Physics 314 (Survey of Astronomy) Exam 3

Please show all significant steps clearly in all problems.
Please give clear, legible, and reasonably complete (although brief) responses to qualitative questions.
See the last page for values of constants.

1. Jeans' criterion for gravitational collapse

Consider a spherical cloud of gas with temperature T and mass density ρ_m (near the center), and with radius L. The cloud is composed of a single molecular species with molecular mass m.

(a) (5) Using the ideal gas law, obtain the outward pressure P in this gas (near the center) in terms of T, ρ_m, m, and the Boltzmann constant k.

$$PV = nRT \quad \Rightarrow \quad PV = NkT \quad \Rightarrow \quad P = \left(\frac{N}{V}\right)kT = \left(\frac{Nm}{V}\right)\left(\frac{kT}{m}\right) == \left(\frac{M}{V}\right)\left(\frac{kT}{m}\right) = \rho_m \frac{kT}{m}$$

(b) (5) There is also an inward pressure due to the gravitational force on a column of gas of height L and cross-sectional area A. If the density is approximated as ρ_m for the entire height of this column, what is the mass of the gas in the column, M_{column}, in terms of ρ_m, L, and A?

$$M_{column} = \rho_m LA$$

(c)(5) Now suppose that the gravitational force on the gas in this column is estimated by

$$F \sim \frac{GM_{column}M_{gas}}{L^2}.$$

Obtain the corresponding estimate for L_J, the minimum size of a cloud that will undergo gravitational collapse at temperature T, and mass density ρ_m, if the molecular mass is m.

$$P = \frac{F}{A} \sim \frac{G(\rho_m LA)}{A}\frac{\left(\frac{4}{3}\pi L^3\right)\rho_m}{L^2} \sim G(\rho_m L)^2$$

$$\rho_m \frac{kT}{m} < G(\rho_m L)^2 \quad \Rightarrow \quad \frac{kT}{G\rho_m m} < L^2 \quad \Rightarrow \quad L > L_J \equiv \left(\frac{kT}{G\rho_m m}\right)^{1/2}$$

(d) (5) Estimate the value of L_J in light years if the temperature is 4000 K, the mass density is 10^{-18} kg/m^3, and the molecular mass is that that of atomic hydrogen, roughly 2×10^{-27} kg.

Answer is about 60 light years.

(e) (5) Estimate L_J if the temperature is 10 K and the density is still the same as in Part (d) (10^{-18} kg/m^3). Then briefly discuss the significance of the answers to Parts (d) and (e) – i.e., briefly describe the kinds of systems that might have formed under gravitational collapse when the temperatures had these values. (The current temperature of the cosmic background radiation is about 3 K.)

Answer is about 3 light years.

Brief answer: Globular clusters when radiation and matter decouple, stellar systems at a later time.

2. (10) **Inflation in the very early universe**

For the Robertson-Walker metric of cosmology, the Einstein field equations for gravity reduce to the two (Friedmann) equations given immediately below and in the next problem:

$$\left(\frac{\dot{R}}{R}\right)^2 + \frac{k}{R^2} = \frac{8\pi G}{3}\rho .$$

Suppose that the following is true in the very early universe (at about 10^{-35} second): 3-space is flat, and ρ consists of just a vacuum energy density ρ_{vac} which is constant in space and time. (I.e., there is not yet any radiation or matter.)

Solve this differential equation for $R(t)$ via integration, showing your steps.

With $k = 0$, we have

$$\frac{\dot{R}}{R} = H \equiv \left(\frac{8\pi G}{3}\rho_{vac}\right)^{1/2} \quad \Rightarrow \quad \frac{dR}{R} = H dt \quad \Rightarrow \quad \ln R = Ht + \ln c \quad \Rightarrow \quad R = ce^{Ht}$$

$$\text{or} \quad R \propto e^{Ht} .$$

3. (10) **Acceleration in the current universe**

The second (Friedmann) equation is

$$\frac{\ddot{R}}{R} = -\frac{4\pi G}{3}(\rho + 3p).$$

According to recent observations, most of the energy density in the current universe is “dark energy”. Let us suppose that this “dark energy” is vacuum energy (again constant in space and time), with the equation of state $p = -\rho$. Finally, let us estimate the behavior of the cosmic scale factor R by considering *only* the vacuum energy.

With $\rho = \rho_{vac}$=constant, find the solution $R(t)$ to the above differential equation.

This time you are allowed to guess a solution and just show that it works (whereas in Problem 2 you are required to derive the solution via integration, showing your steps.).

Substitution of the solution to Problem 2 shows that it works.

4. **Cosmic rays in interstellar space**

A cosmic ray proton of charge e and rest mass m_0 moves in a circular orbit, in a plane perpendicular to a uniform magnetic field $\vec{B}$. Recall that the Lorentz force is given by $\vec{F} = e\,\vec{v} \times \vec{B}$ (in SI units). You may just assume the general formula for the centripetal force for a circular orbit:

$$F = \frac{pv}{r} \quad \text{with} \quad p = mv \ , \ m = \gamma m_0 \ , \ \gamma = \frac{1}{\sqrt{1-(v/c)^2}} \ .$$

(a) (6) Recalling that $p = mv$, and that $v = r\omega$, calculate the angular velocity ω in terms of e, B, γ, and m_0.

$$\frac{mv^2}{r} = evB \quad \Rightarrow \quad \frac{\gamma m_0 v}{r} = eB \quad \Rightarrow \quad \omega = \frac{eB}{\gamma m_0}$$

(b) (6) Calculate the radius r of the orbit in terms of the same quantities and v.

$$r = \frac{\gamma m_0 v}{eB}$$

(c) (4) Calculate the numerical value of ω for an ultrarelativistic proton with $\gamma = 100\,000$, in an interstellar magnetic field with a strength $B = 2 \times 10^{-10}$ tesla. Then determine the period of revolution $T = 2\pi / \omega$ in years.

$$\omega = 2\pi / T, \quad T \approx 1 \text{ year}$$

(d) (4) For the same proton, calculate the numerical value of the radius of the orbit r, giving your final answer in light years.

About 1/6 light year.

(e) (5) Now consider an **electron** in the **Earth's** magnetic field of ~ 1 gauss $= 10^{-4}$ tesla, with $\gamma = 10\,000$. What is the radius of its orbit in km, and how does this compare with the radius of the Earth (which is 6370 km)?

About 200 km.

5. **Galaxies and HII regions**

(a) (6) What are the 3 broad types of galaxies?

Brief answer: elliptical, spiral, irregular.

(b) (4) Draw a rough sketch of a spiral galaxy like our own, and indicate where the very oldest, moderately old, and youngest stars are likely to be found. Which part of a spiral galaxy tends to be more blue and which more yellow or red?

See Fig. 12.4 and related material in Chapter 12.

(c) (5) What typically gives rise to an HII region, or Stromgren sphere?

See, e.g., pp. 216-217.

6. Density of matter, radiation, and vacuum energy as functions of time as the universe expands

If $R(t)$ is the cosmic scale factor, and $\rho(t)$ is the energy density, then the total energy in all of 3-space at a given time t is proportional to ρR^3. According to the first law of thermodynamics, this energy is decreased by an amount equal to the corresponding work done in the expansion of the universe, $P\,dV = P\,d\left(R^3\right)$. (Recall that when a piston with cross-sectional area A expands by an amount dx against a pressure P, the work done is $F\,dx = (F/A)(A\,dx) = P\,dV$.) We then have

$$d\left(\rho R^3\right) = -P d\left(R^3\right) \quad or \quad \frac{d}{dt}\left(\rho R^3\right) = -P\frac{d}{dt}\left(R^3\right).$$

In each of the following parts, the answer should be written as

$$\rho \propto \frac{1}{R^n} \quad or\ else \quad \rho = \text{constant}$$

where you are to determine n.

(a) (5) For matter, $P = 0$ (to a good approximation). How does ρ depend on R in this case?

See solution to Problem 16.6.

(b) (5) For radiation, $P = \rho / 3$. How does ρ depend on R in this case?

See solution to Problem 16.6.

(c) (5) For the vacuum energy, $P = -\rho$. How does ρ depend on R in this case?

After a little calculus and algebra, $\rho_{vac} = \text{constant}$.