function of d. Use a cut-off function which goes to zero as $\omega_k \to \infty$. Just as in the three-dimensional Casimir problem, you will want to use the Euler-Maclaurin formula

$$\sum_{n=0}^{\infty} F(n) - \int_0^{\infty} F(n)\, dn = \frac{1}{2} F(0) - \frac{1}{12} F'(0) + \frac{1}{720} F'''(0) + ... \tag{12}$$

(b) (3) Calculate the force between the "plates" at $x = 0$ and $x = d$.

For a recent experimental measurement of the Casimir effect, see the attached paper.

3. (17) In our long calculation of Compton scattering, we finally arrived at the results

$$\frac{d\sigma}{d\Omega} = \frac{1}{2} e^4 \frac{\omega'^2}{\omega^2} \frac{1}{4m^2} \left[\frac{S_1}{(2k \cdot p_i)^2} + \frac{S_2}{(2k' \cdot p_i)^2} + \frac{2S_3}{(2k \cdot p_i)(2k' \cdot p_i)} \right] \tag{13}$$

with

$$S_1 = 8(k \cdot p_i) \left[p_i \cdot k' + 2(k \cdot \epsilon')^2 \right] \tag{14}$$

$$S_2 = 8(k' \cdot p_i) \left[p_i \cdot k - 2(k' \cdot \epsilon)^2 \right] \tag{15}$$

$$\begin{aligned} S_3 = {} & 8p_i \cdot k \{2\epsilon \cdot \epsilon' [(p_i \cdot k')(\epsilon' \cdot \epsilon) - (p_i \cdot \epsilon')(k' \cdot \epsilon) + (p_i \cdot \epsilon)(k' \cdot \epsilon')] - p_i \cdot k'\} && (16) \\ & -8k \cdot \epsilon' \{[(k \cdot p_i)(k' \cdot \epsilon') - (k \cdot k')(p_i \cdot \epsilon') + (k \cdot \epsilon')(p_i \cdot k')]\} && (17) \\ & +8k' \cdot \epsilon \{[(k' \cdot \epsilon)(k \cdot p_i) - (k' \cdot k)(\epsilon \cdot p_i) + (k' \cdot p_i)(\epsilon \cdot k)]\}. && (18) \end{aligned}$$

(Recall that we choose the Coulomb gauge and the frame in which the electron is initially at rest. The primed quantities refer to the outgoing photon, and the corresponding unprimed quantities to the incoming photon. Finally, p_i is the initial four-momentum of the electron, and m is its rest mass.) Show that this gives the Klein-Nishina formula

$$\frac{d\sigma}{d\Omega} = \frac{e^4}{4m^2} \frac{\omega'^2}{\omega^2} \left[\frac{\omega'}{\omega} + \frac{\omega}{\omega'} + 4(\epsilon \cdot \epsilon')^2 - 2 \right]. \tag{19}$$

(b) (3) Show that the Klein-Nishina cross-section reduces to the nonrelativistic Thomson cross-section for elastic scattering.

4. (a) (5) Write down the lowest-order Feynman diagram for the self-energy of an electron in quantum electrodynamics. Then write down the corresponding mathematical expression (in the momentum representation), and give the expressions for the electron Feynman propagator $S_F(p)$ and the photon propagator $D_{\mu\nu}(q^2)$.

(b) (15) The nonrelativistic Green's function with interactions is

$$G(k) = G\left(\overrightarrow{k}, \omega\right) = \frac{1}{\omega - \hbar^{-1}\varepsilon_0\left(\overrightarrow{k}\right) - \Sigma\left(\overrightarrow{k}, \omega\right)}. \tag{20}$$

Show that the energy $\varepsilon\left(\overrightarrow{k}\right)$ and damping $|\,\gamma\left(\overrightarrow{k}\right)|$ of long-lived single-particle excitations are given to lowest order by

$$\varepsilon\left(\overrightarrow{k}\right) = \varepsilon_0\left(\overrightarrow{k}\right) + Re\,\hbar\,\Sigma\left(\overrightarrow{k}, \varepsilon\left(\overrightarrow{k}\right)/\hbar\right) \tag{21}$$

$$\gamma\left(\overrightarrow{k}\right) = \left[1 - \frac{\partial Re\,\Sigma\left(\overrightarrow{k}, \omega\right)}{\partial\omega}\right]^{-1}_{\varepsilon\left(\overrightarrow{k}\right)/\hbar} Im\,\Sigma\left(\overrightarrow{k}, \varepsilon\left(\overrightarrow{k}\right)/\hbar\right). \tag{22}$$