Astrophysics (Physics 489) Exam 2

Please show all significant steps clearly in all problems.

1. **Fun with dark energy.** Since the dark energy is currently a total mystery, many theoretical cosmologists have had fun assuming some bizarre form of matter with an equation of state which is not exactly the same as that of vacuum energy – namely,

$$p = w\rho \quad , \quad w = -1 \pm \varepsilon$$

where ε is positive but must be small because of the limits already imposed by observations (of Type Ia supernovae, the cosmic microwave background radiation, and the large scale structure of galaxies in the universe). Recall that p is the pressure, which is also important in Einstein gravity.

Since you worked hard in the homework to help derive the following equations, we will simply assume them in doing this problem:

$$\frac{d}{dt}\left(\rho a^3\right) = -p\frac{d}{dt}\left(a^3\right) \qquad\qquad \left(\frac{da/dt}{a}\right)^2 = \frac{8\pi G}{3}\rho \quad .$$

(a) (7) For the positive sign, $w = -1 + \varepsilon$, calculate how the mass density ρ depends on the cosmic scale factor a. [You will obtain a result of the form $\rho \propto a^n$ where n (which is not an integer, of course) depends on ε.]

(b) (7) Again for the positive sign, $w = -1 + \varepsilon$, calculate how a depends on the cosmological time t. [You will again get a simple relation.]

(c) (7) Now let us turn to the more bizarre possibility of a negative sign, $w = -1 - \varepsilon$. For this case, calculate how the mass density ρ depends on the cosmic scale factor a. [Once more, your result should have a simple form.]

(d) (7) Again for the case of a negative sign, $w = -1 - \varepsilon$, calculate how a depends on the cosmological time t. [Be sure to retain the integration constant. The term "Big Rip" was invented by certain Caltech people two or three years ago, and this idea was prominently featured in the news media, because all matter is ripped to pieces as a critical time is approached in the future if $w = -1 - \varepsilon$.]

2. **Fun with time dilation.** Consider a spherical blackbody with constant temperature and mass M whose surface lies at a radial coordinate $r = R$. An observer located at the surface of the sphere and a distant observer both measure the blackbody radiation given off by the sphere.

(a) (7) If the observer at the surface of the sphere measures the luminosity of the blackbody to be L, use the formula for gravitational time dilation to show that the observer at infinity measures

$$L_\infty = L\left(1 - \frac{2GM}{Rc^2}\right).$$

[If you have forgotten the precise formula for time dilation, the form of this result should jog your memory! But be careful to include all relevant effects.]

(b) (7) Both observers use Wien's law (relating temperature and characteristic wavelength $\lambda_{\max}$) to determine the blackbody temperature. Show that

$$T_\infty = T\sqrt{1 - \frac{2GM}{Rc^2}}.$$

(c) (7) Both observers use the Stefan-Boltzmann law (relating radiant flux, or luminosity, to temperature) to determine the radius of this spherical blackbody. Show that

$$R_\infty = \frac{R}{\sqrt{1-\dfrac{2GM}{Rc^2}}}.$$

(d) (3) What sort of star might this be relevant to? I.e., what sort of star will have a gravitational field so strong that neglecting the effect of general relativity will lead to errors in, e.g., determining its size? (Explain in one or two sentences.)

3. **Fun with photodisintegration.**

(a) (10) Calculate the binding energy of the deuterium nucleus, using $m_H = 1.007825$ u, $m_n = 1.008665$ u, and $m_D = 2.014102$ u. (1 u = $1.6605402 \times 10^{-27}$ kg, speed of light = 3.00×10^8 m/s)

(b) (10) Estimate the temperature below which deuterium will hold together in the early universe. (Boltzmann constant = 1.38×10^{-23} J/K)

4. **Fun with stellar pulsation.**

Let us see if we can treat stellar pulsation with a simpler and more general argument than the one we followed in class from the textbook.

(a) (4) Adopt the same one-zone model, in which a spherical shell of mass m and radius R feels an inward force due to the gravitational pull of the mass M of gas inside, and an outward force due to the pressure P of this gas. Write down the net force F in terms of R, G, M, m, and P.

(b) (4) For adiabatic oscillations, show that P can be replaced by $C R^{-3\gamma}$, where C is a constant.

(c) (4) Show that the total force can then be written in the form

$$F = -aR^{-2} + bR^{2-3\gamma}$$

where you will define a and b in terms of the original quantities.

(d) (4) Use the fact that $F = 0$ when $R = R_0$, the equilibrium radius, to obtain b in terms of a and R_0.

(e) (4) Now calculate the force constant $K \equiv -\left[\frac{dF}{dR}\right]_{R=R_0}$ in terms of a and R_0 (and γ), using the result of Part (d) to eliminate b. Try to express your result in the simplest form possible.

(f) (4) Show that one then has the same equation as for a mass m on the end of a spring with force constant K, and obtain a final, simple expression for the angular frequency of oscillation $\omega = \sqrt{K / m}$ in terms of R_0, M, G, and γ.

(g) (4) Estimate the angular frequency and the period of oscillation for this pulsating star if $M \approx 5M_{\odot} \approx 10^{34}$ kg and $R_0 \approx 50R_{\odot} \approx 3.5\times10^{10}$ m. ($G = 6.67\times10^{-11}$ N m^2 / kg^2)

5. (10 extra credit) Describe the most important content of your favorite Mitchell Symposium talk in about 3 substantial sentences. (Be sure to say which talk it was!)