Physics 633 Final Exam

Please show all significant steps clearly in all problems.

1. In class we considered two applications of Bogoliubov transformations: in the BCS theory of superconductivity and in treating weakly-interacting excitations of a superfluid. A third application is particle creation in a rapidly changing gravitational field, either in the expansion of the early universe (considered by Stephen Fulling of our own Texas A&M Math Department) or near a black hole (considered by Stephen Hawking of Cambridge University). References are *Aspects of Quantum Field Theory in Curved Space-Time*, by S. A. Fulling, and *Quantum Fields in Curved Space*, by N. D. Birrell and P. C. W. Davies. The treatment of these problems involves nontrivial applications of Green's functions etc., but the simple ideas below are relevant.

(a) (6) Suppose that we have a simple Bogoliubov transformation for spin 1/2 fermions, of the form

$$\begin{aligned} a_k &= u_k \overline{a}_k + v_k \overline{a}^{\dagger}_{-k} && (1) \\ a^{\dagger}_k &= u^*_k \overline{a}^{\dagger}_k + v^*_k \overline{a}_{-k} && (2) \end{aligned}$$

where $k = (\mathbf{k}, s)$ is a useful notation for nonrelativistic particles in the present context. Here $\hbar\mathbf{k}$ represents the 3-momentum, s the spin orientation, and $\overline{a}_k$ the transformed destruction operator. What condition or conditions must the u_k and v_k satisfy if this is to be a canonical transformation?

(b) (6) For scalar bosons, the transformation is

$$\begin{aligned} a_{\mathbf{k}} &= u_{\mathbf{k}} \overline{a}_{\mathbf{k}} + v_{\mathbf{k}} \overline{a}^{\dagger}_{-\mathbf{k}} && (3) \\ a^{\dagger}_{\mathbf{k}} &= u^*_{\mathbf{k}} \overline{a}^{\dagger}_{\mathbf{k}} + v^*_{\mathbf{k}} \overline{a}_{-\mathbf{k}}. && (4) \end{aligned}$$

What condition or conditions must the $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$ satisfy in this case?

(c) (6) Now consider a more general transformation for bosons, of the form

$$\begin{aligned} a_i &= \sum_j \left(u_{ij} \overline{a}_j + v_{ij} \overline{a}^{\dagger}_j \right) && (5) \\ a^{\dagger}_i &= \sum_j \left(u^*_{ij} \overline{a}^{\dagger}_j + v^*_{ij} \overline{a}_j \right). && (6) \end{aligned}$$

What is the condition on the u_{ij} and v_{ij} for this to be a canonical transformation?

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(d) (6) Suppose that there are two different vacua $|0\rangle$ and $|\overline{0}\rangle$, in two different regions of spacetime. (One might be before a period of expansion in the early universe, and one after. Or one might be near the horizon of a black hole, and the other flat spacetime asymptotically far away.) The respective vacua are defined by

$$\begin{aligned} a_i|0\rangle &= 0 \quad (7) \\ \overline{a}_j|\overline{0}\rangle &= 0. \quad (8) \end{aligned}$$

Now suppose that $|\overline{0}\rangle$ is the vacuum in a region far from you, and that $|0\rangle$ is your local vacuum. An observer in the region far from you sees no particles, so that $\langle\overline{0}|\overline{n}|\overline{0}\rangle = 0$, where $\overline{n} = \overline{N}/\overline{V}$, $\overline{N}$ is the total number operator for all states, and $\overline{V}$ is the normalization volume. What do you see for the number density in that region? I.e., what is $\langle\overline{0}|n|\overline{0}\rangle$ where $n = N/\overline{V}$ and N is your total number operator? Give your answer in terms of the u_{ij} and v_{ij}.

(e) (6) Let $V(r)$ be the electrostatic potential energy of a hypothetical massless fermion with charge $+e$. Suppose that

$$\begin{aligned} V(r) &= 0, \quad r > R \quad (9) \\ V(r) &= V_0, \quad r < R \quad (10) \end{aligned}$$

where $r = |\mathbf{x}|$ is the radial coordinate and $V_0 > 0$. (The region $r < R$ is regarded as large enough that it makes sense to think of it as a different vacuum.) Then let

$$\begin{aligned} \overline{a}_{\mathbf{k}} &= c_{\mathbf{k}} \quad \text{for } \varepsilon_{\mathbf{k}} > V_0 \quad (11) \\ \overline{a}_{\mathbf{k}} &= c^\dagger_{\mathbf{k}} \quad \text{for } \varepsilon_{\mathbf{k}} < V_0 \quad (12) \end{aligned}$$

whereas

$$\begin{aligned} a_{\mathbf{k}} &= c_{\mathbf{k}} \quad \text{for } \varepsilon_{\mathbf{k}} > 0 \quad (13) \\ a_{\mathbf{k}} &= c^\dagger_{\mathbf{k}} \quad \text{for } \varepsilon_{\mathbf{k}} < 0. \quad (14) \end{aligned}$$

Suppose that all the states with energy less than V_0 in the region $r < R$ are occupied, and all those above V_0 are unoccupied – i.e., $\overline{a}_{\mathbf{k}}|\overline{0}\rangle = 0$. If $\overline{V}$ is the volume of the region with a positive potential, what is the number density of particles in this region as measured by you, an observer outside R? I.e., what is $\langle\overline{0}|n|\overline{0}\rangle$? Your answer should be a number, but it can be left in the form of a summation over $\mathbf{k}$ (with a specification of the restriction on $\mathbf{k}$).

This last part is a toy model of proposals to observe electron-positron pair creation during the collision of two heavy nuclei: For a tiny fraction of a second, near the large positive charge, the vacuum level is pulled down for electrons and pushed up for positrons.

2. Let us consider the scattering of an electron off a proton, with the proton approximated as a simple Dirac particle with no internal structure.

(a) (5) Show that the Lippmann-Schwinger equation holds for an electron:

$$\Psi(x) = \psi(x) + \int d^4x\, S_F(x - x')\, e\not{A}(x')\, \Psi(x') \tag{15}$$

where $\Psi(x)$ and $\psi(x)$ are respectively the solutions to the Dirac equation with and without an electromagnetic field, and S_F is the Feynman propagator. (Equation (26) will jog your memory if you have forgotten the Dirac equation.)

(b) (5) Show that the analogous equation holds for the electromagnetic field:

$$A_\mu(x) = \int d^4x\, D^F_{\mu\nu}(x - x')\, J^\nu(x'). \tag{16}$$

(c) (5) The scattering amplitude is defined by

$$S_{fi} = \langle \psi_f | \Psi_i \rangle. \tag{17}$$

I.e., it is defined to be the amplitude that the initial state i will evolve into a particular final state f. Take J^μ to be the transition current of the proton:

$$J^\mu = e_p \overline{\psi}^p_f \gamma^\mu \psi^p_i. \tag{18}$$

Obtain the expression for S_{fi} in terms of $\psi^\dagger_f$, S_F, γ^μ, $D^F_{\mu\nu}$, $\overline{\psi}^p_f$, ψ^p_i, and Ψ_i.

(d) (5) With the normalization

$$\psi(x) = \sqrt{\frac{m}{EV}}\, u(p, s)\, e^{-ip\cdot x} \tag{19}$$

show that

$$S^{(1)}_{fi} = \left[(2\pi)^4 \delta^4(P_f - P_i) \prod_{ext.} \left(\frac{m}{EV} \right)^{1/2} \right] M^{(1)}_{fi} \tag{20}$$

$$M^{(1)}_{fi} = -iee_p \left[\overline{u}(p_f, s_f)\, \gamma^\mu u(p_i, s_i) \right] \frac{g^{\mu\nu}}{(p_f - p_i)^2 + i\eta} \left[\overline{u}\left(p'_f, s'_f\right) \gamma^\mu u(p'_i, s'_i) \right]. \tag{21}$$

(e) (3) Draw a Feynman diagram for this process, indicating the momenta for the electron, proton, and virtual photon.

(f) (4) Show that the above expression is consistent with the Feynman diagram rules. I.e., describe in detail how this expression follows from the rules.

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(g) (3) Finally, let us consider electron-positron, or e^+e^-, scattering (historically called Bhabha scattering after H. J. Bhabha, who considered this problem in 1935). Write down the two Feynman diagrams for this process, labeling the momenta of the electron, positron, and virtual photon. These diagrams respectively involve the "Mandelstam variables"

$$
\begin{aligned}
s &= -\left(p_1 + p_2\right)^2 &(22)\\
t &= -\left(p_1 - p_1'\right)^2 &(23)
\end{aligned}
$$

where 1 and 2 refer to the electron and positron.

3. (20) For a Dirac field, the transformations

$$
\begin{aligned}
\psi(x) &\rightarrow \psi'(x) = e^{i\alpha\gamma_5}\psi(x) &(24)\\
\psi^\dagger(x) &\rightarrow \psi^{\dagger\prime}(x) = \psi^\dagger(x)\, e^{-i\alpha\gamma_5} &(25)
\end{aligned}
$$

are called chiral phase transformations, where α is a real parameter. Show that the Lagrangian density

$$\mathcal{L} = c\overline{\psi}(x)\left[i\hbar\gamma^\mu\partial_\mu - mc\right]\psi(x) \tag{26}$$

is invariant under a chiral phase transformation when $m = 0$, but that the chiral symmetry is broken when the fermions acquire a mass. (This is an important effect in the context of the strong nuclear interaction.) Recall the properties

$$\left[\gamma^5, \gamma^\mu\right]_+ = 0 \quad , \quad \left(\gamma^5\right)^2 = 0 \quad , \quad \gamma^{5\dagger} = \gamma^5. \tag{27}$$

4. (20) From $S_F S_F^{-1} = 1$ and the expression for the Feynman propagator S_F, derive the Ward identity

$$\frac{\partial\Sigma(p)}{\partial p_\mu} = \Lambda^\mu(p, p) \tag{28}$$

where Λ^μ is the vertex correction.