Please show all significant steps clearly in all problems.

1. (15) According to the Saha equation,

$$\frac{N_{i+1}}{N_i} = \frac{2kT\,Z_{i+1}}{P_e Z_i}\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-\chi_i/kT}.$$

In the solar photosphere, the electron pressure P_e is about 15 dyne/cm^2 and the temperature is 5770 K. The ionization energy of hydrogen is 13.6 eV. Calculate the ratio of neutral hydrogen atoms to hydrogen ions in the photosphere of our Sun.

2. Consider a model of a star consisting of a spherical blackbody with a surface temperature of 20,000 K and a radius of 3×10^{9} m, which is located at a distance of 90 parsecs from Earth. Calculate the values of each of the following:

(a) (7) Wavelength λ_{max} at which the blackbody spectrum peaks.

(b) (7) Radiant flux at the star's surface.

(c) (7) Luminosity of the star.

(d) (7) Radiant flux at the Earth's surface.

(e) (7) Apparent bolometric magnitude

$$m = M_{Sun} - 2.5\log_{10}\left(\frac{F}{F_{10,\odot}}\right)$$

where $M_{Sun} = 4.76$ is the absolute magnitude of the Sun, F is the radiant flux received from the star at the Earth, and $F_{10,\odot} = 3.196 \times 10^{-10}$ J/s·cm² is the radiant flux that would be received from the Sun if it were 10 pc away.

3. Consider a (nonrelativistic) binary system with reduced mass μ and total mass M.

(a) (12) In class we used our results for r_p and v_p, the position and velocity at perihelion, to obtain an expression for the total energy E. Here let us start instead with the expressions at aphelion:

$$r_a = a(1+e) \quad \text{and} \quad v_a^2 = \frac{GM}{a}\frac{1-e}{1+e}$$

where a is the semimajor axis, e is the eccentricity, and G is the gravitational constant. Using these expressions, obtain a simple expression for E in terms of G, M, μ, and a.

(b) (12) The time average of a quantity $f(t)$ over an interval τ is

$$\langle f \rangle \equiv \frac{1}{\tau} \int_0^\tau f(t) \, dt \, .$$

Obtain a simple expression for the time average of the gravitational potential energy

$$U = -G \frac{M\mu}{r}$$

by averaging over one period P. You may just assume the results

$$v_\theta = \frac{2\pi a}{P} \frac{1 + e\cos\theta}{\sqrt{1 - e^2}}$$

and

$$\int_0^{2\pi} \frac{d\theta}{1 + e\cos\theta} = \frac{2\pi}{\sqrt{1 - e^2}} \, .$$

(c) (6) Is your result consistent with the virial theorem? Explain, with a relevant equation.

4. (a) (12) In class we obtained
$$\rho \frac{d^2 r}{dt^2} = -G \frac{M_r \rho}{r^2} - \frac{dP}{dr}$$
where ρ is the mass density and M_r is the mass within a sphere of radius r (in a spherically symmetric distribution of mass). We used this equation to obtain the pressure $P(r)$ as a function of the distance r from the center of a white dwarf.

Now assume the following model of a static and degenerate core near the center of a star: The core has a radius R and a nearly constant density. It also has a pressure P_0 at its surface: $P(R) = P_0$. Show that
$$P(r) = \text{constant} \times (R^2 - r^2) + \text{another constant}$$
while at the same time obtaining the values of these constants.

(b) (8) A white dwarf with a radius R_{wd} collapses to form a neutron star with a radius $R_{ns} = R_{wd}/500$. The average magnetic field over the white dwarf's surface is 5000 tesla, and magnetic flux is conserved during the collapse. What is the average magnetic field over the neutron star's surface?