Selection of the wavenumbers \( k \) using an optimization method for the inverse Fourier transform in 2.5D electrical modelling

Shi-zhe Xu,1 Ben-chun Duan2 and Da-hai Zhang1

Abstract

An optimization method is used to select the wavenumbers \( k \) for the inverse Fourier transform in 2.5D electrical modelling. The model tests show that with the wavenumbers \( k \) selected in this way the inverse Fourier transform performs with satisfactory accuracy.

Introduction

A model consisting of a point source on a 2D geoelectric model is often referred to as being 2.5 dimensional. Previous studies (Coggon 1971; Snyder 1976; Pelton, Rijo and Swift 1978; Dey and Morrison 1979; Xu, Zhao and Ni 1998) have used various numerical methods, such as the boundary-integral method, the finite-element method and the boundary-element method, to solve this problem. In these numerical methods, the selection of the wavenumbers \( k \) for the inverse Fourier transform is very important in order to improve the accuracy while reducing the calculation time. A common method for selecting \( k \) is the geometric-ratio method (Zhou and Zhong 1986; Luo and Zhang 1987; Liu and Becker 1992). This study uses an optimization method to select \( k \), which can be performed on a computer. The model tests show that with the wavenumbers \( k \) selected in this way the inverse Fourier transform is sufficiently accurate.

Inverse Fourier transformation

Let \( u(x, y, z) \) be the 3D electrical potential due to a point source on a 2D geoelectric section and let \( U(x, y, k) \) be the 2D potential after Fourier transformation along the strike direction (\( z \)-axis), so that

\[
U(x, y, k) = \int_{0}^{\infty} u(x, y, z) \cos kz \, dz, \tag{1}
\]

Received October 1998, revision accepted January 2000.

1 Department of Geoscience, Zhejiang University, Hangzhou, Zhejiang, 310027, P.R. China.
2 Institute of Geology & Geophysics, Ocean University of Qingdao, 266003, P.R. China.

© 2000 European Association of Geoscientists & Engineers
where \( k \) denotes wavenumber and is a parameter in the 2D transformed potential \( U(x,y,k) \). The inverse Fourier transform is

\[
u(x, y, z) = \frac{2}{\pi} \int_0^\infty U(x, y, k) \cos kz \, dk. \tag{2}\]

In most cases, we are interested in the electric potential on the transverse profile perpendicular to strike and passing through the source point with \( z = 0 \). From (2), the inverse Fourier transform is of the form

\[
u(x, y, 0) = \frac{2}{\pi} \int_0^\infty U(x, y, k) \, dk. \tag{3}\]

This integral is often carried out numerically. Approximating the right-hand side of (3) as a sum gives

\[
u(r) \approx \sum_{j=1}^n U(r, k_j) g_j, \tag{4}\]

where \( r = \sqrt{x^2 + y^2} \), \( k_j \) is the discretized value of \( k \), and \( g_j \) is a weighting coefficient. Values of \( k_j \) and \( g_j \) must now be judiciously selected so as to make (4) as accurate as possible within a certain range of \( r \). However, in the general case, the forms of the functions \( u \) and \( U \) are unknown, so it is impossible to choose \( k \) and \( g \) using (3).

In a homogeneous half-space,

\[u = \frac{1}{2\pi} \frac{1}{\sqrt{x^2 + y^2 + z^2}}.\]

Substituting this into the Fourier transform (1) gives

\[U(x, y, k) = \frac{1}{2\pi} \int_0^\infty \frac{\cos kz}{\sqrt{(k^2 + y^2 + z^2)\pi}} \, dz = \frac{K_0(k\sqrt{x^2 + y^2})}{2\pi} = \frac{1}{2\pi} K_0(kr), \tag{5}\]

where \( r = \sqrt{x^2 + y^2} \) is the distance from the source point to the point on the transverse profile and \( K_0 \) is the zero-order modified Bessel function of the second kind. Substituting (5) into (3) gives

\[\frac{1}{r} = \frac{2}{\pi} \int_0^\infty K_0(kr) \, dk.\]

Rewriting this as a summation yields

\[\frac{1}{r} \approx \sum_{j=1}^n K_0(k_j r) g_j, \tag{6}\]

where \( k_j \) and \( g_j \) are now determined using an optimization method, which makes the above formula as accurate as possible within a certain range of \( r \). The values of \( k_j \) and \( g_j \) determined by this method can be used in the general case. However, several theoretical models should be used to determine the error in this method.
The optimization method

In order to obtain the same relative errors for different values of \( r \), (6) is rewritten as

\[
1 \approx \sum_{j=1}^{n} r K_0(k_j r) g_j = v. \tag{7}
\]

This series of \( r_i \) \((i = 1, \ldots, m)\) yields a system of equations

\[
a_{ij} g_j = v_i \text{ or } A g = v,
\]

where \( a_{ij} = r_i K_0(r_i k_j) \), \( A = (a_{ij}) \), \( g = (g_j) \) and \( v = (v_i) \) for \( i = 1, \ldots, m \) and \( j = 1, \ldots, n \). \( k_j \) and \( g_j \) are now selected such that the objective function,

\[
\varphi = (I - v)^T (I - v) = (I - A g)^T (I - A g),
\]

will reach a minimum. \( I \) is a unit column vector.

The selection of \( k_j \) and \( g_j \) is carried out in two steps.

1. For a given set of \( k_j \), the \( a_{ij} \) are known. Thus, a set of \( g_j \) can be determined by minimizing \( \varphi \), i.e. by differentiating \( \varphi \) with respect to \( g \) and setting the result equal to zero. The differential of the objective function is given by

\[
d \varphi = d(I - A g)^T (I - A g) = 2 d g^T A^T (I - A g) = 0.
\]

Due to the arbitrariness of \( d g^T \), we have

\[
A^T (I - A g) = 0 \text{ or } B g = c,
\]

where \( B = A^T A \), \( c = A^T I \).

The \( g \) obtained from (7) is a set of \( g_j \) corresponding to a given set of \( k_j \) values. If another set of \( k_j \) is given, we will get another set of \( g_j \). The minima of the objective functions corresponding to different sets of \( k_j \) and \( g_j \) differ.

2. This step is performed to determine a set of \( k_j \) which will make the objective function \( \varphi \) reach a global minimum. For this reason, we expand \( v \) in a Taylor series about a set of initial values \( k_j^{(0)} \), and take the first-order term in \( \delta k_j \), so that

\[
v \approx v_0 + \frac{\partial v}{\partial k} \delta k, \tag{8}
\]

where \( v_0 = (v_0) \), \( \frac{\partial v}{\partial k} \) = \( (\partial v_j/\partial k_j) \) and \( \delta k = (\delta k_j) \) for \( i = 1, \ldots, m \) and \( j = 1, \ldots, n \). \( v_0 \) is a set of \( v_i \) corresponding to the initial set of \( k_j^{(0)} \), and \( \delta k_j \) is a small increment of \( k_j \). Substitution of (8) into the objective function \( \varphi \) gives

\[
\varphi = \left( I - v_0 - \frac{\partial v}{\partial k} \delta k \right)^T \left( I - v_0 - \frac{\partial v}{\partial k} \delta k \right).
\]

\( \varphi \) in this equation is a function of \( \delta k \) which can be determined by minimizing \( \varphi \), i.e. by differentiating \( \varphi \) with respect to \( \delta k \) and setting it equal to zero. In a similar way, we get

\[
M \delta k = h, \tag{9}
\]

where
\[ M = \left( \frac{\partial v}{\partial \mathbf{k}} \right)^T \left( \frac{\partial v}{\partial \mathbf{k}} \right) \quad \text{and} \quad h = \left( \frac{\partial v}{\partial \mathbf{k}} \right)^T (I - \mathbf{v}_0). \]

\( \delta \mathbf{k} \) obtained from (9) leads to a new set of \( \mathbf{k}^{(1)} \), given by
\[ \mathbf{k}^{(1)} = \mathbf{k}^{(0)} + \delta \mathbf{k}. \]

Since (9) is an approximate formula, \( \mathbf{k}^{(1)} \) obtained by this method may not reach the global minimum. However by taking \( \mathbf{k}^{(1)} \) as a new set of initial values, expanding \( \mathbf{v} \) at \( \mathbf{k}^{(1)} \) and repeating the process above, we get a second evaluation \( \mathbf{k}^{(2)} \) of \( \mathbf{k} \). After several iterations, the optimum set of \( \mathbf{k} \) is obtained.

**Derivation of the matrix \( \frac{\partial \mathbf{v}}{\partial \mathbf{k}} \)**

Although \( v_i \) is a function of \( k_j \), where \( v_i = v_i(k_1, \ldots, k_n) \), we do not know its analytic expression. However it was shown above that if a set of \( k_1 \ldots k_j \ldots k_n \) is given, then the set \( g_1 \ldots g_j \ldots g_n \) can be determined and \( v_i \) can be calculated. Similarly, another set of \( k_1 \ldots k_j + \Delta k_j \ldots k_n \) will lead to a new \( v'_i \). Thus, the partial derivative is
\[ \frac{\partial v_i}{\partial k_i} = \frac{v'_i - v_i}{\Delta k_j}. \]

Here we use \( \Delta k_j = 0.1 k_j \) for its determination.

**Calculation of \( k \) and \( g \) by the optimization method**

Assuming the range of \( r \) is from 0.2 to 30, we use \( r_i = 0.2, 0.5, 1, 2, 4, 7, 10, 15, 20, 30 \), with \( m \), the total number of points, being 10. The initial \( k_j^{(0)} \) are taken as \( k_j^{(0)} = 0.02, 0.1, 0.5, 2.5 \), with \( n \), the total number of \( k \) values, being 4. The rms is calculated using \( \varepsilon = \sqrt{\frac{\sum w}{m}} \) after each iteration. The relationship between \( \varepsilon \) and the number of iterations is shown in Fig. 1. As can be seen, the rms is reduced to 0.29% after 20 iterations and is kept at this level. The \( k \) and \( g \) values determined by this method are listed in Table 1.

A comparison between the exact \( (1/r) \) and the calculated \( (1/r') \), determined by the formula
\[ \frac{1}{r'} = \sum_{j=1}^{4} K_0(k_j r) g_j \]

using the values of \( k \) and \( g \) listed in Table 1, is shown in Table 2. The accuracy is satisfactory.

We have calculated the minimum rms \( \varepsilon \) corresponding to different ranges of \( r \) (0.1...
to $R$) and different numbers of $k$. The relationships between them are shown in Fig. 2. If the range $R$ and the rms are given in advance, it is easy to find the minimum number of $k$ from Fig. 2. For example, when $R = 50$, $n = 4$ for $k$ will make the rms less than 1%.

**Application to non-uniform models**

The discussion above deals with a model in a homogeneous half-space. For a model in an inhomogeneous region, the selection of $k$ and $g$ is a very difficult problem, and there is no general method. We now apply the values for $k$ and $g$ derived for the homogeneous half-space model to other models and compare the result with the analytical solution in order to determine the accuracy.

**Example 1**

The lower part of Fig. 3 shows a 2D ridge model with an angle of 90° between the two flanks. The optimization method is used to select $k$ and $g$ values and to calculate the potentials on the ground surface using the boundary-element (BEM) method (Xu 1995). The apparent resistivity for the Schlumberger array can then be obtained. The four values of $k$ and $g$ listed in Table 1 are used for the inverse Fourier transformation. The solid line in the upper part of Fig. 3 represents the apparent resistivity calculated from

![Figure 1. The relationship between the number of iterations and the relative rms $\varepsilon$ ($r = 0.2 \pm 30$, four values of $k$).](image)

### Table 1. $k$ and $g$ for $r = 0.2 \pm 30$, $n = 4$.  

<table>
<thead>
<tr>
<th>$k$</th>
<th>0.0217102</th>
<th>0.2161121</th>
<th>1.0608400</th>
<th>5.0765870</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g$</td>
<td>0.0463660</td>
<td>0.2365931</td>
<td>1.0382080</td>
<td>5.3648010</td>
</tr>
</tbody>
</table>

using the BEM method, and the dots represent those obtained using the analytical method. They are basically the same. The error in the apparent resistivity is about 0.5%.

**Example 2**

Figure 4 shows a three-layered geoelectrical section. A sounding curve with the maximum value of $AB/2 = 3000$ m was calculated using the analytic method and is shown by dots. The finite-element (FEM) method (Xu 1994) was used to calculate the sounding curve. Eight values of $k$ were selected for inverse Fourier transformation in the FEM method. The sounding curve obtained using the FEM method is shown by a solid line. The error in the apparent resistivity is about 1%.

**Conclusions**

The number of wavenumbers $k$ selected by the optimization method is reduced. The wavenumbers $k$ selected for the model in a homogeneous half-space can be used as an approximation for the model in inhomogeneous space.

<table>
<thead>
<tr>
<th>$r$</th>
<th>Exact $1/r$</th>
<th>Numerical $1/r'$</th>
<th>$r$</th>
<th>Exact $1/r$</th>
<th>Numerical $1/r'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>5</td>
<td>4.99983</td>
<td>7</td>
<td>0.14286</td>
<td>0.14357</td>
</tr>
<tr>
<td>0.5</td>
<td>2</td>
<td>2.00032</td>
<td>10</td>
<td>0.1</td>
<td>0.09979</td>
</tr>
<tr>
<td>1.0</td>
<td>1</td>
<td>0.99939</td>
<td>15</td>
<td>0.06667</td>
<td>0.06639</td>
</tr>
<tr>
<td>2.0</td>
<td>0.5</td>
<td>0.50071</td>
<td>20</td>
<td>0.05</td>
<td>0.05022</td>
</tr>
<tr>
<td>4.0</td>
<td>0.25</td>
<td>0.24920</td>
<td>30</td>
<td>0.03333</td>
<td>0.03330</td>
</tr>
</tbody>
</table>

Figure 2. The minimum rms $\varepsilon$ corresponding to different ranges of $r$ ($0.1-R$) and different numbers of $k$. 

Figure 3. The apparent resistivities of a Schlumberger array above a 2D ridge with an angle of 90° between the two flanks. The solid line shows results obtained by the BEM method, the dots represent results from the analytic method.

Figure 4. Three-layered geoelectric section and its sounding curve. Results obtained by the FEM method are shown by the solid line, while the dots represent results from the analytic method.
References


