NEURAL CODES AND THE FACTOR COMPLEX

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ABSTRACT. We introduce the *factor complex* of a neural code, and show how intervals and maximal codewords are captured by the combinatorics of factor complexes. We use these results to obtain algebraic and combinatorial characterizations of max-intersection-complete codes, as well as a new combinatorial characterization of intersection-complete codes.

1. INTRODUCTION

A *neural code* on n neurons is a subset of $2^{[n]}$, where $[n] = \{1, 2, \dots, n\}$; determining which neural codes are convex remains a central open problem in this area. The broadest family of codes known to be convex consists of *max-intersection-complete codes*, those codes closed under taking intersections of maximal elements [2, 4]. Recently, Curto et al. [4] asked for an algebraic signature for max-intersection-complete codes.

Here we answer the question of Curto et al. Our main result, Theorem 1.1 below, gives a characterization for when a code is max-intersection-complete in terms of the canonical form of its neural ideal (Definitions 2.3 and 2.4) and the Stanley–Reisner ideal $I(\Delta(C))$ of its simplicial complex $\Delta(C)$ (Definitions 2.7 and 2.8).

Theorem 1.1. *A code C on n neurons is max-intersection-complete if and only if for every non-monomial ϕ in the canonical form of the neural ideal of C, there exists $i \in [n]$ such that*

(i) every associated prime of $I(\Delta(C))$ that contains x_i also contains ϕ, and
(ii) $(1 - x_i) \mid \phi$.

We remark that, if the maximal codewords of a code C as well as its canonical form $\mathrm{CF}(J_C)$ are given as input, Theorem 1.1 can be turned into an algorithm to verify whether a code is max-intersection-complete, whose run time is polynomial in the input size. In order to determine whether this algorithm is more efficient than brute-force checking of intersections of maximal codewords, one needs to understand both the complexity of computing $\mathrm{CF}(J_C)$ (the current best algorithm, in [17], is exponential), and the complexity of $\mathrm{CF}(J_C)$ itself. For instance, it would be useful to know a bound for the number of (non-monomial) pseudomonomials in $\mathrm{CF}(J_C)$ in terms of the number of maximal codewords of C.

To prove Theorem 1.1, which translates a property of a code to a property of its neural ideal, we introduce a new combinatorial object, the *factor complex* of a code. This is a simplicial complex that, like the neural ideal but unlike $\Delta(C)$, captures all the combinatorial information in a code C. We are therefore able to elucidate the relationships among codes, their factor complexes, and their related ideals (neural ideals and Stanley–Reisner ideals) – and then use these results to characterize being max-intersection-complete in terms of the factor complex. Finally, this combinatorial criterion directly translates into an algebraic criterion, Theorem 1.1 above.

Along the way, we give a new characterization of *intersection-complete* codes – those codes that are closed under taking intersections of codewords. Our characterization is combinatorial, via the factor complex, in contrast to a prior algebraic characterization through the neural ideal [4]. Indeed, we expect in the future that the factor complex may help us understand more properties of neural codes.

Our work fits into the literature on neural codes as follows. Like previous works, we are motivated by the question of convexity in neural codes [3, 6, 13, 14, 15, 18, 19], with a specific interest in using neural ideals to study convexity [5, 7, 8, 10, 11, 16]. Also, our factor complexes are motivated by the closely related *polar complexes* introduced recently by Güntürkün et al. [9] (see also [1, 11]).

Outline. This article is organized as follows. Section 2 contains background material, and Section 3 gives our main results. In Section 4, we prove relationships among codes, their factor complexes, and their neural or Stanley-Reisner ideals, and Section 5 relates factor complexes and polar complexes.

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2. Background

Throughout this article, C is a neural code on n neurons, that is, a subset of $2^{[n]}$, where $[n] = \{1, 2, \dots, n\}$. Elements of C are called *codewords*, and may be represented as subsets of $[n]$ or as n-tuples of zeros and ones, where a 1 in position i indicates that i belongs to the codeword.

Given $c \subset d \subset [n]$, the *Boolean interval* between c and d is

$$[c, d] := \{w \in 2^{[n]} \mid c \subset w \subset d\}.$$

The *complement* of a code C on n neurons is the code

$$C' := 2^{[n]} \smallsetminus C. \tag{1}$$

Convention. In this article, we assume that $\varnothing \subsetneq C \subsetneq 2^{[n]}$, so that the neural ideals (defined below) of C and C' have primary decompositions.

Definition 2.1. Let C be a code. The *intervals* of C are the Boolean intervals contained in C. The *maximal intervals* of C are the intervals of C that are maximal with respect to inclusion.

Example 2.2. For the code $C = \{\varnothing, 2, 3, 12, 13\} = \{000, 010, 001, 110, 101\}$, the maximal intervals are $[\varnothing, 2]$, $[\varnothing, 3]$, $[2, 12]$, and $[3, 13]$.

2.1. **Neural ideals and the canonical form.** The main reference for this section is [5].

We denote by $\mathbb{F}_2$ the field with two elements, and let $R = \mathbb{F}_2[x_1, \dots, x_n] = \mathbb{F}_2[x]$. A *pseudomonomial* is a polynomial $\prod_{i\in\sigma} x_i \prod_{j\in\tau}(1 - x_j) \in R$, where $\sigma, \tau \subset [n]$ are disjoint. A *pseudomonomial ideal* is an ideal generated by pseudomonomials. If $c \in 2^{[n]}$, the pseudomonomial

$$\phi_c := \prod_{i\in c} x_i \prod_{j\in[n]\setminus c} (1 - x_j) \tag{2}$$

is called the *indicator polynomial* of c.

Definition 2.3. The *neural ideal* J_C of a code C is the (pseudomonomial) ideal generated by the indicator polynomials of its non-codewords; in symbols,

$$J_C := \langle \phi_c \mid c \in C' \rangle.$$

Note that, using the convention that n-tuples of zeros and ones represent codewords, the zero-set of J_C is C. In other words, the code C and its neural ideal contain the same information. Moreover, any ideal generated by pseudomonomials is the neural ideal of a code [12, Theorem 2.1].

The neural ideal J_C has a unique irredundant decomposition

$$J_C = \bigcap_{h=1}^{g} P_h, \tag{3}$$

where each P_h is a pseudomonomial ideal that is prime [5, Proposition 6.8]. In particular, J_C is a radical ideal. We remark that a pseudomonomial ideal P is prime if and only if it is of the form

$$P = \langle \{x_i \mid i \in \sigma\} \cup \{(1 - x_j) \mid j \in \tau\} \rangle \quad \text{for } \sigma, \tau \text{ disjoint subsets of } [n]. \tag{4}$$

Definition 2.4. Let $J \subset R$ be a pseudomonomial ideal. A pseudomonomial in J is *minimal* if it is minimal with respect to divisibility among all pseudomonomials in J. The *canonical form* of J is the set $\mathrm{CF}(J)$ of all minimal pseudomonomials of J.

The canonical form of a pseudomonomial ideal is a generating set for the ideal [5].

Example 2.5 (Example 2.2, continued)**.** The complement of the code $C = \{\varnothing, 2, 3, 12, 13\}$ is $C' = \{1, 23, 123\}$. Thus, the neural ideal of C is $J_C = \langle x_1(1 - x_2)(1 - x_3),\ x_2x_3(1 - x_1),\ x_1x_2x_3 \rangle$, and the canonical form is $\mathrm{CF}(J_C) = \{x_1(1 - x_2)(1 - x_3),\ x_2x_3\}$.

2.2. **Polarization and squarefree monomial ideals.** Let $S = \mathbb{F}_2[x_1, \dots, x_n, y_1, \dots, y_n] = \mathbb{F}_2[x, y]$.

The following construction was introduced in [9].

Definition 2.6. The *polarization* of a pseudomonomial $\phi = \prod_{i\in\sigma} x_i \prod_{j\in\tau}(1 - x_j) \in R$ is

$$\mathcal{P}(\phi) := \prod_{i\in\sigma} x_i \prod_{j\in\tau} y_j \in S.$$

If $J \subset R$ is a pseudomonomial ideal, the *polarization* of J is the ideal in S obtained by polarizing the pseudomonomials in the canonical form of J, that is,

$$\mathcal{P}(J) := \langle \mathcal{P}(\phi) \mid \phi \in \mathrm{CF}(J) \rangle \subset S.$$

Note that the polarization of a pseudomonomial ideal is a *squarefree* monomial ideal in S, that is, an ideal generated by monomials that are not divisible by the squares of the variables (so, $\mathcal{P}(J)$ is radical). We recall the relationship between squarefree monomial ideals and simplicial complexes.

Definition 2.7. Let Δ be a simplicial complex on $[n]$, and let $\Bbbk$ be a field. The *Stanley–Reisner ideal* of Δ is

$$I(\Delta) := \langle \prod_{i\in\sigma} x_i \mid \sigma \notin \Delta \rangle \subset \Bbbk[x_1, \dots, x_n].$$

The ideal $I(\Delta)$ is radical, with prime decomposition

$$I(\Delta) = \bigcap_{\sigma\in\text{Facets}(\Delta)} \langle x_i \mid i \notin \sigma \rangle. \tag{5}$$

It follows that Δ can be recovered from $I(\Delta)$. In fact, (5) can be used to conclude that any squarefree monomial ideal is the Stanley–Reisner ideal of some simplicial complex.

Definition 2.8. The *simplicial complex of a code* C is $\Delta(C)$, the smallest simplicial complex containing C. Its Stanley–Reisner ideal is denoted by $I(\Delta(C)) \subset R = \mathbb{F}_2[x]$.

It is a fact that $I(\Delta(C))$ is generated by the monomials in $\text{CF}(J_C)$ [5, Lemma 4.4].

Example 2.9 (Example 2.5, continued)**.** For $C = \{\varnothing, 2, 3, 12, 13\}$, the simplicial complex $\Delta(C)$ has two facets, 12 and 13. The corresponding Stanley–Reisner ideal is $I(\Delta(C)) = \langle x_2x_3\rangle$, which is generated by the unique monomial in the canonical form $\text{CF}(J_C) = \{x_1(1-x_2)(1-x_3),\ x_2x_3\}$.

In this article, we work with squarefree monomial ideals in $S = \mathbb{F}_2[x, y]$ that arise from polarization. In order to construct their corresponding simplicial complexes, we use $\{1, \dots, n, \overline{1}, \dots, \overline{n}\}$ as a vertex set, with the understanding that x_i corresponds to i, and y_i corresponds to $\overline{i}$. If $B \subset [n]$, we denote $\overline{B} = \{\overline{i} \mid i \in B\}$. In particular,

$$\overline{[n]} = \{\overline{1}, \dots, \overline{n}\} \quad \text{and} \quad [n] \cup \overline{[n]} = \{1, \dots, n, \overline{1}, \dots, \overline{n}\}.$$

We always use overline notation to denote subsets of $\overline{[n]}$; this is justified, as any subset of $\overline{[n]}$ is of the form $\overline{B}$ for some $B \subset [n]$.

Remark 2.10. As noted above, the ideals that are associated to codes (the neural ideal J_C, the ideal $I(\Delta(C))$, and later the factor ideal $\text{FI}(C)$) are *radical ideals*, that is, they can be expressed as intersections of prime ideals. We emphasize that the sets of associated primes, minimal primes, and primary components of a radical ideal all coincide.

3. Main results

In this section we introduce a new combinatorial tool to study neural codes: the factor complex (Definition 3.1), and state our four main results. Theorems 3.3 and 3.4 summarize the relationships among codes, their factor complexes, and their related ideals (neural ideals and Stanley–Reisner ideals). These results are used to prove Theorems 3.6 and 3.7, which characterize intersection-complete codes and max-intersection-complete codes in two ways: combinatorially and algebraically.

Definition 3.1. Let C be a code on n neurons, and recall the primary decomposition of the neural ideal J_C given in (3). The *factor ideal* of C is obtained by polarizing the components of J_C, namely,

$$\text{FI}(C) := \bigcap_{h=1}^{g} \mathcal{P}(P_h).$$

The *factor complex* $\Delta_\cap(C)$ of C is the simplicial complex on $[n] \cup \overline{[n]}$ whose Stanley–Reisner ideal is $\mathrm{FI}(C)$. A face of $\Delta_\cap(C)$ is *defective* if it contains neither i nor $\bar{i}$ for some $i \in [n]$ (we think of i as a defect, or flaw); faces that are not defective are called *effective*. We say that $\overline{B} \subset \overline{[n]}$ is a *prime-set* of $\Delta_\cap(C)$ if $[n] \cup \overline{B} \notin \Delta_\cap(C)$, and $\overline{B}$ is furthermore *minimal* if $\overline{B}$ is minimal with respect to inclusion among prime-sets. Lemma 4.5 gives the reason why we chose this terminology.

Example 3.2 (Example 2.9, continued)**.** For $C' = \{1, 23, 123\}$, the neural ideal decomposes as follows:

$$J_{C'} = \langle (1-x_1)(1-x_3),\ (1-x_1)(1-x_2),\ x_2(1-x_3),\ x_3(1-x_2) \rangle = \langle x_2, x_3,\ 1-x_1 \rangle \cap \langle 1-x_2,\ 1-x_3 \rangle.$$

The factor ideal is therefore

$$FI(C') = \langle x_2, x_3, y_1 \rangle \cap \langle y_2, y_3 \rangle,$$

and so the two facets of the factor complex $\Delta_\cap(C')$ are $1\bar{2}\bar{3}$ and $123\bar{1}$ (both are effective). The minimal prime-sets of $\Delta_\cap(C')$ are $\{\bar{2}\}$ and $\{\bar{3}\}$.

Theorem 3.3 (Codes, factor complexes, and neural ideals)**.** *Let C be a code on n neurons, and C' its complement code defined in* (1)*. The following two maps are bijections:*

$$\begin{array}{ccccc} \{\textit{pseudomonomials in } J_{C'}\} & \leftarrow & \{\textit{intervals in } C\} & \rightarrow & \{\textit{effective faces of } \Delta_\cap(C)\} \\ \prod_{i\in c} x_i \prod_{j\in[n]\smallsetminus d}(1-x_j) & \leftarrow\!\shortmid & [c,d] & \mapsto & d \cup \overline{[n]\smallsetminus c} \end{array}$$

Moreover, every facet of $\Delta_\cap(C)$ is effective, and the following are equivalent

(1) *$[c,d]$ is a maximal interval in C,*
(2) *$\prod_{i\in c} x_i \prod_{j\in[n]\smallsetminus d}(1-x_j) \in \mathrm{CF}(J_{C'})$, and*
(3) *$d \cup \overline{[n]\smallsetminus c}$ is a facet of $\Delta_\cap(C)$.*

Theorem 3.4 (Codes, factor complexes, and Stanley–Reisner ideals)**.** *Let C be a code on n neurons, with complement code C' and factor complex $\Delta_\cap(C)$. The following two maps are bijections:*

$$\begin{array}{ccccc} \{\textit{minimal primes of } I(\Delta(C))\} & \leftarrow & \{\textit{maximal codewords of } C\} & \rightarrow & \left\{ \begin{array}{c}\textit{minimal prime-sets} \\ \textit{of } \Delta_\cap(C')\end{array} \right\} \\ \langle x_i \mid i \in [n]\smallsetminus M \rangle & \leftarrow\!\shortmid & M & \mapsto & \overline{[n]\smallsetminus M} \end{array}$$

The proofs of Theorems 3.3 and 3.4 are postponed until Sections 4.1 and 4.2 respectively.

Example 3.5 (Example 3.2, continued)**.** According to Theorem 3.3, the facets $1\bar{2}\bar{3}$ and $123\bar{1}$ of $\Delta_\cap(C')$ correspond to the two maximal intervals of C', $[1,1]$ and $[23,123]$, respectively, and also to the two pseudomonomials in $\mathrm{CF}(J_C)$, namely, $x_1(1-x_2)(1-x_3)$ and x_2x_3, respectively.

Similarly, Theorem 3.4 implies that the minimal prime-sets $\{\bar{2}\}$ and $\{\bar{3}\}$ of $\Delta_\cap(C')$ correspond to the minimal primes $\langle x_2 \rangle$ and $\langle x_3 \rangle$ of $I(\Delta(C)) = \langle x_2x_3 \rangle$ and also to the maximal codewords 13 and 12 of C.

The following result translates the algebraic characterization of intersection-complete codes from [4] into a new combinatorial criterion.

Theorem 3.6 (Intersection-complete codes)**.** *Let C be a code on n neurons with neural ideal J_C, and let C' be the complement code of C with factor complex $\Delta_\cap(C')$. The following are equivalent:*

(1) *C is intersection-complete,*

(2) *every pseudomonomial* $\prod_{i\in\sigma} x_i \prod_{j\in\tau}(1-x_j)$ *in* $\mathrm{CF}(J_C)$ *satisfies* $|\tau| \le 1$, *and*
(3) *every facet* F *of* $\Delta_\cap(C')$ *satisfies* $|F \cap [n]| \ge n-1$.

Proof. The equivalence between *(1)* and *(2)* is [4, Theorem 1.9]. By Theorem 3.3, $\prod_{i\in\sigma} x_i \prod_{j\in\tau}(1-x_j)$ belongs to the canonical form of J_C if and only if $F = [n] \smallsetminus \tau \cup \overline{[n] \smallsetminus \sigma}$ is a facet of $\Delta_\cap(C')$. Thus, the condition $|\tau| \le 1$ is equivalent to $|F \cap [n]| \ge n-1$, and so *(2)* is equivalent to *(3)*. □

The following result is an expanded version of Theorem 1.1.

Theorem 3.7 (Max-intersection-complete codes)**.** *Let C be a code on n neurons with neural ideal J_C, and let C' be the complement code of C with factor complex $\Delta_\cap(C')$. The following are equivalent:*

(1) *C is max-intersection-complete,*
(2) *for every facet F of $\Delta_\cap(C')$ that does* <u>*not*</u> *contain $[n]$, there exists $i \in [n]$ such that*
   (i) *every minimal prime-set of $\Delta_\cap(C')$ that contains $\overline{i}$ also contains some $\overline{j}$ such that $\overline{j} \notin F$, and*
   (ii) *$i \notin F$,*
(3) *for every $\phi \in \mathrm{CF}(J_C)$ that is* <u>*not*</u> *a monomial, there exists $i \in [n]$ such that*
   (i) *every minimal prime of $I(\Delta(C))$ that contains x_i also contains ϕ, and*
   (ii) *$(1-x_i) \mid \phi$.*

Proof. We begin by proving *(2)⇔(3)*. By Theorem 3.3, $\phi = \prod_{i\in c} x_i \prod_{j\in[n]\smallsetminus d}(1-x_j) \in \mathrm{CF}(J_C)$ if and only if $F = d \cup \overline{[n] \smallsetminus c}$ is a facet of $\Delta_\cap(C')$. Furthermore, ϕ is non-monomial exactly when $d \not\supseteq [n]$, if and only if F does not contain $[n]$. Thus, by inspection of ϕ and F, *(2)(ii)* is equivalent to *(3)(ii)*, and so we need only show *(2)(i)⇔(3)(i)*.

By Theorem 3.4, the prime ideal $P = \langle x_j \mid j \in B\rangle$ is associated to $I(\Delta(C))$ if and only if $\overline{B}$ is a minimal prime-set of $\Delta_\cap(C')$. Thus, $x_i \in P$ exactly when $\overline{i} \in \overline{B}$. Next, it is straightforward to check that P contains $\phi = \prod_{i\in c} x_i \prod_{j\in[n]\smallsetminus d}(1-x_j)$ if and only if $B \cap c \ne \varnothing$. As ϕ corresponds to the facet $F = d \cup \overline{[n] \smallsetminus c}$ of $\Delta_\cap(C')$, it follows that P contains ϕ if and only if $\overline{j} \notin F$ for some $\overline{j} \in \overline{B}$. This concludes the proof of *(2)⇔(3)*.

We set up notation needed to prove *(1)⇔(2)*. Let $\overline{B}_1, \overline{B}_2, \ldots, \overline{B}_u$ be the minimal prime-sets of $\Delta_\cap(C')$. By Theorem 3.4, the maximal codewords of C are $m_1 = [n] \smallsetminus B_1, \ldots, m_u = [n] \smallsetminus B_u$.

We claim that *(2)* is equivalent to the following:

(2') for every facet F of $\Delta_\cap(C')$ that does not contain $[n]$,

$$([n] \smallsetminus \bigcup_{v\in H_F} B_v) \notin F, \tag{$\star$}$$

where

$$H_F := \{v \in [u] \mid \overline{B}_v \subset F\}.$$

Indeed, condition ($\star$) states that there exists $i \in [n]$ such that $i \notin F$ and $\overline{i}$ is <u>not</u> in any minimal prime-set $\overline{B}_v \subset \{\overline{1}, \overline{2}, \ldots, \overline{n}\}$ for which $\overline{B}_v \subset F$. This latter condition exactly matches *(2)(i)*. Hence, our claim holds, and we may complete this proof by showing *(1)⇔(2')*.

($\Leftarrow$) We prove the contrapositive. Suppose that the intersection of maximal codewords $c = \bigcap_{v\in V} m_v$ (for some $\varnothing \neq V \subset [u]$) is not in C, that is, $c \in C'$. By Theorem 3.3, $c \cup \overline{[n] \setminus c}$ is a face of $\Delta_\cap(C')$. Note that

$$\overline{[n] \setminus c} = \overline{[n] \setminus \bigcap_{v\in V} m_v} = \bigcup_{v\in V} \overline{[n] \setminus m_v} = \bigcup_{v\in V} \overline{B}_v. \tag{6}$$

Let F be a facet of $\Delta_\cap(C')$ containing $c \cup \overline{[n] \setminus c}$. It follows from (6) that F contains the union of minimal prime-sets $\bigcup_{v\in V} \overline{B}_v$, which implies that F does not contain $[n]$ (as, otherwise, each $\overline{B}_v \cup [n]$ is contained in F and hence is a face of $\Delta_\cap(C')$, contradicting the fact that B_v is a prime-set). Since $F \supset \overline{[n] \setminus c} = \bigcup_{v\in V} \overline{B}_v$, we have that $V \subset H_F$. Therefore, $[n] \setminus \bigcup_{v\in H_F} B_v \subset [n] \setminus \bigcup_{v\in V} B_v = c$, where the equality comes from (6). We conclude that F is a facet of $\Delta_\cap(C')$ not containing $[n]$ such that $([n] \setminus \bigcup_{v\in H_F} B_v) \subset c \subset (c \cup \overline{[n] \setminus c}) \subset F$.

($\Rightarrow$) Suppose C is max-intersection-complete. Let F be a facet of $\Delta_\cap(C')$ that does not contain $[n]$. Set $c := [n] \setminus \bigcup_{v\in H_F} B_v$. Our goal is to show that $c \notin F$.

We accomplish this by proving two facts. First, that $c \cup \overline{[n] \setminus c}$ is not a face of $\Delta_\cap(C')$, and second, that $\overline{[n] \setminus c} = \bigcup_{v\in H_F} \overline{B}_v$. The first fact implies that $c \cup \overline{[n] \setminus c} \notin F$ and the second yields $\overline{[n] \setminus c} \subset F$. Our desired relation $c \notin F$ will then follow.

For the first fact, recall that $[n] \setminus B_v = m_v$. Therefore,

$$c = [n] \setminus \bigcup_{v\in H_F} B_v = \bigcap_{v\in H_F} [n] \setminus B_v = \bigcap_{v\in H_F} m_v,$$

so c is the intersection of maximal codewords. As C is max-intersection-complete, $c \in C$, and thus $c \notin C'$. Now Theorem 3.3 implies that $c \cup \overline{[n] \setminus c} \notin \Delta_\cap(C')$.

For the second fact, $\overline{[n] \setminus c} = \overline{[n] \setminus ([n] \setminus \bigcup_{v\in H_F} B_v)} = \overline{\bigcup_{v\in H_F} B_v} = \bigcup_{v\in H_F} \overline{B}_v$. □

Example 3.8 (Example 3.5, continued)**.** The code $C = \{\varnothing, 2, 3, 12, 13\}$ is neither intersection-complete nor max-intersection-complete (as $1 = 12 \cap 13 \notin C$). We can read this information from Theorems 3.6 and 3.7, as follows. For non-intersection-completeness, this can be seen in two ways: first, the pseudomonomial $x_1(1-x_2)(1-x_3)$ is in the canonical form of J_C, and, second, the intersection of the facet $1\bar{2}\bar{3}$ with 123 has size 1, rather than 2 or 3.

For non-max-intersection-completeness, recall that the minimal prime-sets of $\Delta_\cap(C')$ are $\{\bar{2}\}$ and $\{\bar{3}\}$ (equivalently, the minimal primes of $I(\Delta(C))$ are $\langle x_2\rangle$ and $\langle x_3\rangle$). Now, $1\bar{2}\bar{3}$ is a facet of $\Delta_\cap(C')$ that does not contain 123, but for $i \in \{1,2,3\}$, either part *(2)(i)* of Theorem 3.7 is violated (when $i = 2,3$) or part *(2)(ii)* is violated (when $i = 1$). Alternatively, $\mathrm{CF}(J_C)$ contains the non-monomial $x_1(1-x_2)(1-x_3)$, but for $i \in \{1,2,3\}$, either part *(3)(i)* of Theorem 3.7 is violated (when $i = 2,3$) or part *(3)(ii)* is violated (when $i = 1$). Thus, C is not max-intersection-complete.

4. Factor complexes, neural ideals, and codes

In this section, we prove Theorems 3.3 and 3.4.

4.1. **Proof of Theorem 3.3.** We wish to prove that the following maps are bijections:

$$\begin{array}{ccccc} \{\text{pseudomonomials in } J_{C'}\} & \overset{\alpha}{\leftarrow} & \{\text{intervals in } C\} & \overset{\beta}{\rightarrow} & \{\text{effective faces of } \Delta_\cap(C)\} \\ \prod_{i\in c} x_i \prod_{j\in[n]\setminus d}(1-x_j) & \leftarrow\!\shortmid & [c,d] & \mapsto & d\cup\overline{[n]\setminus c} \end{array}$$

The fact that α is a bijection is straightforward from [5, Lemma 5.7]. To show that β is a bijection, we need to better understand the factor ideal and factor complex of C.

Lemma 4.1. *Let C be a code with neural ideal J_C, and let ϕ be a pseudomonomial. Then $\phi \in J_C$ if and only if $\mathcal{P}(\phi) \in \mathrm{FI}(C)$.*

Proof. Recall the decomposition $J_C = \bigcap_{h=1}^{g} P_h$ from (3). Hence, $\phi \in J_C$ if and only if $\phi \in P_h$ for all h. Given the form (4) of each component P_h, it is straightforward to check that $\phi \in P_h$ is equivalent to $\mathcal{P}(\phi) \in \mathcal{P}(P_h)$. Thus, as $\mathrm{FI}(C) = \bigcap \mathcal{P}(P_h)$, the desired result follows. □

Our next results shows how to use the factor complex of a code to read off its codewords.

Lemma 4.2. *Let C be a code on n neurons. Then $c \in 2^{[n]}$ is a codeword of C if and only if $c \cup \overline{[n]\setminus c}$ is a face of $\Delta_\cap(C)$.*

Proof. By [5, Lemma 3.2], $c \in C$ if and only if $\phi_c = \prod_{i\in c} x_i \prod_{j\notin c}(1-x_j) \notin J_C$. This is equivalent to $\mathcal{P}(\phi_c) \notin \mathrm{FI}(C)$ by Lemma 4.1. Since $\mathrm{FI}(C)$ is the Stanley–Reisner ideal of $\Delta_\cap(C)$, we have that $\mathcal{P}(\phi_c) \notin \mathrm{FI}(C)$ exactly when $c \cup \overline{[n]\setminus c}$ is a face of $\Delta_\cap(C)$, which concludes the proof. □

We now extend Lemma 4.2 to show how to extract the intervals of C from its factor complex.

Lemma 4.3. *(Interval-Face Correspondence) Let C be a code on n neurons, and let $c, d \in 2^{[n]}$. Then $[c,d] \subset C$ if and only if $d \cup \overline{[n]\setminus c}$ is a face of $\Delta_\cap(C)$.*

Proof. ($\Leftarrow$) Suppose $d\cup\overline{[n]\setminus c}$ is a face of $\Delta_\cap(C)$, and let $w \in [c,d]$. Then $w\cup\overline{[n]\setminus w} \subset d\cup\overline{[n]\setminus c}$ is a face of $\Delta_\cap(C)$ and thus $w \in C$ by Lemma 4.2.

($\Rightarrow$) We now assume that $d \cup \overline{[n]\setminus c}$ is not a face of $\Delta_\cap(C)$ and show that $[c,d]$ is not an interval of C. As $\mathrm{FI}(C)$ is the Stanley–Reisner ideal of $\Delta_\cap(C)$, the decomposition (5) implies that the ideal

$$\left\langle \{x_i \mid i \notin d \cup \overline{[n]\setminus c}\} \cup \{y_j \mid \bar{j} \notin d \cup \overline{[n]\setminus c}\} \right\rangle = \left\langle \{x_i \mid i \in [n]\setminus d\} \cup \{y_j \mid j \in c\} \right\rangle$$

is <u>not</u> associated to $\mathrm{FI}(C)$, and therefore the following ideal is <u>not</u> associated to J_C:

$$\left\langle \{x_i \mid i \in [n]\setminus d\} \cup \{(1-x_j) \mid j \in c\} \right\rangle. \tag{7}$$

Thus, as $\mathrm{CF}(J_C)$ is a generating set for J_C, there exists a pseudomonomial $\phi = \prod_{i\in\sigma} x_i \prod_{j\in\tau}(1-x_j)$ in $\mathrm{CF}(J_C)$ that is not in the ideal (7), and so $\sigma \subset d$ and $\tau \subset [n]\setminus c$. Note that the indicator pseudomonomial $\phi_{c\cup\sigma}$ is in J_C, as it is divisible by ϕ. We conclude that $\sigma \cup c \in [c,d]\setminus C$, and so $[c,d] \notin C$. □

We can now better understand the facets of $\Delta_\cap(C)$.

Lemma 4.4. *Let C be a code on n neurons. Every facet of $\Delta_\cap(C)$ is effective.*

Proof. By (5), the facets of $\Delta_\cap(C)$ correspond to associated primes of $\mathrm{FI}(C)$, which are polarizations of associated primes of J_C. Since the latter primes cannot contain both x_ℓ and $1 - x_\ell$, it follows that the former primes cannot contain both x_ℓ and y_ℓ, which concludes the proof. □

Proof of Theorem 3.3. By [5, Lemma 5.7], the map α is a bijection, and the correspondence between minimal pseudomonomials and maximal intervals follows from the fact for any two intervals M_1 and M_2 of C, we have $M_1 \subset M_2$ if and only if $\alpha(M_2) \mid \alpha(M_1)$. By Lemma 4.3, plus the fact that effective faces have the form $d \cup \overline{[n] \smallsetminus c}$ for some $c \subset d$, the map β is also a bijection. Lemma 4.4 states that all facets of $\Delta_\cap(C)$ are effective, and thus for each facet F we have $F = \beta(M)$ for some interval M of C. The correspondence between facets and maximal intervals then follows from the fact that for intervals M_1 and M_2 of C, we have $M_1 \subset M_2$ if and only if $\beta(M_1) \subset \beta(M_2)$. □

4.2. **Proof of Theorem 3.4.** We wish to show that the maps

$$\begin{array}{ccccc} \{\text{minimal primes of } I(\Delta(C))\} & \xleftarrow{\gamma} & \{\text{maximal codewords in } C\} & \xrightarrow{\delta} & \left\{\begin{array}{c}\text{minimal prime-sets}\\ \text{of } \Delta_\cap(C')\end{array}\right\} \\ \langle x_i \mid i \in [n] \smallsetminus M\rangle & \leftarrow\!\shortmid & M & \mapsto & \overline{[n] \smallsetminus M} \end{array}$$

are bijections. The main step is to understand the relationship between the prime-sets of $\Delta_\cap(C')$ and the associated primes of $I(\Delta(C))$.

Lemma 4.5. *Let C be a code on n neurons with complement code C'. A subset $\overline{B} \subset \overline{[n]}$ is a prime-set of $\Delta_\cap(C')$ if and only if $\langle x_i \mid i \in B\rangle$ contains $I(\Delta(C))$. Consequently, $\overline{B}$ is a minimal prime-set of $\Delta_\cap(C')$ if and only if $\langle x_i \mid i \in B\rangle$ is a minimal prime of $I(\Delta(C))$.*

Proof. By definition, $\overline{B}$ is a prime-set of $\Delta_\cap(C')$ if and only if $[n] \cup \overline{B}$ is not a face of $\Delta_\cap(C')$. Equivalently, every facet of $\Delta_\cap(C')$ of the form $F = [n] \cup \overline{[n] \smallsetminus c}$ satisfies $B \cap c \neq \varnothing$. By Theorem 3.3, $F = [n] \cup \overline{[n] \smallsetminus c}$ is a facet of $\Delta_\cap(C')$ if and only if the monomial $\prod_{i \in c} x_i$ belongs to $\mathrm{CF}(J_C)$. Also, $B \cap c \neq \varnothing$ if and only if $\prod_{j \in c} x_j \in \langle x_i \mid i \in B\rangle$. Now the result follows, because the monomials in $\mathrm{CF}(J_C)$ generate $I(\Delta(C))$. □

Proof of Theorem 3.4. The map γ is a bijection, by (5) and the fact that maximal codewords of C are facets of $\Delta(C)$, and $I(\Delta(C))$ is its Stanley–Reisner ideal. Given that γ is a bijection, Lemma 4.5 shows that $\delta \circ \gamma^{-1}$ is a bijection, and so, δ is a bijection, completing the proof. □

5. The factor complex and the polar complex

In this section, we explore the relationship between the factor complex and the polar complex introduced in [9]. For a code C, the *polar complex*, denoted by $\Delta_{\mathcal{P}}(C)$, is the simplicial complex whose Stanley–Reisner ideal is $\mathcal{P}(J_C)$, the polarization of the neural ideal of C. The ideal $\mathcal{P}(J_C)$ is the *polar ideal* of C.

We first show in an example that polar and factor complexes associated to a code are, in general, not the same.

Example 5.1 (Example 3.8, continued)**.** For the code $C' = \{1, 23, 123\}$, we polarize the neural ideal $J_{C'} = \langle (1-x_1)(1-x_3),\ (1-x_1)(1-x_2),\ x_2(1-x_3), x_3(1-x_2)\rangle$ to obtain the polar ideal

$$\mathcal{P}(J_{C'}) = \langle y_1y_3,\ y_1y_2,\ x_2y_3,\ x_3y_2\rangle = \langle x_2, x_3, y_1\rangle \cap \langle y_2, y_3\rangle \cap \langle x_3, y_1, y_3\rangle \cap \langle x_2, y_2, y_3\rangle.$$

It follows that the set of facets of the polar complex $\Delta_{\mathcal{P}}(C')$ is $\{1\bar{2}\bar{3}, 123\bar{1}, 12\bar{2}, 13\bar{3}\}$. Thus, the polar complex has 2 more facets than the corresponding factor complex (recall Example 3.2).

On the other hand, the polar ideal and the factor ideal (and their corresponding complexes) share many features. A first observation is that $\mathcal{P}(J_C) \subset \mathrm{FI}(C)$ by construction and Lemma 4.1. Furthermore, Lemma 4.1 is valid when we replace $\mathrm{FI}(C)$ by $\mathcal{P}(J_C)$ [9, Theorem 3.2], and consequently Lemma 4.2 holds for $\Delta_{\mathcal{P}}(C)$. Lemma 4.3 also is valid for $\Delta_{\mathcal{P}}(C)$ [9, Corollary 5.2].

As Example 5.1 illustrates, $\mathrm{FI}(C)$ strictly contains $\mathcal{P}(J_C)$ in general. A larger ideal makes for a smaller simplicial complex. The following result explains the relationship between $\Delta_{\cap}(C)$ and $\Delta_{\mathcal{P}}(C)$.

Proposition 5.2. *For every code C, the factor complex $\Delta_{\cap}(C)$ is the subcomplex of the polar complex $\Delta_{\mathcal{P}}(C)$ whose facets are the effective facets of $\Delta_{\mathcal{P}}(C)$.*

Proof. Lemma 4.4 states that all facets of $\Delta_{\cap}(C)$ are effective, and $\mathcal{P}(J_C) \subset \mathrm{FI}(C)$ implies that $\Delta_{\cap}(C) \subset \Delta_{\mathcal{P}}(C)$. So, it suffices to show that every effective facet of $\Delta_{\mathcal{P}}(C)$ is a face of $\Delta_{\cap}(C)$. By [9, Corollaries 5.2 and 5.3], the effective facets of $\Delta_{\mathcal{P}}(C)$ are of the form $d \cup \overline{[n] \setminus c}$ where $[c, d]$ is a maximal interval of C. Now apply Lemma 4.3. □

The key difference between the factor complex and the polar complex of a code is that the latter can have defective facets. While these facets hold useful information about quotient codes, as shown in [9], the structure of the smaller factor complex was more convenient for our purposes here.

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