Astrophysics (Physics 489) Final Exam

Name______________________________

Please show all significant steps clearly in all problems.

1. Fun with primordial nucleosynthesis.

(a) (3) In the early universe, decaying neutrons were constantly replenished by reactions like

$$p^+ + e^- \to n + ? \quad .$$

What is the particle represented by the question mark?

(b) (5) The electrons for this reaction can come from $\gamma \to e^+ + e^-$, but this reaction will be suppressed when the thermal energy kT falls below the energy required for the production of an e^+e^- pair. Calculate the temperature T when this happens.

(c) (5) Calculate the ratio n_n / n_p at this temperature.

(d) (2) If we take $n_p = 10\,000$ at this temperature, what is the corresponding value of n_n ?

(e) (5) Now suppose that the time $t(T)$ is 1.8 s when the temperature has fallen to T. What is the time $t(T/10)$ when the temperature has fallen still further to $T/10$? (Recall that $RT = \text{constant}$ and that $R \propto t^{1/2}$, where R is the cosmic scale factor, since all this happens in the radiation-dominated epoch.)

(f) (5) The half-life of a neutron is $t_{1/2} = 614\ \text{s} = 10.2\ \text{min}$. Recall that the number of undecayed particles $N(t)$ is given by

$$N(t) = N(0)e^{-\lambda t} \text{ where } \lambda = \ln 2 / t_{1/2}.$$

If you start with the value of n_n calculated in part (c), at the temperature T, what is the value of n_n after the neutrons have decayed over the time interval between the earlier time $t(T)$ and the later time $t(T/10)$?

(g) (5) Now assume that the neutrons are all safely bound in deuterium nuclei after the time $t(T/10)$ (since fusion to form deuterium is efficient and the binding energy of deuterium is greater than the thermal energy below this temperature). Write down a 2-step set of reactions that results in ^4He production (starting with deuterium nuclei, but involving a 3-nucleon intermediate species, because the weak binding of deuterium makes the direct fusion of two deuterium nuclei inefficient). There are two sets of such reactions, and you may choose either one.

(h) (5) Assume that all the neutrons at time $t(T/10)$ end up in ^4He nuclei. Using the result of part (f) for n_n at $t(T/10)$, calculate the abundance of primordial ^4He emerging from the early universe. I.e., calculate

$$\frac{\text{mass of } ^4\text{He nuclei}}{\text{mass of } ^1\text{H nuclei} + \text{mass of } ^4\text{He nuclei}}.$$

(You can ignore the relatively small amounts of other isotopes.)

(h) (5) How does your result compare (approximately) with the observations?

2. Fun with black holes

(a) (5) Superstring theorists spend a lot of time on black holes in hypothetical higher-dimensional spaces where the number of spatial dimensions d is greater than 3. Suppose that we imagine a *lower*-dimensional space with $d = 2$, where the metric for a (2-dimensional) black hole is

$$(ds)^2 = c^2 (d\tau)^2 = \left(1 - \frac{R_S}{r}\right) c^2 (dt)^2 - \left(1 - \frac{R_S}{r}\right)^{-1} (dr)^2 - r^2 (d\theta)^2 .$$

Calculate the value of the circumference of the event horizon. (This is the length that you would measure if you were in this two-dimensional world, and you moved around the circumference with your radial coordinate r infinitesimally greater than R_S, eventually returning to your starting point while laying down a tape measure to record how far you had traveled.)

(b) (20) Now let us return to a 3-dimensional black hole. The rate at which it loses energy due to Hawking radiation is

$$\frac{d\left(Mc^2\right)}{dt} = \frac{dE}{dt} = -\text{luminosity } = -\sigma T_H^4 = -\frac{\hbar c^6}{15360 \pi G^2 M^2} .$$

Calculate the time required for the black hole to lose all its energy through Hawking radiation. Your answer should be in terms of G, c, $\hbar$, π, and the initial mass M_0.

[more room on next page if necessary]

(c) (5) Estimate the mass of a black hole with a lifetime equal to the present lifetime of the universe, roughly 10^{10} years.

(d) (5 points extra credit) Estimate the Schwarzschild radius of the black hole in part (c), in fermi (with $1\ \mathrm{f} = 10^{-15}\ \mathrm{m}$).

3. (25) Fun with gravitational collapse

A rotating cloud of gas and dust has a mass M, an initial (average) angular velocity ω_0, and an initial radius R_0. Let us estimate its final size R_f, after gravitational collapse, by using the following crude picture: An "average" particle with radial coordinate $r = R/\sqrt{2}$ has an acceleration given by

$$\frac{d^2r}{dt^2} + a_c = -G\frac{M/2}{r^2} \qquad \text{where} \qquad a_c = -r\omega^2 = -\frac{(r\omega)^2}{r} = -\frac{L^2}{m^2r^3} \qquad \text{and} \qquad L = mr^2\omega = mr_0^2\omega_0 .$$

(The last equation states that the angular momentum L is conserved for this particle with mass m.)

Using this picture, calculate the final radius R_f after gravitational collapse ceases, in terms of G, M, R_0, and ω_0.

4. (20) Fun with uncertainty

The width or uncertainty $\Delta\lambda$ in the wavelength of a spectral line follows from the version of Heisenberg's uncertainty principle which relates the uncertainties in energy and time. Show that the result is

$$\Delta\lambda = \frac{\lambda^2}{2\pi c}\left(\frac{1}{\Delta t_i} + \frac{1}{\Delta t_f}\right)$$

where Δt_i and Δt_f are respectively the lifetimes of the initial and final states of the electron. (Since you are given the answer, your derivation needs to be particularly clear.)

5. (20) Fun with white dwarfs

We derived the equation

$$\rho \frac{d^2 r}{dt^2} = -G \frac{M_r \rho}{r^2} - \frac{dP}{dr}$$

which is useful in various contexts. Here let us just consider a white dwarf whose mass density ρ is taken to be constant. Show that the pressure varies with depth according to

$$P(r) = \text{constant } \times \left(R^2 - r^2\right)$$

while at the same time determining the constant in terms of ρ and G. Here R is the radius of the white dwarf, and r is the distance from its center.

6. (15) Fun with sunspots

Calculate the magnetic pressure in the center of a large sunspot, assuming that the magnetic field strength is 0.2 T. Compare your answer with a typical value of about 10^4 N/m^2 for the gas pressure at the base of the photosphere. (The magnetic pressure is given by the same mathematical expression as the magnetic energy density, $B^2 / 2\mu_0$ in SI units.)

7. Fun with the quantum measurement problem (based on extra credit talks)

A pair of electrons are prepared in the quantum state

$$\frac{1}{\sqrt{2}}\left(\uparrow\downarrow+\downarrow\uparrow\right).$$

The initial state of Alice before she measures the spin of her electron (the one on the left) is 😐 and the initial state of Bob before he measures the spin of his electron (the one on the right) is also 😐.

If Alice observes her electron as $\uparrow$ her state changes to ☺. But if she observes it as $\downarrow$ her state changes to ☹.

The same for Bob and his electron: ☺ if his electron is $\uparrow$ and ☹ if his electron is $\downarrow$.

(a) (5 points extra credit) Write down the state of everything – pair of electrons, Alice, and Bob – before Alice observes her electron.

(b) (5 points extra credit) Now Alice observes her electron, and Bob subsequently observes his. Write down the state of everything after these observations. (Use either the "wavefunction-collapse" interpretation of quantum mechanics, specifying both of the two possible non-deterministic outcomes, or the "many-worlds" interpretation, in which there is a single deterministic outcome, but say which one you are using.)

Have a good summer!