Physics 222, Modern Physics, Exam 2

NAME____

You are graded on your work, with partial credit where it is deserved.

Please be clear and well-organized in all your steps.

1. (15) Show that

$$\Psi(x,t) = e^{-ipx/\hbar} e^{-iEt/\hbar}$$

satisfies the time-dependent Schrödinger equation for a free particle

$$i\hbar \frac{\partial \Psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi(x,t)}{\partial x^2}$$

and obtain the energy E in terms of the momentum p.

(b) (5) Is this wave traveling in the positive or the negative x direction? Explain.

2. A 3d wavefunction for an electron in a hydrogen atom is

$$\psi(r,\theta,\phi) = A_0\, r^2\, e^{-r/3a_0} \sin^2\theta\, e^{2i\phi}$$

where A_0 is a constant.

(a) (6) What are the values of the principal quantum number n, the angular momentum quantum number ℓ, and the magnetic quantum number m_ℓ?

n = __________ ℓ = __________ m_ℓ = __________

(b) (14) Calculate the value of r for which the probability $4\pi r^2 |R(r)|^2$ reaches a maximum, where $R(r)$ is the radial part of this wavefunction. Is this consistent with the Bohr theory?

3. The potential for the infinite square well is

$$V(x)=0 \text{ for } 0<x<\mathrm{L} \qquad , \qquad V(x)=\infty \text{ for } x<0 \text{ or } x>\mathrm{L} \quad .$$

(a) (5) Show that the time-independent Schrödinger equation for $0<x<\mathrm{L}$ is satisfied by

$$\psi(x)=A\sin(kx)+B\cos(kx).$$

(Recall that the momentum operator is $\hat{p}=-i\hbar\frac{d}{dx}$.)

(b) (5) Use the boundary condition at $x=0$ to eliminate one of the terms in $\psi(x)$.

(c) (5) Use the boundary condition at $x = L$ to determine the allowed values of k in terms of L.

(d) (5) Using the fact that $\int_0^{\pi} du \sin^2 u = \int_0^{\pi} du \cos^2 u = \frac{1}{2}\pi$, obtain the final normalized wavefunction $\psi(x)$.

4. (20) The radial Schrödinger equation for the hydrogen atom is

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\frac{2m}{\hbar^2}\left[E-V(r)-\frac{\hbar^2}{2m}\frac{\ell(\ell+1)}{r^2}\right]R=0 \quad , \quad V(r)=-k\frac{e^2}{r} \quad .$$

Show that

$$R = A\,r\,e^{-\alpha r}$$

is a solution to the radial Schrödinger equation, and at the same time determine the values of ℓ, α, and E in terms of the various constants like m, $\hbar$, k, and e.

(You are supposed to calculate these values of ℓ, α, and E in doing your work. I.e., do not just assume them or quote them.)

5. The ground state wavefunction of the harmonic oscillator is

$$\psi = \left(\frac{\alpha}{\pi}\right)^{1/4} e^{-\alpha x^2/2} \quad , \quad \alpha = \frac{\sqrt{mK}}{\hbar} \quad , \quad -\infty < x < \infty \quad .$$

(a) (4) Calculate $\langle x \rangle$, the expectation value of the coordinate x.

(b) (4) Calculate $\langle x^2 \rangle$.

(c) (4) Calculate $\langle p \rangle$, the expectation value of the momentum p. Recall that the momentum operator is

$$\hat{p} = -i\hbar \frac{d}{dx} \quad .$$

(d) (4) Calculate $\langle p^2 \rangle$.

(e) (4) Using the fact that $(\Delta x)^2 = \langle x^2 \rangle - \langle x \rangle^2$ and $(\Delta p)^2 = \langle p^2 \rangle - \langle p \rangle^2$, calculate the value of $\Delta p \, \Delta x$ in terms of $\hbar$. Is your result consistent with the uncertainty principle?

Have a good weekend (and a happy July 4 next week)!

$$I_n \equiv \int_0^{\infty} x^n e^{-ax^2}\, dx$$

$$I_n = \frac{\left[(n-1)/2\right]!}{2a^{(n+1)/2}} \qquad \text{for odd } n$$

$$I_n = \frac{1\cdot 3\cdot 5\cdots(n-1)}{2^{(n/2)+1}a^{(n/2)}}\sqrt{\frac{\pi}{\mathrm{a}}} \qquad \text{for even } n$$

$$I_0 = \frac{1}{2}\sqrt{\frac{\pi}{\mathrm{a}}}$$

$$I_1 = \frac{1}{2a}$$