Gravitational Physics 647

ABSTRACT

In this course, we develop the subject of General Relativity, and its applications to the study of gravitational physics.

Contents

The material in this course is intended to be more or less self contained. However, here is a list of some books and other reference sources that may be helpful for some parts of the course:

1. S.W. Weinberg, *Gravitation and Cosmology*

2. R.M. Wald, *General Relativity*

3. S.W. Hawking and G.F.R. Ellis, *The Large-Scale Structure of Spacetime*

4. C. Misner, K.S. Thorne and J. Wheeler, *Gravitation*

1 Introduction to General Relativity: The Equivalence Principle

Men occasionally stumble over the truth, but most of them pick themselves up and hurry off as if nothing ever happened — **Sir Winston Churchill**

The experimental underpinning of Special Relativity is the observation that the speed of light is the same in all inertial frames, and that the fundamental laws of physics are the same in all inertial frames. Because the speed of light is so large in comparison to the velocities that we experience in "everyday life," this means that we have very little direct experience of special-relativistics effects, and in consequence special relativity can often seem rather counter-intuitive.

By contrast, and perhaps rather surprisingly, the essential principles on which Einstein's theory of General Relativity are based are not in fact a yet-further abstraction of the already counter-intuitive theory of Special Relativity. In fact, perhaps remarkably, General Relativity has as its cornerstone an observation that is absolutely familiar and intuitively understandable in everyday life. So familiar, in fact, that it took someone with the genius of Einstein to see it for what it really was, and to extract from it a profoundly new way of understanding the world. (Sadly, even though this happened nearly a hundred years ago, not everyone has yet caught up with the revolution in understanding that Einstein achieved. Nowhere is this more apparent than in the teaching of mechanics in a typical undergraduate physics course!)

The cornerstone of Special Relativity is the observation that the speed of light is the same in all inertial frames. From this the consequences of Lorentz contraction, time dilation, and the covariant behaviour of the fundamental physical laws under Lorentz transformations all logically follow. The intuition for understanding Special Relativity is not profound, but it has to be acquired, since it is not the intuition of our everyday experience. In our everyday lives velocities are so small in comparison to the speed of light that we don't notice even a hint of special-relativistic effects, and so we have to train ourselves to imagine how things will behave when the velocities are large. Of course in the laboratory it is now a commonplace to encounter situations where special-relativisitic effects are crucially important.

The cornerstone of General Relativity is the *Principle of Equivalence.* There are many ways of stating this, but perhaps the simplest is the assertion that gravitational mass and inertial mass are the same.

In the framework of Newtonian gravity, the *gravitational mass* of an object is the constant of proportionality $M_{\rm grav}$ in the equation describing the force on an object in the Earth's gravitational field $\vec g$:

$$\vec F = M_{\rm grav}\,\vec g = \frac{G M_{\rm earth}\, M_{\rm grav}\,\vec r}{r^3}\,, \tag{1.1}$$

where $\vec r$ is the position vector of a point on the surface of the Earth.

More generally, if Φ is the Newtonian gravitational potential then an object with gravitational mass $M_{\rm grav}$ experiences a gravitational force given by

$$\vec F = -M_{\rm grav}\,\vec\nabla\Phi\,. \tag{1.2}$$

The *inertial mass* $M_{\rm inertial}$ of an object is the constant of proportionality in Newton's second law, describing the force it experiences if it has an acceleration $\vec a$ relative to an inertial frame:

$$\vec F = M_{\rm inertial}\,\vec a\,. \tag{1.3}$$

It is a matter of everyday observation, and is confirmed to high precision in the laboratory in experiments such as the Eötvös experiment, that

$$M_{\rm grav} = M_{\rm inertial}\,. \tag{1.4}$$

It is an immediate consequence of (1.1) and (1.3) that an object placed in the Earth's gravitational field, with no other forces acting, will have an acceleration (relative to the surface of the Earth) given by

$$\vec a = \frac{M_{\rm grav}}{M_{\rm inertial}}\,\vec g\,. \tag{1.5}$$

From (1.4), we therefore have the famous result

$$\vec a = \vec g\,, \tag{1.6}$$

which says that all objects fall at the same rate. This was allegedly demonstrated by Galileo in Pisa, by dropping objects of different compositions off the leaning tower.

More generally, if the object is placed in a Newtonian gravitational potential Φ then from (1.2) and (1.3) it will suffer an acceleration given by

$$\vec a = -\frac{M_{\rm grav}}{M_{\rm inertial}}\,\vec\nabla\Phi = -\vec\nabla\Phi\,, \tag{1.7}$$

with the second equality holding if the inertial and gravitational masses of the object are equal.

In Newtonian mechanics, this equality of gravitational and inertial mass is noted, the two quantities are set equal and called simply M, and then one moves on to other things.

There is nothing in Newtonian mechanics that *requires* one to equate M_{grav} and M_{inertial}. If experiments had shown that the ratio $M_{\text{grav}}/M_{\text{inertial}}$ were different for different objects, that would be fine too; one would simply make sure to use the right type of mass in the right place. For a Newtonian physicist the equality of gravitational and inertial mass is little more than an amusing coincidence, which allows one to use one symbol instead of two, and which therefore makes some equations a little simpler.

The big failing of the Newtonian approach is that it fails to ask *why is the gravitational mass equal to the inertial mass?* Or, perhaps a better and more scientific way to express the question is *what symmetry in the laws of nature forces the gravitational and inertial masses to be equal?* The more we probe the fundamental laws of nature, the more we find that fundamental "coincidences" just don't happen; if two concepts that *a priori* look to be totally different turn out to be the same, nature is trying to tell us something. This, in turn, should be reflected in the fundamental laws of nature.

Einstein's genius was to recognise that the equality of gravitational and inertial mass is much more than just an amusing coincidence; nature is telling us something very profound about gravity. In particular, it is telling us that *we cannot distinguish, at least by a local experiment, between the "force of gravity," and the force that an object experiences when it accelerates relative to an inertial frame.* For example, an observer in a small closed box cannot tell whether he is sitting on the surface of the Earth, or instead is in outer space in a rocket accelerating at 32 ft. per second per second.

The Newtonian physicist responds to this by going through all manner of circumlocutions, and talks about "fictitious forces" acting on the rocket traveller, etc. Einstein, by contrast, recognises a fundamental truth of nature, and declares that, by definition, *the force of gravity is the force experienced by an object that is accelerated relative to an inertial frame.* Winston Churchill's observation, reproduced under the heading of this chapter, rather accurately describes the reaction of the average teacher of Newtonian physics.

In the Einsteinian way of thinking, once it is recognised that the force experienced by an accelerating object is locally indistinguishable from the force experienced by an object in a gravitational field, the next logical step is to say that they in fact *are* the same thing. Thus, we can say that the "foce of gravity" is nothing but the force experienced by an otherwise isolated object that is accelerating relative to an inertial frame.

Once the point is recognised, all kinds of muddles and confusions in Newtonian physics disappear. The observer in the closed box does not have to sneak a look outside before he is allowed to say whether he is experiencing a gravitational force or not. An observer in free fall, such as an astronaut orbiting the Earth, is genuinely weightless because, by definition, he is in a free-fall frame and thus there is no gravity, locally at least, in his frame of reference. A child sitting on a rotating roundabout (or merry-go-round) in a playground is experiencing an *outward* gravitational force, which can unashamedly be called a centrifugal force (with no need for the quotation marks and the F-word "fictitious" that is so beloved of 218 lecturers!). Swept away completely is the muddling notion of the fictitious "force that dare not speak its name."[1]

Notice that in the new order, there is a radical change of viewpoint about what constitutes an inertial frame. If we neglect any effects due to the Earth's rotation, a Newtonian physicist would say that a person standing on the Earth in a laboratory is in an inertial frame. By contrast, in general relativity we say that a person who has jumped out of the laboratory window is (temporarily!) in an inertial frame. A person standing in the laboratory is accelerating relative to the inertial frame; indeed, that is why he is experiencing the force of gravity.

To be precise, the concept that one introduces in general relativity is that of the *local inertial frame.* This is a free-fall frame, such as that of the person who jumped out of the laboratory, or of the astronaut orbiting the Earth. We must, in general, insist on the word "local," because, as we shall see later, if there is curvature present then one can only define a free-fall frame in a small local region. For example, an observer falling out of a window in College Station is accelerating relative to an observer falling out of a window in Cambridge, since they are moving, with increasing velocities, along lines that are converging on the centre of the Earth. In a small enough region, however, the concept of the free-fall inertial frame makes sense.

Having recognised the equivalence of gravity and acceleration relative to a local inertial frame, it becomes evident that we can formulate the laws of gravity, and indeed *all* the fundamental laws of physics, in a completely frame-independent manner. To be more precise, we can formulate the fundamental laws of physics in such a way that they take the same form in all frames, whether or not they are locally inertial. In fact, another way of stating the equivalence principle is to assert that the fundamental laws of physics take the same form in all frames, i.e. in all coordinate systems. To make this manifest, we need to introduce the formalism of general tensor calculus. Before doing this, it will be helpful first

[1]Actually, having said this, it should be remarked that in fact the concept of a "gravitational force" does not really play a significant role in general relativity, except when discussing the weak-field Newtonian limit. In this limit, the notion of a gravitational force can be made precise, and it indeed has the feature that it is always a consequence of acceleration relative to an inertial frame.

to review some of the basic principles of Special Relativity, and in the process, we shall introduce some notation and conventions that we shall need later.

2 Special Relativity

2.1 Lorentz boosts

The principles of special relativity should be familiar to everyone. From the postulates that the speed of light is the same in all inertial frames, and that the fundamental laws of physics should be the same in all inertial frames, one can derive the *Lorentz Transformations* that describe how the spacetime coordinates of an event seen in one inertial frame are related to those of the event seen in a different inertial frame. If we consider what is called a *pure boost* along the x direction, between a frame S and another frame S' that is moving with constant velocity v along the x direction, then we have the well-known Lorentz transformation

$$t' = \gamma\,(t - \frac{vx}{c^2})\,, \qquad x' = \gamma\,(x - vt)\,, \qquad y' = y\,, \qquad z' = z\,, \tag{2.1}$$

where $\gamma = (1 - v^2/c^2)^{-1/2}$. Let us straight away introduce the simplification of choosing our units for distance and time in such a way that the speed of light c is set equal to 1. This can be done, for example, by measuring time in seconds and distance in light-seconds, where a light-second is the distance travelled by light in an interval of 1 second. It is, of course, straighforward to revert back to "normal" units whenever one wishes, by simply applying the appropriate rescalings as dictated by dimensional analysis. Thus, the pure Lorentz boost along the x direction is now given by

$$t' = \gamma\,(t - vx)\,, \qquad x' = \gamma\,(x - vt)\,, \qquad y' = y\,, \qquad z' = z\,, \qquad \gamma = (1 - v^2)^{-1/2}\,. \tag{2.2}$$

It is straightforward to generalise the pure boost along x to the case where the velocity $\vec v$ is in an arbitrary direction in the three-dimensional space. This can be done by exploiting the rotational symmetry of the three-dimensional space, and using the three-dimensional vector notation that makes this manifest. It is easy to check that the transformation rules

$$t' = \gamma\,(t - \vec v\cdot\vec r)\,, \qquad \vec r\,' = \vec r + \frac{\gamma - 1}{v^2}\,(\vec v\cdot\vec r)\,\vec v - \gamma\,\vec v\,t\,, \qquad \gamma = (1 - v^2)^{-1/2} \tag{2.3}$$

reduce to the previous result (2.2) in the special case that $\vec v$ lies along the x diection, i.e. if $\vec v = (v, 0, 0)$. (Note that here $\vec r = (x, y, z)$ denotes the position-vector describing the spatial location of the event under discussion.) Since (2.3) is written in 3-vector notation, it is then the unique 3-covariant expression that generalises (2.2).

One can easily check that the primed and the unprimed coordinates appearing in (2.2) or in (2.3) satisfy the relation

$$x^2 + y^2 + z^2 - t^2 = x'^2 + y'^2 + z'^2 - t'^2 \,. \tag{2.4}$$

Furthermore, if we consider two infinitesimally separated spacetime events, at locations (t, x, y, z) and $(t + dt, x + dx, y + dy, z + dz)$, then it follows that we shall also have

$$dx^2 + dy^2 + dz^2 - dt^2 = dx'^2 + dy'^2 + dz'^2 - dt'^2 \,. \tag{2.5}$$

This quantity, which is thus invariant under Lorentz boosts, is the spacetime generalisation of the infinitesimal spatial distance between two neighbouring points in Euclidean 3-space. We may define the spacetime *interval* ds, given by

$$ds^2 = dx^2 + dy^2 + dz^2 - dt^2 \,. \tag{2.6}$$

This quantity, which gives the rule for measuring the interval between neigbouring points in spacetime, is known as the Minkowski spacetime *metric*. As seen above, it is invariant under arbitrary Lorentz boosts.

2.2 Lorentz 4-vectors and 4-tensors

It is convenient now to introduce a 4-dimensional notation. The Lorentz boosts (2.3) can be written more succinctly if we first define the set of four spacetime coordinates denoted by x^μ, where μ is an index, or label, that ranges over the values 0, 1, 2 and 3. The case $\mu = 0$ corresponds to the time coordinate t, while $\mu = 1$, 2 and 3 corresponds to the space coordinates x, y and z respectively. Thus we have[2]

$$(x^0, x^1, x^2, x^3) = (t, x, y, z) \,. \tag{2.7}$$

Of course, once the abstract index label μ is replaced, as here, by the specific index values 0, 1, 2 and 3, one has to be very careful when reading a formula to distinguish between, for example, x^2 meaning the symbol x carrying the spacetime index $\mu = 2$, and x^2 meaning the square of x. It should generally be obvious from the context which is meant.

The invariant quadratic form appearing on the left-hand side of (2.5) can now be written in a nice way, if we first introduce the 2-index quantity $\eta_{\mu\nu}$, defined to be given by

$$\eta_{00} = -1 \,, \qquad \eta_{11} = \eta_{22} = \eta_{33} = 1 \,, \tag{2.8}$$

[2]The choice to put the index label μ as a superscript, rather than a subscript, is purely conventional. But, unlike the situation with many arbitrary conventions, in this case the coordinate index is placed upstairs in *all* modern literature.

with $\eta_{\mu\nu} = 0$ if $\mu \neq \nu$. Note that $\eta_{\mu\nu}$ is symmetric:

$$\eta_{\mu\nu} = \eta_{\nu\mu}\,. \tag{2.9}$$

Using $\eta_{\mu\nu}$, the metric ds^2 defined in (2.6) can be rewritten as

$$ds^2 = dx^2 + dy^2 + dz^2 - dt^2 = \sum_{\mu=0}^{3}\sum_{\nu=0}^{3} \eta_{\mu\nu}\, dx^\mu dx^\nu\,. \tag{2.10}$$

It is often convenient to represent 2-index tensors such as $\eta_{\mu\nu}$ in a matrix notation, by defining

$$\eta = \begin{pmatrix} \eta_{00} & \eta_{01} & \eta_{02} & \eta_{03} \\ \eta_{10} & \eta_{11} & \eta_{12} & \eta_{13} \\ \eta_{20} & \eta_{21} & \eta_{22} & \eta_{23} \\ \eta_{30} & \eta_{31} & \eta_{32} & \eta_{33} \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.11}$$

The 4-dimensional notation in (2.10) is still somewhat clumsy, but it can be simplified considerably by adopting the *Einstein Summation Convention*, whereby the explicit summation symbols are omitted, and we simply write (2.10) as

$$ds^2 = \eta_{\mu\nu}\, dx^\mu dx^\nu\,. \tag{2.12}$$

We can do this because in *any* valid covariant expression, if an index occurs exactly twice in a given term, then it will always be summed over. Conversely, there will never be any occasion when an index that appears other than exactly twice in a given term is summed over, in any valid covariant expression. Thus there is no ambiguity involved in omitting the explicit summation symbols, with the understanding that the Einstein summation convention applies.

So far, we have discussed Lorentz boosts, and we have observed that they have the property that the Minkowski metric ds^2 is invariant. Note that the Lorentz boosts (2.3) are *linear* transformations of the spacetime coordinates. We may define the general class of *Lorentz transformations* as strictly linear transformations of the spacetime coordinates that leave ds^2 invariant. The most general such linear transformation can be written as[3]

$$x'^\mu = \Lambda^\mu{}_\nu\, x^\nu\,, \tag{2.13}$$

where $\Lambda^\mu{}_\nu$ form a set of $4 \times 4 = 16$ constants. We also therefore have $dx'^\mu = \Lambda^\mu{}_\nu\, dx^\nu$. Requiring that these transformations leave ds^2 invariant, we therefore must have

$$ds^2 = \eta_{\mu\nu}\, \Lambda^\mu{}_\rho\, \Lambda^\nu{}_\sigma\, dx^\rho\, dx^\sigma = \eta_{\rho\sigma}\, dx^\rho\, dx^\sigma\,, \tag{2.14}$$

[3]We are using the term "linear" here to mean a relation in which the x'^μ are expressed as linear combinations of the original coordinates x^ν, with constant coefficients. We shall meet transformations later on where there are further terms involving purely constant shifts of the coordinates.

and hence we must have

$$\eta_{\mu\nu}\,\Lambda^\mu{}_\rho\,\Lambda^\nu{}_\sigma = \eta_{\rho\sigma}\,. \tag{2.15}$$

Note that we can write this equation in a matrix form, by introducing the 4×4 matrix Λ given by

$$\Lambda = \begin{pmatrix} \Lambda^0{}_0 & \Lambda^0{}_1 & \Lambda^0{}_2 & \Lambda^0{}_3 \\ \Lambda^1{}_0 & \Lambda^1{}_1 & \Lambda^1{}_2 & \Lambda^1{}_3 \\ \Lambda^2{}_0 & \Lambda^2{}_1 & \Lambda^2{}_2 & \Lambda^2{}_3 \\ \Lambda^3{}_0 & \Lambda^3{}_1 & \Lambda^3{}_2 & \Lambda^3{}_3 \end{pmatrix}\,. \tag{2.16}$$

Equation (2.15) then becomes (see (2.11))

$$\Lambda^T\,\eta\,\Lambda = \eta\,. \tag{2.17}$$

We can easily count up the number of *independent* components in a general Lorentz transformation by counting the number of independent conditions that (2.15) imposes on the 16 components of $\Lambda^\mu{}_\nu$. Since μ and ν in (2.15) each range over 4 values, there are 16 equations, but we must take note of the fact that the equations in (2.15) are automatically *symmetric* in μ and ν. Thus there are only $(4\times 5)/2 = 10$ *independent* conditions in (2.15), and so the number of *independent* components in the most general $\Lambda^\mu{}_\nu$ that satisfies (2.15) is $16 - 10 = 6$.

We have already encountered the pure Lorentz boosts, described by the transformations (2.3). By comparing (2.3) and (2.13), we see that for the pure boost, $\Lambda^\mu{}_\nu$ is given by the components $\Lambda^\mu{}_\nu$ are given by

$$\begin{aligned} \Lambda^0{}_0 &= \gamma\,, \qquad \Lambda^0{}_i = -\gamma v_i\,, \\ \Lambda^i{}_0 &= -\gamma\,v_i\,, \qquad \Lambda^i{}_j = \delta_{ij} + \frac{\gamma-1}{v^2} v_i v_j\,, \end{aligned} \tag{2.18}$$

where δ_{ij} is the Kronecker delta symbol,

$$\delta_{ij} = 1 \quad \text{if} \quad i = j\,, \qquad \delta_{ij} = 0 \quad \text{if} \quad i \ne j\,. \tag{2.19}$$

Note that here, and subsequently, we use Greek indices $\mu, \nu, \ldots$ for spacetime indices ranging over $0, 1, 2$ and 3, and Latin indices $i, j, \ldots$ for spatial indices ranging over $1, 2$ and 3. Clearly, the pure boosts are characterised by three independent parameters, namely the three independent components of the boost velocity $\vec v$.

The remaining three parameters of a general Lorentz transformation are easily identified. Consider rotations entirely within the three spatial directions (x, y, z), leaving time untransformed:

$$t' = t\,, \qquad x'^i = M_{ij}\,x^j\,, \qquad \text{where} \quad M_{ki} M_{kj} = \delta_{ij}\,. \tag{2.20}$$

(Note that in Minkowski spacetime we can freely put the spatial indices upstairs or downstairs as we wish.) The last equation in (2.20) is the orthogonality condition $M^T\,M = 1\!\!1$ on M, viewed as a 3×3 matrix with components M_{ij}. It ensures that the transformation leaves $x^i\,x^i$ invariant, as a rotation should. It is easy to see that a general 3-dimensional rotation is described by three indepedent parameters. This may be done by the same method we used above to count the parameters in a Lorentz transformation. Thus a general 3×3 matrix M has $3\times 3 = 9$ components, but the equation $M^T\,M = 1\!\!1$ imposes $(3\times 4)/2 = 6$ independent conditions (since it is a symmetric equation), leading $9 - 6 = 3$ independent parameters in a general 3-dinemsional rotation.

A general Lorentz transformation may in fact be written as the product of a general Lorentz boost $\Lambda^\mu_{(B)\nu}$ and a general 3-dimensional rotation $\Lambda^\mu_{(R)\nu}$:

$$\Lambda^\mu{}_\nu = \Lambda^\mu_{(B)\rho}\,\Lambda^\rho_{(R)\nu}\,, \tag{2.21}$$

where $\Lambda^\mu_{(B)\nu}$ is given by the expressions in (2.18) and $\Lambda^\mu_{(R)\nu}$ is given by

$$\Lambda^0_{(R)0} = 1\,, \qquad \Lambda^i_{(R)j} = M_{ij}\,, \qquad \Lambda^i_{(R)0} = \Lambda^0_{(R)i} = 0\,. \tag{2.22}$$

Note that if the two factors in (2.21) were written in the opposite order, then this would be another equally good, although inequivalent, factorisation of a general Lorentz transformation.

It should be remarked here that we have actually been a little cavalier in our discussion of the Lorentz group, and indeed the three-dimensional rotation group, as far as discrete symmetries are concerned. The general 3×3 matrix M satisfying the orthogonality condition $M^T\,M = 1\!\!1$ is an element of the orthogonal group $O(3)$. Taking the determinant of $M^T\,M = 1\!\!1$ and using that $\det M^T = \det M$, one deduces that $(\det M)^2 = 1$ and hence $\det M = \pm 1$. The set of $O(3)$ matrices with $\det M = +1$ themselves form a group, known as $SO(3)$, and it is these that describe a general pure rotation. These are continuously connected to the identity. Matrices M wth $\det M = -1$ correspond to a composition of a spatial reflection and a pure rotation. Because of the reflection, these transformations are not continuously connected to the identity. Likewise, for the full Lorentz group, which is generated by matrices Λ satisfying (2.17) (i.e. $\Lambda^T\,\eta\,\Lambda = \eta$), one has $(\det\Lambda) = \pm 1$. The description of Λ in the form (2.21), where $\Lambda_{(B)}$ is a pure boost of the form (2.18) and $\Lambda_{(R)}$ is a pure rotation, comprises a subset of the full Lorentz group, where there are no spatial reflections and there is no reversal of the time direction. One can compose these transformations with a time reversal and/or a space reflection, in order to obtain the full Lorentz group.

The group of transformations that preserves the Minkowski metric is actually larger than just the Lorentz group. To find the full group, we can begin by considering what are called *General Coordinate Transformations*, of the form

$$x'^\mu = x'^\mu(x^\nu)\,, \tag{2.23}$$

that is, arbitrary redefinitions to give a new set of coordinates x'^μ that are arbitrary functions of the original coordinates. By the chain rule for differentiation, we shall have

$$ds'^2 \equiv \eta_{\mu\nu}\, dx'^\mu\, dx'^\nu = \eta_{\mu\nu}\, \frac{\partial x'^\mu}{\partial x^\rho}\, \frac{\partial x'^\nu}{\partial x^\sigma}\, dx^\rho\, dx^\sigma\,. \tag{2.24}$$

Since we want this to equal ds^2, which we may write as $ds^2 = \eta_{\rho\sigma}\, dx^\rho\, dx^\sigma$, we therefore have that

$$\eta_{\rho\sigma} = \eta_{\mu\nu}\, \frac{\partial x'^\mu}{\partial x^\rho}\, \frac{\partial x'^\nu}{\partial x^\sigma}\,. \tag{2.25}$$

Differentiating with respect to x^λ then gives

$$0 = \eta_{\mu\nu}\, \frac{\partial^2 x'^\mu}{\partial x^\lambda\, \partial x^\rho}\, \frac{\partial x'^\nu}{\partial x^\sigma} + \eta_{\mu\nu}\, \frac{\partial x'^\mu}{\partial x^\rho}\, \frac{\partial^2 x'^\nu}{\partial x^\lambda\, \partial x^\sigma}\,. \tag{2.26}$$

If we add to this the equation with ρ and λ exchanged, and subtract the equation with σ and λ exchanged, then making use of the fact that seconed partial derivatives commute, we find that four of the total of six terms cancel, and the remaining two are equal, leading to

$$0 = 2\eta_{\mu\nu}\, \frac{\partial^2 x'^\mu}{\partial x^\lambda\, \partial x^\rho}\, \frac{\partial x'^\nu}{\partial x^\sigma}\,. \tag{2.27}$$

Now $\eta_{\mu\nu}$ is non-singular, and we may assume also that $\frac{\partial x'^\nu}{\partial x^\sigma}$ is a non-singular, and hence invertible, 4×4 matrix. (We wish to restrict our transformations to ones that are non-singular and invertible.) Hence we conclude that

$$\frac{\partial^2 x'^\mu}{\partial x^\lambda\, \partial x^\rho} = 0\,. \tag{2.28}$$

This implies that x'^μ must be of the form

$$x'^\mu = C^\mu{}_\nu\, x^\nu + a^\mu\,, \tag{2.29}$$

where $C^\mu{}_\nu$ and a^μ are independent of the x^ρ coordinates, i.e. they are constants. We have already established, by considering transformations of the form $dx'^\mu = \Lambda^\mu{}_\nu\, dx^\nu$ that $\Lambda^\mu{}_\nu$ must satisfy the condtions (2.15) in order for ds^2 to be invariant. Thus we conclude that the most general transformations that preserve the Minkowski metric are given by

$$x'^\mu = \Lambda^\mu{}_\nu\, x^\nu + a^\mu\,, \tag{2.30}$$

where $\Lambda^\mu{}_\nu$ are Lorentz transformations, obeying (2.15), and a^μ are constants. The transformations (2.30) generate the *Poincaré Group*, which has 10 parameters, comprising 6 for the Lorentz subgroup and 4 for translations generated by a^μ.

Now let us introduce the general notion of Lorentz vectors and tensors. The Lorentz transformation rule of the coordinate differential dx^μ, i.e.

$$dx'^\mu = \Lambda^\mu{}_\nu \, dx^\nu \,, \tag{2.31}$$

can be taken as the prototype for more general 4-vectors. Thus, we may define any set of four quantities U^μ, for $\mu = 0$, 1, 2 and 3, to be the components of a Lorentz 4-vector (often, we shall just abbreviate this to simply a 4-vector) if they transform, under Lorentz transformations, according to the rule

$$U'^\mu = \Lambda^\mu{}_\nu \, U^\nu \,. \tag{2.32}$$

The Minkowski metric $\eta_{\mu\nu}$ may be thought of as a 4×4 matrix, whose rows are labelled by μ and columns labelled by ν, as in (2.11). Clearly, the inverse of this matrix takes the same form as the matrix itself. We denote the components of the inverse matrix by $\eta^{\mu\nu}$. This is called, not surprisingly, the *inverse Minkowksi metric*. Clearly it satisfies the relation

$$\eta_{\mu\nu} \, \eta^{\nu\rho} = \delta_\mu^\rho \,, \tag{2.33}$$

where the 4-dimensional Kronecker delta is defined to equal 1 if $\mu = \rho$, and to equal 0 if $\mu \neq \rho$. Note that like $\eta_{\mu\nu}$, the inverse $\eta^{\mu\nu}$ is symmetric also: $\eta^{\mu\nu} = \eta^{\nu\mu}$.

The Minkowksi metric and its inverse may be used to lower or raise the indices on other quantities. Thus, for example, if U^μ are the components of a Lorentz 4-vector, then we may define

$$U_\mu = \eta_{\mu\nu} \, U^\nu \,. \tag{2.34}$$

This is another type of Lorentz 4-vector. Two distinguish the two, we call a 4-vector with an upstairs index a *contravariant* 4-vector, while one with a downstairs index is called a *covariant* 4-vector. Note that if we raise the lowered index in (2.34) again using $\eta^{\mu\nu}$, then we get back to the starting point:

$$\eta^{\mu\nu} \, U_\nu = \eta^{\mu\nu} \, \eta_{\nu\rho} \, U^\rho = \delta^\mu_\rho \, U^\rho = U^\mu \,. \tag{2.35}$$

It is for this reason that we can use the same symbol U for the covariant 4-vector $U_\mu = \eta_{\mu\nu} \, U^\nu$ as we used for the contravariant 4-vector U^μ.

In a similar fashion, we may define the quantities $\Lambda_\mu{}^\nu$ by

$$\Lambda_\mu{}^\nu = \eta_{\mu\rho}\, \eta^{\nu\sigma}\, \Lambda^\rho{}_\sigma\,. \tag{2.36}$$

It is then clear that (2.15) can be restated as

$$\Lambda^\mu{}_\nu\, \Lambda_\mu{}^\rho = \delta^\rho_\nu\,. \tag{2.37}$$

We can also then invert the Lorentz transformation $x'^\mu = \Lambda^\mu{}_\nu\, x^\nu$ to give

$$x^\mu = \Lambda_\nu{}^\mu\, x'^\nu\,. \tag{2.38}$$

It now follows from (2.32) that the components of the covariant 4-vector U_μ defined by (2.34) transform under Lorentz transformations according to the rule

$$U'_\mu = \Lambda_\mu{}^\nu\, U_\nu\,. \tag{2.39}$$

Any set of 4 quantities U_μ which transform in this way under Lorentz transformations will be called a covariant Lorentz 4-vector.

Using (2.38), we can see that the gradient operator $\partial/\partial x^\mu$ transforms as a covariant 4-vector. Using the chain rule for partial differentiation we have

$$\frac{\partial}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu}\, \frac{\partial}{\partial x^\nu}\,. \tag{2.40}$$

But from (2.38) we have (after a relabelling of indices) that

$$\frac{\partial x^\nu}{\partial x'^\mu} = \Lambda_\mu{}^\nu\,, \tag{2.41}$$

and hence (2.40) gives

$$\frac{\partial}{\partial x'^\mu} = \Lambda_\mu{}^\nu\, \frac{\partial}{\partial x^\nu}\,. \tag{2.42}$$

As can be seen from (2.39), this is precisely the transformation rule for a a covariant Lorentz 4-vector. The gradient operator arises sufficiently often that it is useful to use a special symbol to denote it. We therefore define

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu}\,. \tag{2.43}$$

Thus the Lorentz transformation rule (2.42) is now written as

$$\partial'_\mu = \Lambda_\mu{}^\nu\, \partial_\nu\,. \tag{2.44}$$

Having seen how contravariant and covariant 4-vectors transform under Lorentz transformations (as given in (2.32) and (2.39) respectively), we can now define the transformation rules for more general objects called Lorentz tensors. These objects carry multiple indices, and each one transforms with a Λ factor, of either the (2.32) type if the index is upstairs, or of the (2.39) type if the index is downstairs. Thus, for example, a tensor $T_{\mu\nu}$ transforms under Lorentz transformations according to the rule

$$T'_{\mu\nu} = \Lambda_\mu{}^\rho\, \Lambda_\nu{}^\sigma\, T_{\rho\sigma}\,. \tag{2.45}$$

More generally, a tensor $T^{\mu_1\cdots\mu_p}{}_{\nu_1\cdots\nu_q}$ will transform according to the rule

$$T'^{\mu_1\cdots\mu_p}{}_{\nu_1\cdots\nu_q} = \Lambda^{\mu_1}{}_{\rho_1}\cdots\Lambda^{\mu_p}{}_{\rho_p}\,\Lambda_{\nu_1}{}^{\sigma_1}\cdots\Lambda_{\nu_q}{}^{\sigma_q}\,T^{\rho_1\cdots\rho_p}{}_{\sigma_1\cdots\sigma_q}\,. \tag{2.46}$$

We may refer to such a tensor as a (p,q) Lorentz tensor. Note that scalars are just special cases of tensors of type $(0,0)$ with no indices, while vectors are special cases with just one index, $(1,0)$ or $(0,1)$.

It is easy to see that the *outer product* of two tensors gives rise to another tensor. For example, if U^μ and V^μ are two contravariant vectors then $T^{\mu\nu} \equiv U^\mu V^\nu$ is a tensor, since, using the known transformation rules for U and V we have

$$\begin{aligned} T'^{\mu\nu} &= U'^\mu V'^\nu = \Lambda^\mu{}_\rho\, U^\rho\, \Lambda^\nu{}_\sigma\, V^\sigma\,, \\ &= \Lambda^\mu{}_\rho\, \Lambda^\nu{}_\sigma\, T^{\rho\sigma}\,. \end{aligned} \tag{2.47}$$

Note that the gradient operator ∂_μ can also be used to map a tensor into another tensor. For example, if U_μ is a vector field (i.e. a vector that changes from place to place in spacetime) then $S_{\mu\nu} \equiv \partial_\mu U_\nu$ is a Lorentz tensor field, as may be verified by looking at its transformation rule under Lorentz transformations.

We make also define the operation of *Contraction*, which reduces a tensor to one with a smaller number of indices. A contraction is performed by setting an upstairs index on a tensor equal to a downstairs index. The Einstein summation convention then automatically comes into play, and the result is that one has an object with one fewer upstairs indices and one fewer downstairs indices. Furthermore, a simple calculation shows that the new object is itself a tensor. Consider, for example, a tensor $T^\mu{}_\nu$. This, of course, transforms as

$$T'^\mu{}_\nu = \Lambda^\mu{}_\rho\, \Lambda_\nu{}^\sigma\, T^\rho{}_\sigma \tag{2.48}$$

under Lorentz transformations. If we form the contraction and define $\phi \equiv T^\mu{}_\mu$, then we see that under Lorentz transformations we shall have

$$\begin{aligned} \phi' &\equiv T'^\mu{}_\mu = \Lambda^\mu{}_\rho\, \Lambda_\mu{}^\sigma\, T^\rho{}_\sigma\,, \\ &= \delta^\sigma_\rho\, T^\rho{}_\sigma = \phi\,. \end{aligned} \tag{2.49}$$

Since $\phi' = \phi$, it follows, by definition, that ϕ is a scalar.

An essentially identical calculation shows that for a tensor with a arbitrary numbers of upstairs and downstairs indices, if one makes an index contraction of one upstairs with one downstairs index, the result is a tensor with the corresponding reduced numbers of indices. Of course multiple contractions work in the same way.

The Minkowski metric $\eta_{\mu\nu}$ is itself a Lorentz tensor, but of a rather special type, known as an *invariant* tensor. This is because, unlike a generic 2-index tensor, the Minkowski metric is identical in all Lorentz frames. This can be seen by applying the tensor transformation rule (2.46) to the case of $\eta_{\mu\nu}$, giving

$$\eta'_{\mu\nu} = \Lambda_\mu{}^\rho \, \Lambda_\nu{}^\sigma \, \eta_{\rho\sigma} \,. \tag{2.50}$$

However, it follows from the condition (2.15) that the right-hand side of (2.50) is actually equal to $\eta_{\mu\nu}$, and hence we have $\eta'_{\mu\nu} = \eta_{\mu\nu}$, implying that $\eta_{\mu\nu}$ is an invariant tensor. This can be seen by first writing (2.15) in matrix language, as in (2.17): $\Lambda^T \, \eta \, \Lambda = \eta$. Then right-multiply by Λ^{-1} and left-multiply by η^{-1}; this gives $\eta^{-1} \, \Lambda^T \, \eta = \Lambda^{-1}$. Next left-multiply by Λ and right-multiply by η^{-1}, which gives $\Lambda \, \eta^{-1} \, \Lambda^T = \eta^{-1}$. (This is the analogue for the Lorentz transformations of the proof, for rotations, that $M^T \, M = 1$ implies $M \, M^T = 1$.) Converting back to index notation gives $\Lambda^\mu{}_\rho \, \Lambda^\nu{}_\sigma \, \eta^{\rho\sigma} = \eta^{\mu\nu}$. After some index raising and lowering, this gives $\Lambda_\mu{}^\rho \, \Lambda_\nu{}^\sigma \, \eta_{\rho\sigma} = \eta_{\mu\nu}$, which is the required result. The inverse metric $\eta^{\mu\nu}$ is also an invariant tensor.

We already saw that the gradient operator $\partial_\mu \equiv \partial/\partial x^\mu$ transforms as a covariant vector. If we define, in the standard way, $\partial^\mu \equiv \eta^{\mu\nu} \, \partial_\nu$, then it is evident from what we have seen above that the operator

$$\Box \equiv \partial^\mu \partial_\mu = \eta^{\mu\nu} \, \partial_\mu \partial_\nu \tag{2.51}$$

transforms as a scalar under Lorentz transformations. This is a very important operator, which is otherwise known as the wave operator, or d'Alembertian:

$$\Box = -\partial_0 \partial_0 + \partial_i \partial_i = -\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \,. \tag{2.52}$$

It is worth commenting further at this stage about a remark that was made earlier. Notice that in (2.52) we have been cavalier about the location of the Latin indices, which of course range only over the three spatial directions $i = 1$, 2 and 3. We can get away with this because the metric that is used to raise or lower the Latin indices is just the Minkowski metric restricted to the index values 1, 2 and 3. But since we have

$$\eta_{00} = -1 \,, \qquad \eta_{ij} = \delta_{ij} \,, \qquad \eta_{0i} = \eta_{i0} = 0 \,, \tag{2.53}$$

this means that Latin indices are lowered and raised using the Kronecker delta δ_{ij} and its inverse δ^{ij}. But these are just the components of the unit matrix, and so raising or lowering Latin indices has no effect. It is because of the minus sign associated with the η_{00} component of the Minkowski metric that we have to pay careful attention to the process of raising and lowering Greek indices. Thus, we can get away with writing $\partial_i\partial_i$, but we cannot write $\partial_\mu\partial_\mu$. Note, however, that once we move on to discussing general relativity, we shall need to be much more careful about always distinguishing between upstairs and downstairs indices.

We defined the Lorentz-invariant interval ds between infinitesimally-separated spacetime events by

$$ds^2 = \eta_{\mu\nu}\, dx^\mu dx^\nu = -dt^2 + dx^2 + dy^2 + dz^2\,. \tag{2.54}$$

This is the Minkowskian generalisation of the spatial interval in Euclidean space. Note that ds^2 can be positive, negative or zero. These cases correspond to what are called spacelike, timelike or null separations, respectively.

Note that neighbouring spacetime points on the worldline of a light ray are null separated. Consider, for example, a light front propagating along the x direction, with $x = t$ (recall that the speed of light is 1). Thus neighbouring points on the light front have the separations $dx = dt$, $dy = 0$ and $dz = 0$, and hence $ds^2 = 0$.

On occasion, it is useful to define the negative of ds^2, and write

$$d\tau^2 = -ds^2 = -\eta_{\mu\nu}\, dx^\mu dx^\nu = dt^2 - dx^2 - dy^2 - dz^2\,. \tag{2.55}$$

This is called the *Proper Time* interval, and τ is the proper time. Since ds is a Lorentz scalar, it is obvious that $d\tau$ is a scalar too.

We know that dx^μ transforms as a contravariant 4-vector. Since $d\tau$ is a scalar, it follows that

$$U^\mu \equiv \frac{dx^\mu}{d\tau} \tag{2.56}$$

is a contravariant 4-vector also. If we think of a particle following a path, or *worldline* in spacetime parameterised by the proper time τ, i.e. it follows the path $x^\mu = x^\mu(\tau)$, then U^μ defined in (2.56) is called the *4-velocity* of the particle.

Assuming that the particle is massive, and so it travels at less than the speed of light, one can parameterise its path using the proper time. For such a particle, we then have

$$U^\mu U_\mu = \eta_{\mu\nu}\, \frac{dx^\mu}{d\tau}\, \frac{dx^\nu}{d\tau} = \frac{1}{d\tau^2}\, \eta_{\mu\nu}\, dx^\mu dx^\nu = \frac{ds^2}{d\tau^2} = -1\,, \tag{2.57}$$

so the 4-velocity of any massive particle satisfies $U^\mu U_\mu = -1$.

If we divide (2.55) by dt^2 and rearrange the terms, we get

$$\frac{dt}{d\tau} = (1-u^2)^{-1/2} \equiv \gamma\,, \qquad \text{where} \quad \vec{u} = \Big(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\Big) \tag{2.58}$$

is the 3-velocity of the particle. Thus its 4-velocity can be written as

$$U^\mu = \frac{dx^\mu}{d\tau} = \gamma \frac{dx^\mu}{dt} = (\gamma, \gamma\,\vec{u})\,. \tag{2.59}$$

2.3 Electrodynamics in special relativity

Maxwell's equations, written in Gaussian units, and in additiona with the speed of light set to 1, take the form

$$\begin{aligned} \vec{\nabla}\cdot\vec{E} &= 4\pi\rho\,, \qquad \vec{\nabla}\times\vec{B} - \frac{\partial\vec{E}}{\partial t} = 4\pi\,\vec{J}\,, \\ \vec{\nabla}\cdot\vec{B} &= 0\,, \qquad \vec{\nabla}\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0\,. \end{aligned} \tag{2.60}$$

Introducing the 2-index antisymmetric Lorentz tensor $F_{\mu\nu} = -F_{\nu\mu}$, with components given by

$$F_{0i} = -E_i\,, \qquad F_{i0} = E_i\,, \qquad F_{ij} = \epsilon_{ijk}\,B_k\,, \tag{2.61}$$

it is then straightforward to see that the Maxwell equations (2.60) can be written in terms of $F_{\mu\nu}$ in the four-dimensional forms

$$\partial_\mu F^{\mu\nu} = -4\pi J^\nu\,, \tag{2.62}$$

$$\partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} = 0\,, \tag{2.63}$$

where $J^0 = \rho$ and J^i are just the components of the 3-current density $\vec{J}$. This can be seen by specialising the free index ν in (2.62) to be either $\nu = 0$, which then leads to the $\vec{\nabla}\cdot\vec{E}$ equation, or to $\nu = i$, which leads to the $\vec{\nabla}\times\vec{B}$ equation. In (2.63), specialising to $(\mu\nu\rho) = (0,i,j)$ gives the $\vec{\nabla}\times\vec{E}$ equation, while taking $(\mu,\nu,\rho) = (i,j,k)$ leads to the $\vec{\nabla}\cdot\vec{B}$ equation. (Look in my EM611 lecture notes if you need to see more details of the calculations.)

It is useful to look at the form of the 4-current density for a moving point particle with electric charge q. We have

$$\rho(\vec{r},t) = q\,\delta^3(\vec{r} - \vec{\bar{r}}(t))\,, \qquad \vec{J}(\vec{r},t) = q\,\delta^3(\vec{r} - \vec{\bar{r}}(t))\,\frac{d\vec{\bar{r}}(t))}{dt}\,, \tag{2.64}$$

where the particle is moving along the path $\vec{\bar{r}}(t)$. If we define $\bar{x}^0 = t$ and write $\vec{\bar{r}} = (\bar{x}^1, \bar{x}^2, \bar{x}^3)$, we can therefore write the 4-current density as

$$J^\mu(\vec{r},t) = q\,\delta^3(\vec{r} - \vec{\bar{r}}(t))\,\frac{d\bar{x}^\mu(t)}{dt}\,, \tag{2.65}$$

and so, by adding in an additional delta function factor in the time direction, together whether an integration over time, we can write[4]

$$J^\mu(\vec{r},t) = q\int dt'\,\delta^4(x^\nu - \bar{x}^\nu(t'))\,\frac{d\bar{x}^\mu(t')}{dt'}\,. \tag{2.66}$$

The differentials dt' cancel, and so we can just as well write the 4-current as

$$J^\mu(\vec{r},t) = q\int d\tau\,\delta^4(x^\nu - \bar{x}^\nu(\tau))\,\frac{d\bar{x}^\mu(\tau)}{d\tau} = q\int d\tau\,\delta^4(x^\nu - \bar{x}^\nu(\tau))\,U^\mu\,, \tag{2.67}$$

where τ is the proper time on the path of the particle, and $U^\mu = d\bar{x}^\mu(\tau)/d\tau$ is its 4-velocity. For a set of N charges q_n, following worldlines $x^\mu = \bar{x}^\mu_n$, we just take a sum of terms of the form (2.67):

$$J^\mu(\vec{r},t) = \sum_n q_n\int d\tau\,\delta^4(x^\nu - \bar{x}^\nu_n(\tau))\,\frac{d\bar{x}^\mu_n(\tau)}{d\tau}\,. \tag{2.68}$$

With the 4-current density J^μ written in the form (2.67), it is manifest that it is a Lorentz 4-vector. This follows since q is a scalar, τ is a scalar, U^μ is a 4-vector and $\delta^4(x^\nu - \bar{x}^\nu(\tau))$ is a scalar. (Integrating the four-dimensional delta function, using the (Lorentz-invariant) volume element d^4x yields a scalar, so it itself must be a scalar.) Using this fact, it can be seen from the Maxwell equations written in the form (2.62) and (2.63) that $F_{\mu\nu}$ is a Lorentz 4-tensor.

2.4 Energy-momentum tensor

Suppose we consider a point particle of mass m, following the worldline $x^\mu = \bar{x}^\mu(\tau)$. Its 4-momentum p^μ is defined in terms of its 4-velocity by

$$p^\upsilon = mU^\mu = m\frac{d\bar{x}^\mu(\tau)}{d\tau}\,. \tag{2.69}$$

Analogously to the current density discussed previously, we may define the *momentum density* of the particle:

$$T^{\mu 0} = p^\mu(t)\,\delta^3(\vec{r} - \vec{\bar{r}}(t))\,. \tag{2.70}$$

(We are temporarily using 3-dimensional notation.) Note that the momentum density is not a 4-vector, but rather, the components $T^{\mu 0}$ of a certain 2-index tensor, as we shall see below.

[4]Note that the notation $\delta^4(x^\nu - \bar{x}^\nu(t'))$ means the product of four delta functions, $\delta(x^0 - \bar{x}^0(t'))\,\delta(x^1 - \bar{x}^1(t'))\,\delta(x^2 - \bar{x}^2(t'))\,\delta(x^3 - \bar{x}^3(t'))$. The notation with the ν index is not ideal here. A similar kind of notational issue arises when we wish to indicate that a function, e.g. f, depends on the four spacetime coordinates (x^0, x^1, x^2, x^3). For precision, one would write $f(x^0, x^1, x^2, x^3)$, but sometimes one adopts a rather sloppy notation and writes $f(x^\nu)$.

We may also define the *momentum current* for the particle, as

$$T^{\mu i} = p^\mu(t)\,\delta^3(\vec{r} - \vec{\bar{r}}(t))\,\frac{d\bar{x}^i(t)}{dt}\,. \tag{2.71}$$

Putting the above definitions together, we have

$$T^{\mu\nu} = p^\mu(t)\,\delta^3(\vec{r} - \vec{\bar{r}}(t))\,\frac{d\bar{x}^\nu(t)}{dt}\,. \tag{2.72}$$

This is the energy-momentum tensor for the point particle. Two things are not manifestly apparent here, but are in fact true: Firstly, $T^{\mu\nu}$ is symmetric in its indices, that is to say, $T^{\mu\nu} = T^{\nu\mu}$. Secondly, $T^{\mu\nu}$ is a Lorentz 4-tensor.

Taking the first point first, the 4-momentum may be written as

$$p^\mu = (\mathcal{E}, \vec{p})\,, \tag{2.73}$$

where $\mathcal{E} = p^0$ is the energy, and and $\vec{p}$ is the relativistic 3-momentum. Thus we have

$$p^\mu = \Big(m\frac{dt}{d\tau}, m\frac{dx^i}{d\tau}\Big) = \Big(m\frac{dt}{d\tau}, m\frac{dt}{d\tau}\,\frac{dx^i}{dt}\Big) = \mathcal{E}\,\frac{dx^\mu}{dt}\,, \tag{2.74}$$

and so we may rewrite (2.72) as

$$T^{\mu\nu} = \frac{1}{\mathcal{E}}\,p^\mu(t)\,p^\nu(t)\,\delta^3(\vec{r} - \vec{\bar{r}}(t))\,. \tag{2.75}$$

This makes manifest that it is symmetric in μ and ν.

To show that $T^{\mu\nu}$ is a Lorentz tensor, we use the same trick as for the case of the current density, and add in an additional integration over time, together with a delta function in the time direction. Thus we write

$$T^{\mu\nu} = \int dt' p^\mu(t')\,\delta^4(x^\alpha - \bar{x}^\alpha(t'))\,\frac{d\bar{x}^\nu(t')}{dt'} = \int d\tau\, p^\mu(\tau)\,\delta^4(x^\alpha - \bar{x}^\alpha(\tau))\,\frac{d\bar{x}^\nu(\tau)}{d\tau}\,. \tag{2.76}$$

Everything in the final expression is constructed from Lorentz scalars and 4-vectors, and hence $T^{\mu\nu}$ must be a Lorentz 4-tensor.

Clearly, for a system of N non-interacting particles of masses m_n following worldlines $x^\mu = \bar{x}_n^\mu$, the total energy-momentum tensor will be just the sum of contributions of the form discussed above:

$$T^{\mu\nu} = \sum_n \int d\tau\, p_n^\mu(\tau)\,\delta^4(x^\nu - \bar{x}_n^\nu(\tau))\,\frac{d\bar{x}_n^\mu(\tau)}{d\tau}\,. \tag{2.77}$$

The energy-momentum tensor for a closed system is conserved, namely

$$\partial_\nu T^{\mu\nu} = 0\,. \tag{2.78}$$

This is the analogue of the charge-conservation equation $\partial_\mu J^\mu = 0$ in electrodynamics, except that now the analogous conservation law is that the total 4-momentum is conserved. The proof for an isolated particle, or non-interacting system of particles, goes in the same way that one proves conservation of J^μ for a charged particle or system of particles. Thus we have

$$\begin{aligned}\partial_\nu T^{\mu\nu} &= \partial_0 T^{\mu 0} + \partial_i T^{\mu i}\,, \\ &= \sum_n \frac{dp_n^\mu(t)}{dt}\,\delta^3(\vec r - \vec r_n(t)) + \sum_n p_n^\mu(t)\,\frac{\partial}{\partial t}\delta^3(\vec r - \vec r_n(t)) \\ &\quad + \sum_n p_n^\mu(t)\,\Big(\frac{\partial}{\partial x^i}\,\delta^3(\vec r - \vec r_n(t))\Big)\,\frac{d\bar x_n^i(t)}{dt}\,. \end{aligned} \tag{2.79}$$

The last term can be rewritten as

$$-\sum_n p_n^\mu(t)\,\Big(\frac{\partial}{\partial \bar x_n^i}\,\delta^3(\vec r - \vec r_n(t))\Big)\,\frac{d\bar x_n^i(t)}{dt}\,, \tag{2.80}$$

which, by the chain rule for differentiation, gives

$$-\sum_n p_n^\mu(t)\,\frac{\partial}{\partial t}\delta^3(\vec r - \vec r_n(t))\,. \tag{2.81}$$

This therefore cancels the second term in (2.79), leaving the result

$$\partial_\nu T^{\mu\nu} = \sum_n \frac{dp_n^\mu(t)}{dt}\,\delta^3(\vec r - \vec r_n(t))\,. \tag{2.82}$$

Thus, if no external forces act on the particles, so that $dp_n^\mu(t)/dt = 0$, the the energy -momentum tensor will be conserved.

Suppose now that we consider particles that are electrically charged, and that they are in the presence of an electromagnetic field $F_{\mu\nu}$. The Lorentz force law for a particle of charge q is[5]

$$\frac{dp^\mu}{d\tau} = q F^\mu{}_\nu\,\frac{dx^\nu}{d\tau}\,, \tag{2.83}$$

and hence, multiplying by $d\tau/dt$,

$$\frac{dp^\mu}{dt} = q F^\mu{}_\nu\,\frac{dx^\nu}{dt}\,. \tag{2.84}$$

For a system of N particles, with masses m_n and charges q_n, it follows from (2.82), and the definition (2.65), that the energy-momentum tensor for the particles will now satisfy

$$\partial_\nu T^{\mu\nu}_{\rm part.} = F^\mu{}_\nu\,J^\nu\,, \tag{2.85}$$

[5]By taking $\mu = i$ and using (2.61), is easy to see that this is equivalent to the 3-vector equation

$$\frac{d\vec p}{dt} = e\,(\vec E + \vec v \times \vec B)$$

where J^ν is the sum of the current-density contributions for the N particles. We have added a subscript "part." to the energy-momentum tensor, to indicate that this is specifically the energy-momentum tensor of the particles alone. Not surprisingly, it is not conserved, because the particles are being acted on by the electromagnetic field.

In order to have a closed system in this example, we must include also the energy-momentum tensor of the electromagnetic field. For now, we shall just present the answer, since later in the course when we consider electromagnetism in general relativity, we shall have a very simple method available to us for computing it. The answer, for the electromagnetic field, is that its energy-momentum tensor is given by[6]

$$T^{\mu\nu}_{\rm em} = \frac{1}{4\pi}\Big(F^{\mu\rho}\, F^\nu{}_\rho - \tfrac14 F^{\rho\sigma}\, F_{\rho\sigma}\, \eta^{\mu\nu}\Big)\,. \tag{2.86}$$

Note that it is symmetric in μ and ν. If we take the divergence, we find

$$\begin{aligned}\partial_\nu T^{\mu\nu}_{\rm em} &= \frac{1}{4\pi}\Big(F^{\mu\rho}\,\partial_\nu F^\nu{}_\rho + (\partial_\nu F^{\mu\rho})F^\nu{}_\rho - \tfrac12 F^{\rho\sigma}\,\partial_\nu F_{\rho\sigma}\,\eta^{\mu\nu}\Big)\,,\\ &= \frac{1}{4\pi}\Big(F^{\mu\rho}\,(-4\pi J_\rho) + (\partial_\nu F^{\mu\rho})F^\nu{}_\rho + \tfrac12 F^{\rho\sigma}(\partial_\rho F_\sigma{}^\mu + \partial_\sigma F^\mu{}_\rho)\Big)\,. \end{aligned} \tag{2.87}$$

In getting from the first line to the second line we have used (2.62) in the first term, and (2.63) in the last term. It is now easy to see with some relabelling of dummy indices, and making use of the antisymmetry of $F_{\mu\nu}$, that the last three terms in the second line add to zero, thus leaving us with the result

$$\partial_\nu T^{\mu\nu}_{\rm em} = -F^{\mu\rho}\, J_\rho\,. \tag{2.88}$$

This implies that in the absence of sources $\partial_\nu T^{\mu\nu}_{\rm em} = 0$, as it should for an isolated system. Going back to our discussion of a system of charged particles in an electromagnetic field, we see from from (2.85) and (2.88) that the total energy-momentum tensor for this system, i.e.

$$T^{\mu\nu} \equiv T^{\mu\nu}_{\rm part.} + T^{\mu\nu}_{\rm em} \tag{2.89}$$

is indeed conserved, $\partial_\nu T^{\mu\nu} = 0$.

An important point to note is that the T^{00} component of the energy-momentum tensor is the energy density. This can be seen for the point particle case from (2.70), which implies $T^{00} = \mathcal{E}\,\delta^3(\vec r - \vec{\bar r}(t))$, with $\mathcal{E}$ being the energy of the particle. In the case of electromagnetism, one can easily see from (2.86), using the definitions (2.61), that $T^{00}_{\rm em} = (E^2 + B^2)/(8\pi)$, which is indeed the well-known result for the energy density of the electromagnetic field.

[6]See, for example, my E&M 611 lecture notes for a derivation of the energy-momentum tensor for the electromagnetic field in Minkowski spacetime.

As a final important example of an energy-momentum tensor, we may consider a perfect fluid. When the fluid is at rest at a particular spacetime point, the energy-momentum tensor at that point will be given by

$$\bar{T}^{00} = \rho\,, \qquad \bar{T}^{ij} = p\,\delta^{ij}\,, \qquad \bar{T}^{i0} = \bar{T}^{0i} = 0\,, \tag{2.90}$$

where ρ is the energy density and p is the pressure. From the viewpoint of an arbitrary Lorentz frame we just have to find a Lorentz 4-tensor expression that reduces to $\bar{T}^{\mu\nu}$ when the 3-velocity vanishes. The answer is

$$T^{\mu\nu} = p\,\eta^{\mu\nu} + (p+\rho)\,U^\mu U^\nu\,, \tag{2.91}$$

since in the rest frame the 4-velocity $U^\mu = dx^\mu/d\tau$ reduces to $U^0 = 1$ and $U^i = 0$.

3 Gravitational Fields in Minkowski Spacetime

As mentioned in the introduction, to the extent that one can still talk about a "gravitational force" in general relativity (essentially, in the weak-field Newtonian limit), it is a phenomenon that is viewed as resulting from being in a frame that is accelerating with respect to a local inertial frame. This might, for example, be because one is standing on the surface of the earth. Or it might be because one is in a spacecraft with its rocket engine running, that is accelerating while out in free space far away from any stars or planets. We can gain many insights into the principles of general relativity by thinking first about these simple kinds of situation where the effects of "ponderable matter" can be neglected.

Suppose that there is a particle moving in Minkowski spacetime, with no external forces acting on it. Viewed from a frame $\widetilde{S}$ in which the spacetime metric is literally the Minkowski metric

$$ds^2 = \eta_{\mu\nu}\,d\tilde{x}^\mu d\tilde{x}^\nu = -d\tilde{t}^2 + d\tilde{x}^2 + d\tilde{y}^2 + d\tilde{z}^2\,, \tag{3.1}$$

the particle will be moving along a worldline $\tilde{x}^\mu = \tilde{x}^\mu(\tau)$ that is just a straight line, which may be characterised by the equation

$$\frac{d^2\tilde{x}^\mu}{d\tau^2} = 0\,. \tag{3.2}$$

Now suppose that we make a completely general coordinate transformation to a frame S whose coordinates are related to the $\tilde{x}^\mu$ coordinates by $x^\mu = x^\mu(\tilde{x}^\nu)$. We shall assume that the Jacobian of the transformation is non-zero, so that we can invert the relation, and write

$$\tilde{x}^\mu = \tilde{x}^\mu(x^\nu)\,. \tag{3.3}$$

Using the chain rule for differentiation, we shall therefore have

$$\frac{d\tilde{x}^\mu}{d\tau} = \frac{\partial \tilde{x}^\mu}{\partial x^\nu}\, \frac{dx^\nu}{d\tau}\,, \tag{3.4}$$

and hence

$$\frac{d^2\tilde{x}^\mu}{d\tau^2} = \frac{d^2 x^\nu}{d\tau^2}\, \frac{\partial \tilde{x}^\mu}{\partial x^\nu} + \frac{\partial^2 \tilde{x}^\mu}{\partial x^\rho\, \partial x^\nu}\, \frac{dx^\rho}{d\tau}\, \frac{dx^\nu}{d\tau} = 0\,, \tag{3.5}$$

where the vanishing of this expression follows from (3.2). Using the assumed invertibility of the transformation, and the result from the chain rule that

$$\frac{\partial \tilde{x}^\mu}{\partial x_\nu}\, \frac{\partial x^\sigma}{\partial \tilde{x}^\mu} = \delta^\sigma_\nu\,, \tag{3.6}$$

we therefore have that

$$\frac{d^2 x^\sigma}{d\tau^2} + \frac{\partial x^\sigma}{\partial \tilde{x}^\mu}\, \frac{\partial^2 \tilde{x}^\mu}{\partial x^\rho\, \partial x^\nu}\, \frac{dx^\rho}{d\tau}\, \frac{dx^\nu}{d\tau} = 0\,. \tag{3.7}$$

We may write this equation, after a relabelling of indices to neaten it up a bit, in the form

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu{}_{\nu\rho}\, \frac{dx^\nu}{d\tau}\, \frac{dx^\rho}{d\tau} = 0\,, \tag{3.8}$$

where

$$\Gamma^\mu{}_{\nu\rho} = \frac{\partial x^\mu}{\partial \tilde{x}^\sigma}\, \frac{\partial^2 \tilde{x}^\sigma}{\partial x^\nu\, \partial x^\rho}\,. \tag{3.9}$$

Note that $\Gamma^\mu{}_{\nu\rho}$ is symmetric in ν and ρ. Equation (3.8) is known as the *Geodesic Equation*, and $\Gamma^\mu{}_{\nu\rho}$ is called the *Christoffel Connection*. It should be emphasised that even though the affine connection is an object with spacetime indices on it, it is *not* a tensor.

Equation (3.8) describes the worldline of the particle, as seen from the frame S. Observe that it is not, in general, moving along a straight line, because of the second term involving the quantity $\Gamma^\mu{}_{\nu\rho}$ defined in (3.9). What we are seeing is that the particle is moving in general along a curved path, on account of the "gravitational force" that it experiences due to the fact that the frame S is not an inertial frame. Of course, if we hade made a restricted coordinate transformation that caused $\Gamma^\mu{}_{\nu\rho}$ to be zero, then the motion of the particle would still be in a straight line. The condition for $\Gamma^\mu{}_{\nu\rho}$ to vanish would be that

$$\frac{\partial^2 \tilde{x}^\sigma}{\partial x^\nu\, \partial x^\rho} = 0\,. \tag{3.10}$$

This is exactly the condition that we derived in (2.28) when looking for the most general possible coordinate transformations that left the Minkowski metric $ds^2 = \eta_{\mu\nu}\, dx^\mu dx^\nu$ invariant. The solution to those equations gave us the Poincaré transformations (2.30).

To summarise, we have seen above that if we make an arbitrary Poincaré transformation of the original Minkowski frame $\widetilde{S}$, we end up in a new frame where the metric is still the

Minkowski metric, and the free particle continues to move in a straight line. This is the arena of Special Relativity. If, on the other hand, we make a general coordinate transformation that leads to a non-vanishing $\Gamma^\mu{}_{\nu\rho}$, the particle will no longer move in a straight line, and we may attribute this to the "force of gravity" in that frame. Furthermore, the metric will no longer be the Minkowski metric. We are heading towards the arean of general relativity, although we are still, for now discussing the subclass of metrics that are merely coordinate transformations of the flat Minkowksi metric.

It is instructive now to calculate the metric that we obtain when we make the general coordinate transformation of the original Minkowski metric. Using the chain rule we have $d\tilde{x}^\mu = (\partial\tilde{x}^\mu/\partial x^\nu)\, dx^\nu$, and so the Minkowski metric becomes

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu\,, \tag{3.11}$$

where

$$g_{\mu\nu} = \eta_{\rho\sigma}\, \frac{\partial\tilde{x}^\rho}{\partial x^\mu}\, \frac{\partial\tilde{x}^\sigma}{\partial x^\nu}\,. \tag{3.12}$$

We can in fact express the quantities $\Gamma^\mu{}_{\nu\rho}$ given in (3.9) in terms of the metric tensor $g_{\mu\nu}$. To do this, we begin by multiplying (3.9) by $(\partial\tilde{x}^\lambda/\partial x^\mu)$, making use of the relation, which follows from the chain rule, that

$$\frac{\partial\tilde{x}^\lambda}{\partial x^\mu}\, \frac{\partial x^\mu}{\partial\tilde{x}^\sigma} = \delta^\lambda_\sigma\,. \tag{3.13}$$

Thus we get

$$\frac{\partial^2\tilde{x}^\lambda}{\partial x^\nu\partial x^\rho} = \frac{\partial\tilde{x}^\lambda}{\partial x^\mu}\, \Gamma^\mu{}_{\nu\rho}\,. \tag{3.14}$$

Now differentiate (3.12) with respect to x^λ:

$$\begin{aligned}
\frac{\partial g_{\mu\nu}}{\partial x^\lambda} &= \eta_{\rho\sigma}\, \frac{\partial^2\tilde{x}^\rho}{\partial x^\lambda\partial x^\mu}\, \frac{\partial\tilde{x}^\sigma}{\partial x^\nu} + \eta_{\rho\sigma}\, \frac{\partial\tilde{x}^\rho}{\partial x^\mu}\, \frac{\partial^2\tilde{x}^\sigma}{\partial x^\lambda\partial x^\nu}\,,\\
&= \eta_{\rho\sigma}\, \frac{\partial\tilde{x}^\rho}{\partial x^\alpha}\, \Gamma^\alpha{}_{\lambda\mu}\, \frac{\partial\tilde{x}^\sigma}{\partial x^\nu} + \eta_{\rho\sigma}\, \frac{\partial\tilde{x}^\rho}{\partial x^\mu}\, \frac{\partial\tilde{x}^\sigma}{\partial x^\alpha}\, \Gamma^\alpha{}_{\lambda\nu}\,,\\
&= g_{\alpha\nu}\, \Gamma^\alpha{}_{\lambda\mu} + g_{\alpha\mu}\, \Gamma^\alpha{}_{\lambda\nu}\,,
\end{aligned} \tag{3.15}$$

where we have used (3.14) in getting to the second line, and then (3.12) in getting to the third line. We now take this equation, add the equation with μ and λ interchanged, and subtract the equation with ν and λ exchanged. This gives

$$\begin{aligned}
\frac{\partial g_{\mu\nu}}{\partial x^\lambda} + \frac{\partial g_{\lambda\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\lambda}}{\partial x^\nu} &= g_{\alpha\nu}\, \Gamma^\alpha{}_{\lambda\mu} + g_{\alpha\mu}\, \Gamma^\alpha{}_{\lambda\nu} + g_{\alpha\nu}\, \Gamma^\alpha{}_{\mu\lambda} + g_{\alpha\lambda}\, \Gamma^\alpha{}_{\mu\nu} - g_{\alpha\lambda}\, \Gamma^\alpha{}_{\nu\mu} - g_{\alpha\mu}\, \Gamma^\alpha{}_{\nu\lambda}\,,\\
&= 2g_{\alpha\nu}\, \Gamma^\alpha{}_{\mu\lambda}\,,
\end{aligned} \tag{3.16}$$

after making use of the fact, which is evident from (3.9), that $\Gamma^\alpha{}_{\mu\nu} = \Gamma^\alpha{}_{\nu\mu}$. Defining the inverse metric $g^{\mu\nu}$ by the requirement that

$$g^{\mu\nu}\, g_{\nu\rho} = \delta^\mu_\rho\,, \tag{3.17}$$

we finally arrive at the result that

$$\Gamma^\mu{}_{\nu\rho} = \tfrac{1}{2} g^{\mu\lambda} \Big(\frac{\partial g_{\lambda\rho}}{\partial x^\nu} + \frac{\partial g_{\nu\lambda}}{\partial x^\rho} - \frac{\partial g_{\nu\rho}}{\partial x^\lambda} \Big) . \tag{3.18}$$

4 General-Coordinate Tensor Analysis in General Relativity

In the previous section we examined some aspects of special relativity when viewed within the enlarged framework of coordinate systems that are related to an original intertial system by means of completely arbitrary transformations of the coordinates. Of course, these transformations lie outside the restricted set of transformations normally considered in special relativity, since they did not preserve the form of the Minkowski metric $\eta_{\mu\nu}$. Only the very restricted subset of Poincaré transformations (2.30) would leave $\eta_{\mu\nu}$ invariant. Instead, the general coordinate transformations we considered mapped the system to a non-intertial frame, and we could see the way in which "gravitational forces" appeared in these frames, as reflected in the fact that the geodesic equation (3.8) demonstrated that a particle with no external forces acting would no longer move in linear motion, on account of the non-vanishing affine connection $\Gamma^\mu{}_{\nu\rho}$.

The non-Minkowskian metric $g_{\mu\nu}$ in the spacetime viewed from the frame S in the previous discussion was nothing but a corodinate transformation of the Minkowski metric. Now, we shall "kick away the ladder" of the construction in the previous section, and begin afresh with the proposal that a spacetime in general can have have a metric $g_{\mu\nu}$ that is not necessarily related to the Minkowski metric by a coordinate transformation. In general, $g_{\mu\nu}$ may be a metric on a curved spacetime, as opposed to Minkowski spacetime, which is flat. The precise way in which the curvature of a spacetime is characterised will emerge as we go along. In the spirit of the earlier discussion, the idea will be that we allow completely arbitrary transformations from one coordinate system to another. The goal will be to develop an appropriate tensor calculus that will allow us to formulate the fundamental laws of physics in such a way that they take the same form in *all* coordinate frames. This extends the notion in special relativity that the fundamental laws of physics should take the same form in all inertial frames.

The framework that we shall be developing here falls under the general rubric of *Riemannian Geometry.* In fact, since we shall be concerned with spacetimes where the metric tensor, like the Minkowski metric, has one negative eigenvalue and three positive, the more precise terminolgy is *pseudo-Riemannian Geometry.* (The term Riemannian Geometry is used when the metric is of positive-definite signature; i.e. when all its eigenvalues are positive.)

The starting point for our discussion will be to introduce the notion of quantities that are vectors or tensors under general coordinate transformations.

4.1 Vector and co-vector fields

When discussing vector fields in curved spaces, or indeed whenever we use a non-Minkowskian or non-Cartesian system of coordinates, we have to be rather more careful about how we think of a vector. In Cartesian or Minkowski space, we can think of a vector as corresponding to an arrow joining one point to another point, which could be nearby or it could be far away. In a curved space or even in a flat space written in a non-Cartesian cordinate system, it makes no sense to think of a line joining two non-infinitesimally separated points as representing a vector. For example, on the surface of the earth we can think of a very short arrow on the surface as representing a vector, but not a long arrow such as one joining London to New York. The precise notion of a vector requires that we should consider just arrows joining infinitesimally-separated points.

To implement this idea, we may consider a curve in spacetime, that is to say, a worldline. We may suppose that points along the worldline are parameterised by a parameter λ that increases monotonically along the worldline. If we consider neighbouring points on the curve, parameterised by λ and $\lambda + d\lambda$, then the infinitesimal interval on the curve between the two points will be like a little straight-line segment, which defines the tangent to the curve at the point λ. By Taylor's theorem, the derivative operator

$$V = \frac{d}{d\lambda} \tag{4.1}$$

is the generator of the translation along the tangent to the curve:

$$f(\lambda + d\lambda) = f(\lambda) + \frac{df(\lambda)}{d\lambda}\, d\lambda + \cdots\,. \tag{4.2}$$

Thus we may think of $V = d/d\lambda$ as defining the tangent vector to the curve. Notice that this has been defined without reference to any particular coordinate system.

Suppose now that we choose some coordinate system x^μ that is defined in a region that includes the neighbourhood of the point λ on the curve. The curve may now be specified by giving the coordinates of each point, as functions of λ:

$$x^\mu = x^\mu(\lambda)\,. \tag{4.3}$$

Using the chain rule, can now write the vector V as

$$V = \frac{d}{d\lambda} = \frac{dx^\mu}{d\lambda}\,\frac{\partial}{\partial x^\mu}\,. \tag{4.4}$$

In fact, we can view the quantities $dx^\mu/d\lambda$ as the *components* of V with respect to the coordinate system x^μ:

$$V = V^\mu \frac{\partial}{\partial x^\mu}\,, \qquad \text{with} \quad V^\mu = \frac{dx^\mu}{d\lambda}\,. \tag{4.5}$$

In order to abbreviate the writing a bit, we shall henceforth use the same shorthand for partial coordinate derivatives that we introduced earlier when discussing special relativity, and write

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu}\,. \tag{4.6}$$

Thus the vector V can be written in terms of its components V^μ in the x^μ coordinate frame as

$$V = V^\mu\,\partial_\mu\,. \tag{4.7}$$

If we now consider another coordinate system $x'\mu$ that is defined in a region that also includes the neighbourhood of the point λ on the curve, then we may also write the vector V as

$$V = V'^\mu\,\partial'_\mu\,, \tag{4.8}$$

where, of course, ∂'_μ means $\partial/\partial x'^\mu$. Notice that the vector V itself is exactly the same in the two cases, since as emphasised above, it is itself defined without reference to any coordinate system at all. However, when we write V in terms of its components in a coordinate basis, then those components will differ as between one coordinate basis and another. Using the chain rule, we clearly have

$$\frac{\partial}{\partial x^\nu} = \frac{\partial x'^\mu}{\partial x^\nu}\,\frac{\partial}{\partial x'^\mu}\,, \qquad \text{i.e.} \quad \partial_\nu = \frac{\partial x'^\mu}{\partial x^\nu}\,\partial'_\mu\,, \tag{4.9}$$

and so from

$$V = V'^\mu\,\partial'_\mu = V^\nu\,\partial_\nu = V^\nu\,\frac{\partial x'^\mu}{\partial x^\nu}\,\partial'_\mu \tag{4.10}$$

we can read off that the components of V with respect to the primed and the unprimed coordinate systems are related by

$$V'^\mu = \frac{\partial x'^\mu}{\partial x^\nu}\,V^\nu\,. \tag{4.11}$$

In fact we don't really need to introduce the notion of the curve parameterised by λ in order to discuss the vector field. Such a curve, or indeed a whole family of curves filling the whole spacetime, could always be set up if desired. But we can carry away from this construction the essential underlying idea, that a vector field can always be viewed as a derivative operator, which can then be expanded in terms of its components in a coordinate basis, as in eqn (4.6). Under a change of coordinate basis induced by the general coordinate transformation $x'^\mu = x'^\mu(x^\nu)$, the components will transform according to the transformation rule (4.11). Thus, by definition, we shall say that a vector is a geometrical object whose components transform as in (4.11).

In practice, there is often a tendency to abbreviate the statement slightly, and to speak of the components V^μ themselves as being the vector. One would then say that V^μ is a vector under general coodinate transformations if it transforms in the manner given in eqn (4.11). Note that this extends the notion of the *Lorentz Vector* that we discussed in special relativity, where it was only required to transform in the given manner (eqn (2.32)) under the highly restricted subset of coordinate transformations that were Lorentz transformations.

As we saw above, a vector field can be thought of as a differential operator that generates a translation along a tangent to a curve. For this reason, vector fields are said to live in the *tangent space* of the manifold or spacetime. One can then define the dual space of the tangent space, which is known as the *co-tangent space*. This is done by establishing a pairing between a tangent vector and a co-tangent vector, resulting in a scalar field which, by definition, does not transform under general coordinate transformations. If V is a vector and ω is a co-tangent vector, the pairing is denoted by

$$\langle \omega | V \rangle \,. \tag{4.12}$$

This pairing is also known as the *inner product* of ω and V. The co-tangent vector ω is defined in terms of its components ω_μ in a coordinate frame by

$$\omega = \omega_\mu \, dx^\mu \,. \tag{4.13}$$

The pairing is defined in the coordinate basis by

$$\langle dx^\mu | \frac{\partial}{\partial x^\nu} \rangle = \delta^\mu_\nu \,, \tag{4.14}$$

and so we shall have

$$\langle \omega | V \rangle = \langle \omega_\mu dx^\mu | V^\nu \frac{\partial}{\partial x^\nu} \rangle = \omega_\mu \, V^\nu \, \langle dx^\mu | \frac{\partial}{\partial x^\nu} \rangle = \omega_\mu \, V^\nu \, \delta^\mu_\nu = \omega_\mu \, V^\mu \,. \tag{4.15}$$

Just as the vector V itself is independent of the choice of coordinate system, so too is the co-vector ω, and so by using the chain rule we can calculate how its components change under a general coordinate transformation. Thus we shall have

$$\omega = \omega'_\mu \, dx'^\mu = \omega_\nu \, dx^\nu = \omega_\nu \, \frac{\partial x^\nu}{\partial x'^\mu} \, dx'^\mu \,, \tag{4.16}$$

from which we can read off that

$$\omega'_\mu = \frac{\partial x^\nu}{\partial x'^\mu} \, \omega_\nu \,. \tag{4.17}$$

We can now verify that indeed the inner product $\langle\omega|V\rangle$ is a general coordinate scalar, since we know how the components V^μ of V transform (4.11) and how the components ω_μ of ω transform (4.17). Thus in the primed coordinate system we have

$$\langle\omega|V\rangle = \omega'_\mu\, V'^\mu = \frac{\partial x^\nu}{\partial x'^\mu}\,\omega_\nu\,\frac{\partial x'^\mu}{\partial x^\rho}\,V^\rho = \omega_\nu\,V^\rho\,\delta^\nu_\rho = \omega_\nu\,V^\nu\,, \tag{4.18}$$

thus showing that it equals $\langle\omega|V\rangle$ in the unprimed coordinate system, and hence it is a general coordinate scalar. Not that in deriving this we used the result, which follows from the chain rule and the definition of partial differentiation, that

$$\frac{\partial x^\nu}{\partial x'^\mu}\,\frac{\partial x'^\mu}{\partial x^\rho} = \delta^\nu_\rho\,. \tag{4.19}$$

4.2 General-coordinate tensors

Having obtained the transformation rule of the components V^μ of a vector field in (4.11), and the components ω_μ of a co-vector field in (4.17), we can now immediately give the extension to transformation of an arbitrary tensor field. Such a field will have components with some number p of vector indices, and some number q of co-vector indices (otherwise known as upstairs and downstairs indices resepectively), and will transform as

$$T'^{\mu_1\cdots\mu_p}{}_{\nu_1\cdots\nu_q} = \frac{\partial x'^{\mu_1}}{\partial x^{\rho_1}}\cdots\frac{\partial x'^{\mu_p}}{\partial x^{\rho_p}}\,\frac{\partial x^{\sigma_1}}{\partial x'^{\nu_1}}\cdots\frac{\partial x^{\sigma_q}}{\partial x'^{\nu_q}}\,T^{\rho_1\cdots\rho_p}{}_{\sigma_1\cdots\sigma_q}\,. \tag{4.20}$$

Thus there are p factors of $(\partial x')/\partial x)$ and q factors of $(\partial x)/(\partial x')$ in the transformation. The actual "geometrical object" T of which $T^{\mu_1\cdots\mu_p}{}_{\nu_1\cdots\nu_q}$ are the components in a coordinate frame would be written as

$$T = T^{\mu_1\cdots\mu_p}{}_{\nu_1\cdots\nu_q}\,\partial_{\mu_1}\otimes\cdots\otimes\partial_{\mu_p}\otimes dx^{\nu_1}\otimes\cdots\otimes dx^{\nu_q}\,. \tag{4.21}$$

T then lives in the p-fold tensor product of the tangent space times the q-fold tensor product of the co-tangent space. T itself is coordinate-independent, but its components $T^{\mu_1\cdots\mu_p}{}_{\nu_1\cdots\nu_q}$ transform under general coordinate transgformations according to (4.20). We may refer to T as being a (p,q) general-coordinate tensor. A vector is the special case of a $(1,0)$ tensor, and a co-vector is the special case of a $(0,1)$ tensor. Of course a scalar field is a $(0,0)$ tensor. As in the case of vectors, which we remarked upon earlier, it is common to adopt a slightly sloppy terminology and and to refer to $T^{\mu_1\cdots\mu_p}{}_{\nu_1\cdots\nu_q}$ as a (p,q) tensor, rather than giving it the rather more proper but cumbersome description of being "the components of the (p,q) tensor T with respect to a coordinate frame." Of course, if there is no ambiguity as to which tensor one is talking about, one might very well omit the (p,q) part of the description.

General-coordinate vectors, co-vectors and tensors satisfy all the obvious properties that follow from their defined transformation rules. For example, if T and S are any two (p,q) tensors, then $T+S$ is also a (p,q) tensor. If T is a (p,q) tensor, then ϕT is also a (p,q) tensor, where ϕ is any scalar field. This is really a special case of a more general result, that if T is a (p_1,q_1) tensor and S is a (p_2,q_2) tensor, then the tensor product (in the sense of the tensor products in (4.21)) $T\otimes S$ is a (p_1+p_2,q_1+q_2) tensor. Restated in more human language, and as an example, if U and V are vectors then $W=U\otimes V$ is a $(2,0)$ tensor, with components

$$W^{\mu\nu} = U^\mu\, V^\nu\,. \tag{4.22}$$

As one can immediately see from the transformation rule (4.11) applied to U and to V, one indeed has

$$W'^{\mu\nu} = \frac{\partial x'^\mu}{\partial x^\rho}\,\frac{\partial x'^\nu}{\partial x^\sigma}\,W^{\rho\sigma}\,, \tag{4.23}$$

which is in accordance with the general transformation rule (4.20) for the special case of a $(2,0)$ tensor.

Another very important property of general-coordinate tensors is that if an upstairs and a downstairs index on a (p,q) tensor are *contracted*, then the result is a $(p-1,q-1)$ tensor. Here, the operation of *contraction* means setting the upstairs index equal to the downstairs index, which then means, by virtue of the Einstein summation convention, that this repeated index is now understood to be summed over. For example, if we start from the (p,q) tensor T we considered above, and if we set the upper index μ_1 equal to the lower index ν_1, then we obtain the quantity

$$S^{\mu_2\cdots\mu_p}{}_{\nu_2\cdots\nu_q} \equiv T^{\nu_1\mu_2\cdots\mu_p}{}_{\nu_1\nu_2\cdots\nu_q}\,. \tag{4.24}$$

We check its transformation properties by using the known transformations (4.20) to calculate it in the primed frame:

$$\begin{aligned}
S'^{\mu_2\cdots\mu_p}{}_{\nu_2\cdots\nu_q} &= T'^{\nu_1\mu_2\cdots\mu_p}{}_{\nu_1\nu_2\cdots\nu_q}\,,\\
&= \frac{\partial x'^{\nu_1}}{\partial x^{\rho_1}}\,\frac{\partial x'^{\nu_2}}{\partial x^{\rho_2}}\cdots\frac{\partial x'^{\mu_p}}{\partial x^{\rho_p}}\,\frac{\partial x^{\sigma_1}}{\partial x'^{\nu_1}}\,\frac{\partial x^{\sigma_2}}{\partial x'^{\nu_2}}\cdots\frac{\partial x^{\sigma_q}}{\partial x'^{\nu_q}}\,T^{\rho_1\rho_2\cdots\rho_p}{}_{\sigma_1\sigma_2\cdots\sigma_q}\,,\\
&= \frac{\partial x'^{\nu_2}}{\partial x^{\rho_2}}\cdots\frac{\partial x'^{\mu_p}}{\partial x^{\rho_p}}\,\frac{\partial x^{\sigma_2}}{\partial x'^{\nu_2}}\cdots\frac{\partial x^{\sigma_q}}{\partial x'^{\nu_q}}\,T^{\rho_1\rho_2\cdots\rho_p}{}_{\sigma_1\sigma_2\cdots\sigma_q}\,\delta^{\sigma_1}_{\rho_1}\,,\\
&= \frac{\partial x'^{\nu_2}}{\partial x^{\rho_2}}\cdots\frac{\partial x'^{\mu_p}}{\partial x^{\rho_p}}\,\frac{\partial x^{\sigma_2}}{\partial x'^{\nu_2}}\cdots\frac{\partial x^{\sigma_q}}{\partial x'^{\nu_q}}\,T^{\sigma_1\rho_2\cdots\rho_p}{}_{\sigma_1\sigma_2\cdots\sigma_q}\,,\\
&= \frac{\partial x'^{\nu_2}}{\partial x^{\rho_2}}\cdots\frac{\partial x'^{\mu_p}}{\partial x^{\rho_p}}\,\frac{\partial x^{\sigma_2}}{\partial x'^{\nu_2}}\cdots\frac{\partial x^{\sigma_q}}{\partial x'^{\nu_q}}\,S^{\rho_2\cdots\rho_p}{}_{\sigma_2\cdots\sigma_q}\,,
\end{aligned} \tag{4.25}$$

thus showing that S transforms in the way that a $(p-1,q-1)$ tensor should. The crucial step in the above calculation was the one between lines two and three, where the contracted pair of transformation matrices gave rise to the Kronecker delta:

$$\frac{\partial x'^{\nu_1}}{\partial x^{\rho_1}}\,\frac{\partial x^{\sigma_1}}{\partial x'^{\nu_1}} = \delta^{\sigma_1}_{\rho_1}\,. \tag{4.26}$$

We already saw a simple example of this property above, when we showed that $\omega_\mu\, V^\mu$ was a scalar field. This was just the special case of starting from a $(1,1)$ tensor formed as the outer product of ω and V, with components $\omega_\mu\, V^\nu$, and then making the index contraction $\mu=\nu$ to obtain the $(0,0)$ tensor (i.e. scalar field) $\omega_\mu\, V^\mu$.

4.3 Covariant differentiation

In special relativity, we saw that if $T^{\mu_1\cdots\mu_p}{}_{\nu_1\cdots\nu_q}$ are the components of a Lorentz (p,q) tensor, then

$$\partial_\rho\, T^{\mu_1\cdots\mu_p}{}_{\nu_1\cdots\nu_q} \tag{4.27}$$

are the components of a $(p,q+1)$ Lorentz tensor. However, the situation is very different in the case of general-coordinate tensors. To see this, it suffices for a preliminary discussion to consider the case of a vector field $V = V^\mu\,\partial_\mu$, i.e. a $(1,0)$ tensor. Let us define

$$Z_\mu{}^\nu \equiv \partial_\mu\, V^\nu\,. \tag{4.28}$$

We now test whether $Z_\mu{}^\nu$ are the components of a $(1,1)$ general-coordinate tensor, which can be done by calculating it in the primed frame, making use of the known transformation rules for ∂_μ and V^ν:

$$\begin{aligned} Z'_\mu{}^\nu &= \partial'_\mu\, V'^\nu = \frac{\partial x^\rho}{\partial x'^\mu}\,\partial_\rho\Big(\frac{\partial x'^\nu}{\partial x^\sigma}\,V^\sigma\Big)\,,\\ &= \frac{\partial x^\rho}{\partial x'^\mu}\,\frac{\partial x'^\nu}{\partial x^\sigma}\,\partial_\rho\, V^\sigma + \frac{\partial x^\rho}{\partial x'^\mu}\,\frac{\partial^2 x'^\nu}{\partial x^\rho\,\partial x^\sigma}\,V^\sigma\,,\\ &= \frac{\partial x^\rho}{\partial x'^\mu}\,\frac{\partial x'^\nu}{\partial x^\sigma}\,Z_\rho{}^\sigma + \frac{\partial x^\rho}{\partial x'^\mu}\,\frac{\partial^2 x'^\nu}{\partial x^\rho\,\partial x^\sigma}\,V^\sigma\,. \end{aligned} \tag{4.29}$$

If the result had produced only the first term on the last line we would be happy, since that would then be the correct transformation rule for a $(1,1)$ general-coordinate tensor. However, the occurrence of the second term spoils the transformation behaviour. Notice that this problem would not have occurred in the case of Lorentz tensors, since for Lorentz transformations the second derivatives $\frac{\partial^2 x'^\nu}{\partial x^\rho\,\partial x^\sigma}$ of the coordinates x'^ν would be zero (see (2.28)). The problem, in the case of general-coordinate tensors, is that the transformation matrix

$$\frac{\partial x'^\nu}{\partial x^\sigma} \tag{4.30}$$

is not constant.

In order to overcome this problem, we need to introduce a new kind of derivative ∇_μ, known as a *covariant derivative*, to replace the partial derivative ∂_μ. We achieve this by defining

$$\nabla_\mu V^\nu \equiv \partial_\mu V^\nu + \Gamma^\nu{}_{\mu\rho} V^\rho \,, \tag{4.31}$$

where the object $\Gamma^\mu{}_{\nu\rho}$ is *defined* to transform under general coordinate transformations in precisely the right way to ensure that

$$W_\mu{}^\nu \equiv \nabla_\mu V^\nu \tag{4.32}$$

is a $(1,1)$ general-coordinate tensor. That is to say, by definition we will have

$$\frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x'^\nu}{\partial x^\sigma} \nabla_\rho V^\sigma = \nabla'_\mu V'^\nu \equiv \partial'_\mu V'^\nu + \Gamma'^\nu{}_{\mu\rho} V'^\rho \,. \tag{4.33}$$

Writing out the two sides here, we therefore have

$$\frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x'^\nu}{\partial x^\sigma} (\partial_\rho V^\sigma + \Gamma^\sigma{}_{\rho\lambda} V^\lambda) = \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x'^\nu}{\partial x^\sigma} \partial_\rho V^\sigma + \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial^2 x'^\nu}{\partial x^\rho \partial x^\lambda} V^\lambda + \Gamma'^\nu{}_{\mu\rho} \frac{\partial x'^\rho}{\partial x^\lambda} V^\lambda \,. \tag{4.34}$$

The $\partial_\rho V^\sigma$ terms cancel on the two sides. The remaining terms all involved the undifferentiated V^λ (we relabelled dummy indices on the right-hand side so that in each remaining term we have V^λ). Since the equation is required to hold for all possible V^λ, we can deduce that

$$\frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x'^\nu}{\partial x^\sigma} \Gamma^\sigma{}_{\rho\lambda} = \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial^2 x'^\nu}{\partial x^\rho \partial x^\lambda} + \Gamma'^\nu{}_{\mu\rho} \frac{\partial x'^\rho}{\partial x^\lambda} \,, \tag{4.35}$$

and this allows us to read off the required transformation rule for $\Gamma^\mu{}_{\nu\rho}$. Multiplying by $\partial x^\lambda / \partial x'^\alpha$, we find

$$\Gamma'^\nu{}_{\mu\alpha} = \frac{\partial x'^\nu}{\partial x^\sigma} \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\lambda}{\partial x'^\alpha} \Gamma^\sigma{}_{\rho\lambda} - \frac{\partial x^\lambda}{\partial x'^\alpha} \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial^2 x'^\nu}{\partial x^\rho \partial x^\lambda} \,. \tag{4.36}$$

The first term on the right-hand side of (4.36) is exactly the transformation we would expect for a $(1,2)$ general-coordinate tensor. The second term on the right-hand side is a mess, and the fact that it is there means that $\Gamma^\mu{}_{\nu\rho}$ is *not* a general-coordinate tensor. This should be no surprise, since it was introduced with the express purpose of cleaning up the mess that arose when we looked at the transformation properties of $\partial_\mu V^\nu$.

It is actually quite easy to construct an object $\Gamma^\mu{}_{\nu\rho}$ that has exactly the right properties under general-coordinate transformations, and in fact the expression for $\Gamma^\mu{}_{\nu\rho}$ will be quite simple. In order to do this we will now need to introduce, for the first time in our discussion of general-coordinate tensors, the metric tensor $g_{\mu\nu}$. This will be an arbitrary 2-index symmetric tensor, whose components are allowed to depend on the spacetime coordinates in an arbitrary way. In order to pin down an explicit expression for $\Gamma^\mu{}_{\nu\rho}$ in terms of the metric, it will be necessary first to extend the definition of the covariant derivative, which so far we defined only when acting on vectors V^μ, to arbitrary (p,q) tensors.

To extend the definition of the covariant derivative we shall impose two requirements. Firstly, that the covariant derivative of a scalar field will just be the ordinary partial derivative ∂_μ. This is reasonable, since $\partial_\mu\phi$ already transforms like the components of a co-vector, for any scalar field ϕ, and so no covariant correction term is needed in this case. The second requirement of the covariant derivative will be that it should obey the Leibnitz rule for the differentiation of products. Thus, for example, it should be such that

$$\nabla_\mu(V^\nu\, U_\rho) = (\nabla_\mu V^\nu)\, U_\rho + V^\nu\, \nabla_\mu U_\rho\,. \tag{4.37}$$

With these two assumptions, we can next calculate the covariant derivative of a co-vector, by writing

$$\nabla_\mu(V^\nu\, U_\nu) = (\nabla_\mu\, V^\nu)\, U_\nu + V^\nu\, \nabla_\mu\, U_\nu\,. \tag{4.38}$$

Now the left-hand side can be written as $\partial_\mu(V^\nu\, U_\nu)$ since $V^\nu\, U_\nu$ is a general-coordinate scalar. On the right-hand side we already know how to write $\nabla_\mu\, V^\nu$, using (4.31). Thus we have

$$(\partial_\mu\, V^\nu)\, U_\nu + V^\nu\, \partial_\mu\, U_\nu = (\partial_\mu\, V^\nu + \Gamma^\nu{}_{\mu\rho}\, V^\rho)\, U_\nu + V^\nu\, \nabla_\mu\, U_\nu\,. \tag{4.39}$$

The $(\partial_\mu\, V^\nu)\, U_\nu$ terms cancel on the two sides, and the remaining terms can be written as

$$V^\nu\, \partial_\mu\, U_\nu = \Gamma^\rho{}_{\mu\nu}\, V^\nu\, U_\rho + V^\nu\, \nabla_\mu\, U_\nu\,. \tag{4.40}$$

(We have relabelled dummy indices in the first term on the right, so that the index on V on all three terms is a ν.) The equation should hold for *any* vector V^ν, and so we can deduce that

$$\nabla_\mu\, U_\nu = \partial_\mu\, U_\nu - \Gamma^\rho{}_{\mu\nu}\, U_\rho\,. \tag{4.41}$$

This gives us the expression for the covariant derivative of a co-vector.

By repeating this process, of using Leibnitz rule together with the use of the known covariant derivatives, one can iteratively calculate the action of the covariant derivative on a general-coordinate tensor with any number of upstairs and downstairs indices. The answer is simple: for each upstairs index there is a Γ term as in (4.31), and for each downstairs index there is a Γ term as in (4.41). The example of the covariant derivative of a $(2,2)$ general-coordinate tensor should be sufficient to make the pattern clear. We shall have

$$\nabla_\mu\, T^{\nu\rho}{}_{\sigma\lambda} = \partial_\mu\, T^{\nu\rho}{}_{\sigma\lambda} + \Gamma^\nu{}_{\mu\alpha}\, T^{\alpha\rho}{}_{\sigma\lambda} + \Gamma^\rho{}_{\mu\alpha}\, T^{\nu\alpha}{}_{\sigma\lambda} - \Gamma^\alpha{}_{\mu\sigma}\, T^{\nu\rho}{}_{\alpha\lambda} - \Gamma^\alpha{}_{\mu\lambda}\, T^{\nu\rho}{}_{\sigma\alpha}\,. \tag{4.42}$$

It now remains to find a nice expression for $\Gamma^\mu{}_{\nu\rho}$. We do this by introducing the metric tensor $g_{\mu\nu}$ in the spacetime. All we shall require for now is that is a 2-index symmetric tensor, whose components could be arbitrary functions of the spacetime coordinates. We shall also require that it be invertible, i.e. that, viewed as a matrix, its determinant should be non-zero. The inverse metric tensor will be represented by $g^{\mu\nu}$. By definition, it must satisfy

$$g^{\mu\nu}\, g_{\nu\rho} = \delta^\mu_\rho\,. \tag{4.43}$$

Just as we saw with the Minkowski metric in special relativity, here in general relativity we can use the metric and its inverse to lower and raise indices. Thus

$$V_\mu = g_{\mu\nu}\, V^\nu\,, \qquad V^\mu = g^{\mu\nu}\, V_\nu\,, \tag{4.44}$$

etc. Raising a lowered index gets us back to where we started, because of (4.43), which is why we can use the same symbol for the vector or tensor with raised or lowered indices.

Of course, since $g_{\mu\nu}$ is itself a tensor, it follows also that if we lower or raise indices with $g_{\mu\nu}$ or $g^{\mu\nu}$, we map a tensor into another tensor.

We are now ready to obtain an expression for $\Gamma^\mu{}_{\nu\rho}$. We do this by making two further assumptions:

1. The metric tensor is covariantly constant, i.e. $\nabla_\mu\, g_{\nu\rho} = 0$.

2. $\Gamma^\mu{}_{\nu\rho} = \Gamma^\mu{}_{\rho\nu}$.

It turns out that we can always find a solution for a $\Gamma^\mu{}_{\nu\rho}$ with these properties, and in fact the solution is unique. Clearly the covariant constancy of the metric is a nice property to have, since it then means that the process of raising and lowering indices commutes with covariant differentiation. For example, we have

$$\nabla_\mu\, V_\nu = \nabla_\mu (g_{\nu\rho}\, V^\rho) = g_{\nu\rho}\, \nabla_\mu\, V^\rho\,. \tag{4.45}$$

The symmetry of $\Gamma^\mu{}_{\nu\rho}$ in its lower indices is an additional bonus, and leads to further simplifications, as we shall see.

The covariant constancy of the metric means that

$$0 = \nabla_\mu\, g_{\nu\rho} = \partial_\mu\, g_{\nu\rho} - \Gamma^\alpha{}_{\mu\nu}\, g_{\alpha\rho} - \Gamma^\alpha{}_{\mu\rho}\, g_{\nu\alpha}\,, \tag{4.46}$$

where we have used the expression for the covariant derivative of a $(0,2)$ tensor, which can be seen from (4.42). We now add the same equation with μ and ν exchanged, and subtract the equation with μ and ρ exchanged. Using the symmetry of the metric tensor, and the symmetry of Γ in its lower two indices, we then find that of the six Γ terms 4 cancel in pairs, and the remaining 2 add up, giving

$$2\Gamma^{\alpha}{}_{\mu\nu}\, g_{\alpha\rho} = \partial_\mu\, g_{\nu\rho} + \partial_\nu\, g_{\mu\rho} - \partial_\rho\, g_{\mu\nu}\,. \qquad (4.47)$$

Multiplying by the inverse metric $g^{\rho\lambda}$ then gives, after relabelling indices for convenience,

$$\Gamma^{\mu}{}_{\nu\rho} = \tfrac{1}{2} g^{\mu\sigma}\,(\partial_\nu\, g_{\sigma\rho} + \partial_\rho\, g_{\sigma\nu} - \partial_\sigma\, g_{\nu\rho})\,. \qquad (4.48)$$

The $\Gamma^{\mu}{}_{\nu\rho}$ so defined is known as the *Christoffel Connection.* Notice that it coincides with the equation (3.18) that we found when we studied the motion of a particle in Minkowski spacetime, seen from the viewpoint of a non-inertial frame of reference. That was in fact a special case of what we are studying now, in which the metric had the special feature of being merely a coordinate transformation of the Minkowski metric. Our present derivation of $\Gamma^{\mu}{}_{\nu\rho}$ is much more general, since $g_{\mu\nu}$ is now an arbitrary metric, which may be curved.

4.4 Some properties of the covariant derivative

As we have seen, the covariant derivative ∇_μ has the key property that when acting on a general-coordinate tensor of type (p, q) it gives another general-coordinate tensor, of type $(p, q + 1)$. It therefore plays the same role for general-coordinate tensors as the partial derivative ∂_μ plays for Lorentz tensors. And in fact, as can easily be seen from (4.48), if the metric $g_{\mu\nu}$ is just equal to the Minkowski metric $\eta_{\mu\nu}$, then $\Gamma^{\mu}{}_{\nu\rho}$ will vanish and the covariant derivative reduces to the partial derivative. We shall now examine a few more properties of the covariant derivative:

Curl:

A common occurrence is that one needs to evaluate the anti-symmetrised covariant derivative of a co-vector. Using (4.41), we have

$$\nabla_\mu\, V_\nu - \nabla_\nu\, V_\mu = \partial_\mu\, V_\nu - \Gamma^{\rho}{}_{\mu\nu}\, V_\rho - \partial_\nu\, V_\mu + \Gamma^{\rho}{}_{\nu\mu}\, V_\rho\,. \qquad (4.49)$$

Recalling that $\Gamma^{\rho}{}_{\mu\nu}$ is symmetric in μ and ν (as can be seen from (4.48), it therefore follows that

$$\nabla_\mu\, V_\nu - \nabla_\nu\, V_\mu = \partial_\mu\, V_\nu - \partial_\nu\, V_\mu\,. \qquad (4.50)$$

This antisymmetrised derivative of a co-vector is a generalisation of the curl operation in three-dimensional Cartesian vector analysis, where one has

$$(\text{curl}\vec{V})_i = (\vec{\nabla}\times\vec{V})_i = \epsilon_{ijk}\,\partial_j V_k\,. \qquad (4.51)$$

(In this three-dimensional case, the fact that the epsilon tensor has three indices is utilised in order to map the 2-index antisymmetric tensor $\partial_i V_j - \partial_j V_i$ into a vector.)

Divergence:

Another useful operation is to take the divergence of a vector. This is given by

$$\nabla_\mu V^\mu = \partial_\mu V^\mu + \Gamma^\mu{}_{\mu\nu} V^\nu \,. \tag{4.52}$$

From (4.48) we have

$$\Gamma^\mu{}_{\mu\nu} = \tfrac{1}{2} g^{\mu\sigma} (\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}) = \tfrac{1}{2} g^{\mu\sigma} \partial_\nu g_{\mu\sigma} \,. \tag{4.53}$$

Note that the first and the third terms cancelled because of the symmetry of $g^{\mu\sigma}$. If we define $\mathbf{g}$ to be the matrix whose components are $g_{\mu\nu}$, with its inverse $\mathbf{g}^{-1}$ whose components are $g^{\mu\nu}$, then we see that

$$\Gamma^\mu{}_{\mu\nu} = \tfrac{1}{2} \mathrm{tr}(\mathbf{g}^{-1} \partial_\nu \mathbf{g}) \,. \tag{4.54}$$

Suppose that M is any non-degenerate matrix. One can straightforwardly show that

$$\log \det M = \mathrm{tr} \log M \,. \tag{4.55}$$

This is most clear for a symmetric matrix, since one can always diagonalise the matrix, and then the identity is obvious. If we now make an infinitesimal variation of (4.55) we find

$$\begin{aligned} (\det M)^{-1} \delta(\det M) &= \mathrm{tr} \log(M + \delta M) - \mathrm{tr} \log M = \mathrm{tr} \log[M^{-1} (M + \delta M)] \\ &= \mathrm{tr} \log(1 + M^{-1} \delta M) \\ &= \mathrm{tr}[M^{-1} \delta M - (M^{-1} \delta M)^2 + \cdots] \\ &= \mathrm{tr}(M^{-1} \delta M) \,, \end{aligned} \tag{4.56}$$

since the terms at order $(\delta M)^2$ and above can be neglected in the infinitesimal limit. Thus we have $(\det M)^{-1} \partial_\mu (\det M) = \mathrm{tr}(M^{-1} \partial_\mu M)$. Applying this result to (4.54), we can therefore write $\Gamma^\mu{}_{\mu\nu}$ as

$$\Gamma^\mu{}_{\mu\nu} = \tfrac{1}{2} g^{-1} \partial_\nu g \,, \tag{4.57}$$

where we have defined g to be the determinant of the metric,

$$g \equiv \det \mathbf{g} \,. \tag{4.58}$$

We are considering spacetimes with one time direction and three space directions. Although the metric $g_{\mu\nu}$ is not in general the Minkowski metric $\eta_{\mu\nu}$, it will have in common with the Minkowski metric the feature that it has one negative eigenvalue (associated with the time direction) and three positive eigenvalues (asociated with the spatial directions). Therefore the determinant g will be negative. We can write (4.57) as

$$\Gamma^\mu{}_{\mu\nu} = \frac{1}{\sqrt{-g}}\, \partial_\nu \sqrt{-g}\,, \tag{4.59}$$

and so from (4.52) we shall have

$$\nabla_\mu V^\mu = \frac{1}{\sqrt{-g}}\, \partial_\mu(\sqrt{-g}\, V^\mu)\,. \tag{4.60}$$

This is a useful expression, since it allows one to calculate the divergence of a vector without first having to calculate and tabulate all the components of the Christoffel connection.

A further result along the same lines is as follows. If $F^{\mu_1\cdots\mu_p}$ is a *totally-antisymmetric* $(p,0)$ tensor, then

$$\nabla_\mu\, F^{\mu\nu_2\cdots\nu_p} = \frac{1}{\sqrt{-g}}\, \partial_\mu(\sqrt{-g}\, F^{\mu\nu_2\cdots\nu_p})\,. \tag{4.61}$$

The proof, which we leave as an exercise to the reader, makes use of the symmetry of $\Gamma^\mu{}_{\nu\rho}$ in its two lower indices. It is important to note that (4.61) is valid *only* when the indices on F are all upstairs, and *only* when in addition F is totally antisymmetric in all its indices.

4.5 Riemann curvature tensor

We are now ready to introduce a key feature of (pseudo)-Riemannian geometry, namely the concept of curvature. To begin, we make the simple observation that the commutator of covariant derivatives acting on a scalar field gives zero:

$$[\nabla_\mu, \nabla_\nu]\phi = \nabla_\mu\partial_\nu\phi - \nabla_\nu\partial_\mu\phi = \partial_\mu\partial_\nu\phi - \partial_\nu\partial_\mu\phi = 0\,. \tag{4.62}$$

Note that the second equality, where the covariant derivatives are replaced by partial derivatives, follows from the result (4.50) for the antisymmetrised covariant derivative of a co-vector, applied to the special case of the co-vector $V_\mu = \partial_\mu\phi$.

The situation is more interesting if we look instead at the commutator of covariant derivatives of a vector field:

$$\begin{aligned}
[\nabla_\mu, \nabla_\nu]\, V^\rho &= \partial_\mu(\nabla_\nu V^\rho) - \Gamma^\sigma{}_{\mu\nu}\, \nabla_\sigma V^\rho + \Gamma^\rho{}_{\mu\sigma}\, \nabla_\nu V^\sigma - (\mu \leftrightarrow \nu)\,, \\
= \quad & \partial_\mu(\partial_\nu V^\rho + \Gamma^\rho{}_{\nu\sigma}\, V^\sigma) - \Gamma^\sigma{}_{\mu\nu}\, (\partial_\sigma V^\rho + \Gamma^\rho{}_{\sigma\lambda}\, V^\lambda) + \Gamma^\rho{}_{\mu\sigma}\, (\partial_\nu V^\sigma + \Gamma^\sigma{}_{\nu\lambda}\, V^\lambda) \\
& -\partial_\nu(\partial_\mu V^\rho + \Gamma^\rho{}_{\mu\sigma}\, V^\sigma) + \Gamma^\sigma{}_{\nu\mu}\, (\partial_\sigma V^\rho + \Gamma^\rho{}_{\sigma\lambda}\, V^\lambda) - \Gamma^\rho{}_{\nu\sigma}\, (\partial_\mu V^\sigma + \Gamma^\sigma{}_{\mu\lambda}\, V^\lambda)\,.
\end{aligned} \tag{4.63}$$

It is evident from this that all of the terms where either one or two partial derivatives land on V cancel out completely. Of the remaining terms, a pair of $\Gamma\Gamma$ terms cancel because of the symmetry of $\Gamma^\sigma{}_{\mu\nu}$ in its lower indices, and the remaining terms can then be written, after an index relabelling, as

$$[\nabla_\rho, \nabla_\sigma]\, V^\mu = R^\mu{}_{\nu\rho\sigma}\, V^\nu\,, \tag{4.64}$$

where,

$$R^\mu{}_{\nu\rho\sigma} = \partial_\rho \Gamma^\mu{}_{\sigma\nu} - \partial_\sigma \Gamma^\mu{}_{\rho\nu} + \Gamma^\mu{}_{\rho\lambda}\, \Gamma^\lambda{}_{\sigma\nu} - \Gamma^\mu{}_{\sigma\lambda}\, \Gamma^\lambda{}_{\rho\nu}\,. \tag{4.65}$$

The left-hand side of (4.64) is clearly a $(1,2)$ general-coordinate tensor, since, by construction, we know that the covariant derivative of a tensor is another tensor. On the right-hand side we know that V^σ is a general-coordinate vector. By an application of the quotient theorem (an example of which was established for Lorentz tensors in homework 1, and for general-coordinate tensors in homework 2), it follows that $R^\mu{}_{\nu\rho\sigma}$ must be a $(1,3)$ general-coordinate tensor. This very important object is called the *Riemann Tensor*, and it characterises the curvature of the spacetime.

Symmetries of the Riemann tensor:

The Riemann tensor has some important symmetry properties. First of all, as can be seen from (4.65), $R^\mu{}_{\nu\rho\sigma}$ is antisymmetric in ρ and σ. It also has further symmetries that are not immediately apparent by inspecting (4.65). They become more apparent if one first obtains an expression for[7]

$$R_{\alpha\sigma\mu\nu} \equiv g_{\alpha\rho}\, R^\rho{}_{\sigma\mu\nu}\,. \tag{4.66}$$

To do this, it is convenient also to define

$$\Gamma_{\mu\rho\sigma} \equiv g_{\mu\lambda}\, \Gamma^\lambda{}_{\rho\sigma} = \tfrac{1}{2}(\partial_\rho\, g_{\mu\sigma} + \partial_\sigma\, g_{\mu\rho} - \partial_\mu\, g_{\rho\sigma})\,. \tag{4.67}$$

(Note that the *first* index on $\Gamma_{\mu\rho\sigma}$ is the one that has been lowered!) Thus, from (4.65), we have

$$R_{\alpha\sigma\mu\nu} = g_{\alpha\rho}\, \partial_\mu (g^{\rho\lambda}\, \Gamma_{\lambda\nu\sigma}) - g_{\alpha\rho}\, \partial_\nu (g^{\rho\lambda}\, \Gamma_{\lambda\mu\sigma}) + \Gamma_{\alpha\mu\lambda}\, \Gamma^\lambda{}_{\nu\sigma} - \Gamma_{\alpha\nu\lambda}\, \Gamma^\lambda{}_{\mu\sigma}\,. \tag{4.68}$$

Since $g_{\alpha\rho}\, g^{\rho\lambda} = \delta^\lambda_\alpha$, which is constant, it follows that

$$g_{\alpha\rho}\, \partial_\mu\, g^{\rho\lambda} = -g^{\rho\lambda}\, \partial_\mu\, g_{\alpha\rho}\,. \tag{4.69}$$

[7]For historical reasons I had relabelled the indices in what follows, so that the Riemman tensor on the right-hand side of (4.66) is labelled as $R^\rho{}_{\sigma\mu\nu}$ rather than $R^\mu{}_{\nu\rho\sigma}$. I have stuck with this rather than risk introducing errors by relabelling at this stage. Sorry!

Using this, together with the expression that we can read off from (4.46) for the partial derivative of the metric in terms of the Christoffel connection, we find from (4.68) that

$$
\begin{aligned}
R_{\alpha\sigma\mu\nu} &= \partial_\mu \Gamma_{\alpha\nu\sigma} - \partial_\nu \Gamma_{\alpha\mu\sigma} - g^{\rho\lambda}\,(\Gamma^\gamma{}_{\mu\alpha}\, g_{\gamma\rho} + \Gamma^\gamma{}_{\mu\rho}\, g_{\gamma\alpha})\,\Gamma_{\lambda\nu\sigma} + g^{\rho\lambda}\,(\Gamma^\gamma{}_{\nu\alpha} + \Gamma^\gamma{}_{\nu\rho})\,\Gamma_{\lambda\mu\sigma} \\
&\quad + \Gamma_{\alpha\mu\lambda}\,\Gamma^\lambda{}_{\nu\sigma} - \Gamma_{\alpha\nu\lambda}\,\Gamma^\lambda{}_{\mu\sigma}\,. \qquad (4.70)
\end{aligned}
$$

Most of the $\Gamma\,\Gamma$ terms cancel, and after plugging in the expression (4.67) in the $\partial\Gamma$ terms, one finds the remarkably simple result, after a convenient relabelling of indices,

$$
R_{\mu\nu\rho\sigma} = \tfrac{1}{2}(\partial_\mu\partial_\sigma g_{\nu\rho} - \partial_\mu\partial_\rho g_{\nu\sigma} + \partial_\nu\partial_\rho g_{\mu\sigma} - \partial_\nu\partial_\sigma g_{\mu\rho}) + g_{\alpha\beta}\,(\Gamma^\alpha{}_{\mu\sigma}\Gamma^\beta{}_{\nu\rho} - \Gamma^\alpha{}_{\mu\rho}\Gamma^\beta{}_{\nu\sigma})\,. \quad (4.71)
$$

From this, the following symmetries are immediately apparent:

$$R_{\mu\nu\rho\sigma} = -R_{\mu\nu\sigma\rho}\,; \qquad \text{antisymmetry on second index pair} \qquad (4.72)$$

$$R_{\mu\nu\rho\sigma} = -R_{\nu\mu\rho\sigma}\,; \qquad \text{antisymmetry on first index pair} \qquad (4.73)$$

$$R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu}\,. \qquad \text{exchange of first and second index pair} \qquad (4.74)$$

$$R_{\mu\nu\rho\sigma} + R_{\mu\rho\sigma\nu} + R_{\mu\sigma\nu\rho} = 0\,; \qquad \text{cyclic identity} \qquad (4.75)$$

The antisymmetry in (4.72) was obvious from the original construction of the Riemann tensor in (4.64). However, the antisymmetry in (4.73), the symmetry under the exchange of the first and second index pair in (4.74), and the cyclic symmetry in (4.75) only became manifest after obtaining the expression (4.71) for $R_{\mu\nu\rho\sigma}$.

It is interesting to compare the derivation of these symmetries in different textbooks. The most common approach involves establishing that one can choose a special coordinate frame, at an arbitrarily selected point in spacetime, where $g_{\mu\nu} = \eta_{\mu\nu}$ and $\Gamma^\mu{}_{\nu\rho} = 0$ (only at the single point). A rather simpler calculation then shows that at the selected point, in the special coordinate frame, the Riemann tensor $R_{\mu\nu\rho\sigma}$ is given by just the $\partial\partial g$ terms in (4.71). The symmetries discussed above are then manifest at that point, and so when combined with the argument that any arbitrary point could have been chosen for the calculation, the general results then follow. As far as I am aware, only Weinberg in his textbook has taken the rather brutal approach of a head-on sledge-hammer attack obtaining the formula (4.71) that is valid in any coordinate frame. I can recommend checking all the details of the Weinberg calculation, as outlined above, because it is one of those rather satisfying calculations where the end result is remarkably simpler than one might expect during the intermediate stages.

It is worth remarking that although the conventional way to calculate the components of the Riemann tensor is by using eqn (4.65) to calculate $R^\mu{}_{\nu\rho\sigma}$, in some cases it can be considerably easy to calculate $R_{\mu\nu\rho\sigma}$ using eqn (4.71). This may not be such a big difference if one is using an algebraic computing program to do the calculation, since computers don't mind grinding through a lot of tedious and rather repetitive steps for lots of cases. But for a human, the expression (4.71) has the advantage that one does not have to evaluate derivatives of the Christoffel connection (which in many cases may be a lot more complicated than the individual metric components). Also, precisely because the various symmetries detailed above are already present in (4.71), one can straightforwardly exploit these in order to minimise the number of distinct calculations one has to perform.

It is useful at this point to introduce a convenient piece of notation, to denote antisymmetrisations or symmetrisations over sets of indices on a tensor. For antisymmetrisation, we write

$$T_{[\mu_1\ldots\mu_p]} \equiv \frac{1}{p!}\Big[T_{\mu_1\cdots\mu_p} + (\text{even permutations}) - (\text{odd permutations})\Big]\,, \tag{4.76}$$

where we include terms with all the possible permutations of the p indices, with a plus sign or a minus sign according to whether the permutation is an even or an odd permuation of the original ordering of indices $\mu_1\cdots\mu_p$. There will be $p!$ terms in total. Thus, for example,

$$\begin{aligned} T_{[\mu\nu]} &= \tfrac{1}{2}(T_{\mu\nu} - T_{\nu\mu})\,,\\ T_{[\mu\nu\rho]} &= \tfrac{1}{6}\,(T_{\mu\nu\rho} + T_{\nu\rho\mu} + T_{\rho\mu\nu} - T_{\nu\mu\rho} - T_{\mu\rho\nu} - T_{\rho\nu\mu})\,, \end{aligned} \tag{4.77}$$

and so on. For symmetrisation we use round brackets instead of square brackets, and define

$$T_{(\mu_1\ldots\mu_p)} \equiv \frac{1}{p!}\Big[T_{\mu_1\cdots\mu_p} + (\text{even permutations}) + (\text{odd permutations})\Big]\,. \tag{4.78}$$

Thus we have

$$\begin{aligned} T_{(\mu\nu)} &= \tfrac{1}{2}(T_{\mu\nu} + T_{\nu\mu})\,,\\ T_{(\mu\nu\rho)} &= \tfrac{1}{6}\,(T_{\mu\nu\rho} + T_{\nu\rho\mu} + T_{\rho\mu\nu} + T_{\nu\mu\rho} + T_{\mu\rho\nu} + T_{\rho\nu\mu})\,, \end{aligned} \tag{4.79}$$

and so on. Note that the normalisations in (4.76) and (4.78) are such that

$$T_{[[\mu_1\cdots\mu_p]]} = T_{[\mu_1\cdots\mu_p]}\,,\qquad T_{((\mu_1\cdots\mu_p))} = T_{(\mu_1\cdots\mu_p)}\,. \tag{4.80}$$

Using the notation for antisymmetrisation, and in view of the fact that the antisymmetry (4.72) holds, it is easy to check that the cyclic identity (4.75) can be written as

$$R_{\rho[\sigma\mu\nu]} = 0\,. \tag{4.81}$$

In fact, one can also see, after making use of the three symmetry properties (4.72), (4.73) and (4.74), that the cyclic identity is implied by (and trivially, it implies)

$$R_{[\rho\sigma\mu\nu]} = 0\,. \tag{4.82}$$

We are now in a position to count how many algebraicially-independent components are contained in the Riemann tensor. The antisymmetry (4.73) on the first index-pair and the antisymmetry (4.72) on the second index-pair, together with the symmetry (4.74) on the exchange of the first index-pair with the second index-pair, mean that we could think of the Riemann tensor as a symmetric matrix of dimension $(4\times 3)/2$ by $(4\times 3)/2$. This will have

$$\tfrac{1}{2}[4\times 3)/2][4\times 3)/2+1] = 21 \tag{4.83}$$

independent components. But we must still impose the remaining conditions from the cyclic identity, which are described by (4.82). This gives $(4\times 3\times 2\times 1)/4! = 1$ further condition. Thus in four dimensions the Riemann tensor has $21-1=20$ algebraically-independent components. It is straightforward to repeat this calculation in an arbitrary spacetime dimension n, and one finds the Riemann tensor then has

$$\tfrac{1}{12}\, n^2(n^2-1) \tag{4.84}$$

algebraically-independent components.

In addition to the four algebraic symmetries (4.72), (4.73), (4.74) and (4.75), there is also a differential symmetry known as the *Bianchi Identity*, which takes the form

$$\nabla_\lambda R^\mu{}_{\nu\rho\sigma} + \nabla_\rho R^\mu{}_{\nu\sigma\lambda} + \nabla_\sigma R^\mu{}_{\nu\lambda\rho} = 0\,. \tag{4.85}$$

This could in principle be derived from the expression (4.65) for the Riemann tensor by simply writing out all the terms in (4.85), with the covariant derivatives expressed in terms of partial derivatives and Christoffel connections, but the calculation would be even more brutal than the one given above for the derivation of the algebraic symmetries. On this occasion, it is probably better to make use of special choice of coordinate frame alluded to above, in which one can set $g_{\mu\nu}=\eta_{\mu\nu}$ at an arbitrarily selected point, and in addition one can set $\Gamma^\mu{}_{\nu\rho}=0$ at that point. Of course, one cannot also set derivatives of $\Gamma^\mu{}_{\nu\rho}=0$ to zero at that point. Using the expression (4.71) for $R_{\rho\sigma\mu\nu}$, it is easy to see that *at the selected point*, we shall simply have

$$\nabla_\lambda\, R_{\mu\nu\rho\sigma} = \tfrac{1}{2}\partial_\lambda(\partial_\mu\partial_\sigma g_{\nu\rho} - \partial_\mu\partial_\rho g_{\nu\sigma} + \partial_\nu\partial_\rho g_{\mu\sigma} - \partial_\nu\partial_\sigma g_{\mu\rho})\,. \tag{4.86}$$

This is because all undifferentiated Γ terms will be zero at that point. It is now immediately clear from (4.86) that the Bianchi identity (4.85) is satisfied at the selected point.[8] Since that point could have been chosen to be anywhere, and since a tensor that vanishes in one frame vanishes in all frames, it follows that (4.85) is satisfied everywhere.

As a side remark here, we note that one sometimes encounters a different notation for partial derivatives and for covariant derivatives. In this notation, a partial derivative ∂_μ is denoted by a comma, and so, for example, one would write

$$\partial_\mu V_\nu = V_{\nu,\mu}\,. \tag{4.87}$$

A covariant derivative is denoted by a semi-colon, and so one writes

$$\nabla_\mu V_\nu = V_{\nu;\mu}\,. \tag{4.88}$$

In this notation, the Bianchi identity (4.85) is written as

$$R^\mu{}_{\nu\rho\sigma;\lambda} + R^\mu{}_{\nu\sigma\lambda;\rho} + R^\mu{}_{\nu\lambda\rho;\sigma} = 0\,. \tag{4.89}$$

Using the notation for antisymmetrisation given by (4.76), and recalling the antisymmetry of the Riemann tensor on its second index-pair (4.72), we see that (4.89) can be written as

$$R^\mu{}_{\nu[\rho\sigma;\lambda]} = 0\,. \tag{4.90}$$

Ricci tensor and Ricci scalar:

There are two very important contractions of the Riemann tensor, which we now define. The first is the *Ricci tensor* $R_{\mu\nu}$, which is defined by

$$R_{\mu\nu} = R^\rho{}_{\mu\rho\nu}\,. \tag{4.91}$$

As a consequence of the symmetry (4.74) of the Riemann tensor, the Ricci tensor is symmetric in its two indices, $R_{\mu\nu} = R_{\nu\mu}$. One can also make a further contraction to obtain the *Ricci Scalar* R, defined by

$$R = g^{\mu\nu}\, R_{\mu\nu}\,. \tag{4.92}$$

The Ricci tensor satisfies a differential identity that can be derived from the Bianchi identity (4.85) for the Riemann tensor. Contracting (4.85) by setting $\lambda = \rho$, and using the

[8]Of course we can freely lower the ρ index in (4.85), since as emphasised earlier, the covariant constancy of the metric means that raising and lowering indices commutes with covariant differentiation.

algebraic symmetries of the Riemann tensor and definition of the Ricci tensor in (4.91), gives

$$\nabla_\mu R^\mu{}_{\nu\rho\sigma} = \nabla_\rho R_{\sigma\nu} - \nabla_\sigma R_{\rho\nu}\,. \tag{4.93}$$

If we now contract this equation with $g^{\nu\sigma}$, we get

$$\nabla_\mu R^\mu{}_\nu = \tfrac{1}{2}\partial_\nu\, R\,, \tag{4.94}$$

after an index relabelling. Notice that this means that the tensor $G_{\mu\nu}$, defined by

$$G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2}R\, g_{\mu\nu}\,, \tag{4.95}$$

obeys the divergence-free condition

$$\nabla^\mu\, G_{\mu\nu} = 0\,. \tag{4.96}$$

The tensor $G_{\mu\nu}$ is a very important one in general relativity. It is called the *Einstein Tensor*, and it arises in the gravitational field equations in Einstein gravity, as we shall see shortly.

Parallel transport and the meaning of curvature

Let us return, for a moment, to Minkowski spacetime, with coordinates $\tilde{x}^\mu$. Suppose we have vector V, with components $\widetilde{V}^\mu$ with respect to this coordinate basis, and that we wish to parallely transport it long some curve $\tilde{x}(\lambda)$, where λ is a parameter that monotonically increases along the curve. Clearly, in Minkowski spacetime, parallel transport means the direction of the vector stays unchanged as it is carried along the curve, so

$$\frac{d\widetilde{V}^\mu}{d\lambda} = 0\,. \tag{4.97}$$

Now let us make an arbitrary general-coordinate transformation, as we discussed in chapter 3, to coordinate system x^μ. The components of the vector V will be related in the two frames by

$$\widetilde{V}^\mu = \frac{\partial\tilde{x}^\mu}{\partial x^\nu}\, V^\nu\,, \tag{4.98}$$

and so (4.97) becomes

$$\frac{\partial\tilde{x}^\mu}{\partial x^\nu}\,\frac{dV^\nu}{d\lambda} + \frac{d}{d\lambda}\Big(\frac{\partial\tilde{x}^\mu}{\partial x^\nu}\Big)\, V^\nu = 0\,. \tag{4.99}$$

Using the chain rule in the second term gives

$$\frac{\partial\tilde{x}^\mu}{\partial x^\nu}\,\frac{dV^\nu}{d\lambda} + \frac{dx^\rho}{d\lambda}\,\frac{\partial^2\tilde{x}^\mu}{\partial x^\rho\partial x^\nu}\, V^\nu = 0\,. \tag{4.100}$$

Multiplying by $(\partial x^\sigma/\partial \tilde{x}^\mu)$ gives

$$\begin{aligned}0 &= \frac{dV^\sigma}{d\lambda} + \frac{dx^\rho}{d\lambda}\,\frac{\partial x^\sigma}{\partial\tilde{x}^\mu}\,\frac{\partial^2\tilde{x}^\mu}{\partial x^\rho\partial x^\nu}\,V^\nu\,,\\ &= \frac{dV^\sigma}{d\lambda} + \frac{dx^\rho}{d\lambda}\,\Gamma^\sigma{}_{\rho\nu}\,V^\nu\,,\\ &= \frac{dx^\rho}{d\lambda}\Big(\partial_\rho\,V^\sigma + \Gamma^\sigma{}_{\rho\nu}\,V^\nu\Big)\,,\\ &= \frac{dx^\rho}{d\lambda}\,\nabla_\rho\,V^\sigma\,,\end{aligned} \tag{4.101}$$

where, in getting to the second line, we have used (3.9); the third line follows from the use of the chain rule in the first term; and finally the last line follows from the definition (4.31) of the covariant derivative on a vector field. Thus, the equation of parallel transport for a vector in Minkowski spacetime, but described from an arbitrary coordinate frame, is

$$\frac{dx^\mu}{d\lambda}\,\nabla_\mu\,V^\nu = 0\,. \tag{4.102}$$

The equation is sometimes written as

$$\frac{DV^\mu}{D\lambda} \equiv \frac{dx^\mu}{d\lambda}\,\nabla_\mu\,V^\nu = 0\,. \tag{4.103}$$

Although we derived (4.102) within the framework of special relativity viewed from an arbitrary coordinate frame, it is equally valid in the more general context of general relativity, for a completely arbitrary curved metric, where the covariant derivative ∇_μ is defined by (4.31) and the Christoffel connection is given by (4.48). It is a manifestly general-covariant equation, since dx^μ transforms as a general-cooordinate vector and λ is coordinate invariant (i.e. a scalar).

Consider now a displacement, by parallel transport, along an infinitesimal segment of a curve $x^\mu(\lambda)$. Multiplying the parallel transport equation in the second line of (4.101) by $\delta\lambda$, and relabelling indices, we have

$$\delta V^\mu(x) = -\Gamma^\mu{}_{\nu\rho}(x)\,V^\rho(x)\,\delta x^\nu\,. \tag{4.104}$$

We can now use this expression to calculate the result of parallel propagating the vector V around a very small closed loop C. For convenience, and without any loss of generality, we can choose the origin of the coordinate system to that the loop begins and ends at $x^\mu = 0$. For small values of x^μ it follows (4.104) that

$$V^\mu(x) = V^\mu(0) - \Gamma^\mu{}_{\nu\rho}(0)\,V^\rho(0)\,x^\nu + \mathcal{O}(x^2)\,. \tag{4.105}$$

We can also Taylor expand $\Gamma^\mu{}_{\nu\rho}$ around $x^\mu = 0$, which gives

$$\Gamma^\mu{}_{\nu\rho}(x) = \Gamma^\mu{}_{\nu\rho}(0) + \partial_\sigma\Gamma^\mu{}_{\nu\rho}(0)\,x^\sigma + \mathcal{O}(x^2)\,, \tag{4.106}$$

where $\partial_\sigma\Gamma^\mu{}_{\nu\rho}(0)$ means first evaluate $\partial_\sigma\Gamma^\mu{}_{\nu\rho}(x)$ and then set $x^\mu = 0$. Thus from (4.104), the result of integrating up around the small loop will be given by

$$\begin{aligned}\Delta V^\mu &= \oint_C \delta V^\mu = -\oint_C \Gamma^\mu{}_{\nu\rho}(x)\, V^\rho(x)\, dx^\nu\,,\\ &= -\oint_C (\Gamma^\mu{}_{\nu\rho}(0) + \partial_\sigma\Gamma^\mu{}_{\nu\rho}(0)\, x^\sigma)(V^\rho(0) - \Gamma^\rho{}_{\alpha\beta}(0)\, V^\beta(0)\, x^\alpha) dx^\nu\,,\\ &= -\Gamma^\mu{}_{\nu\rho} V^\rho \oint_C dx^\nu - \partial_\sigma\Gamma^\mu{}_{\nu\rho}\, V^\rho \oint_C x^\sigma\, dx^\nu + \Gamma^\mu{}_{\nu\rho}\Gamma^\rho{}_{\alpha\beta}\, V^\beta \oint_C x^\alpha\, dx^\nu + \cdots\,, \end{aligned} \tag{4.107}$$

where the ellipses denote terms of higher order in powers of x^μ, which can be neglected when the closed loop is sufficiently small. (In the last line, and from now on, we suppress the $x^\mu = 0$ argument of all quantities outside the integrals.) Now since the intgral of an exact differential around a closed loop gives zero, we shall have

$$\oint_C dx^\nu = 0\,, \tag{4.108}$$

and

$$\oint_C x^\sigma\, dx^\nu = \oint_C [d(x^\sigma\, x^\nu) - x^\nu\, dx^\sigma] = -\oint_C x^\nu\, dx^\sigma\,, \tag{4.109}$$

and hence

$$\oint_C x^\sigma\, dx^\nu = \tfrac{1}{2}\oint_C (x^\sigma\, dx^\nu - x^\nu\, dx^\sigma)\,. \tag{4.110}$$

After some index relabelling, (4.107) gives

$$\begin{aligned}\Delta V^\mu &= -[\partial_\sigma\Gamma^\mu{}_{\nu\beta} - \Gamma^\mu{}_{\nu\rho}\,\Gamma^\rho{}_{\sigma\beta}]\, V^\beta \oint_C x^\sigma\, dx^\nu\,,\\ &= -\tfrac{1}{2}[\partial_\sigma\Gamma^\mu{}_{\nu\beta} - \partial_\nu\Gamma^\mu{}_{\sigma\beta} - \Gamma^\mu{}_{\nu\rho}\,\Gamma^\rho{}_{\sigma\beta} + \Gamma^\mu{}_{\sigma\rho}\,\Gamma^\rho{}_{\nu\beta}]\, V^\beta \oint_C x^\sigma\, dx^\nu\,, \end{aligned} \tag{4.111}$$

where in getting to the second line, we have used the antisymmetry of the integral under the exchange of σ and ν. Comparing with the definition of the Riemann tensor, given by (4.65), we see, after a relabelling of indices, that

$$\Delta V^\mu = -\tfrac{1}{2} R^\mu{}_{\nu\rho\sigma}\, V^\nu \oint x^\rho\, dx^\sigma\,. \tag{4.112}$$

(Recall that the Riemann tensor is evaluated at $x^\mu = 0$ here.)

The integral $\oint x^\rho dx^\sigma$ is equal to the area $\Delta A^{\rho\sigma}$ that is bounded by the small closed loop C. To be more precise, this area lies in a 2-plane, and the orientation of that 2-plane is specified by ρ and σ. Suppose, for example, that $x^1 = x$ and $x^2 = y$, and that the loop consists of a small square of side ϵ in the xy plane. Then

$$\Delta A^{12} = \oint_C x dy = \int_0^\epsilon \epsilon dy + \int_\epsilon^0 0 dy = \epsilon^2 \tag{4.113}$$

which is indeed the area of the square bounded by C.

Thus we have

$$\Delta V^\mu = -\tfrac{1}{2} R^\mu{}_{\nu\rho\sigma}\, V^\nu\, \Delta A^{\rho\sigma}\,. \tag{4.114}$$

Thus we see that the Riemann curvature tensor characterises the change that a vector undergoes when it is parallel propagated around a closed loop. In flat space, where the Riemann tensor vanishes, the vector would, by contrast, return completely unchanged after its trip around the closed loop.

4.6 An example: The 2-sphere

It is instructive to look at a simple example of a curved space, and the simplest is probably the 2-sphere (like the surface of the earth).[9] We can define a 2-sphere of radius a via its embedding in Euclidean 3-space, by means of the equation $x^2 + y^2 + z^2 = a^2$. The points (x, y, z) on the spherical surface can be parameterised by writing

$$x = a\, \sin\theta\, \cos\varphi\,, \qquad y = a\, \sin\theta\, \sin\varphi\,, \qquad z = a\, \cos\theta\,. \tag{4.115}$$

The metric on the sphere is the one inherited from the metric $ds_3^2 = dx^2 + dy^2 + dz^2$ on the Euclidean 3-space by making the substitutions (4.115), which gives

$$ds^2 = a^2\,(d\theta^2 + \sin^2\theta\, d\varphi^2)\,. \tag{4.116}$$

If we define the coordinates $x^1 = \theta$ and $x^2 = \varphi$, then we see that the metric and its inverse are diagonal, with

$$g_{11} = a^2\,, \quad g_{22} = a^2\, \sin^2\theta\,, \qquad g^{11} = \frac{1}{a^2}\,, \quad g^{22} = \frac{1}{a^2\, \sin^2\theta}\,. \tag{4.117}$$

Calculating the various components of $\Gamma^\mu{}_{\nu\rho}$ using (4.48), one finds that the only non-vanishing components are

$$\Gamma^1{}_{22} = -\sin\theta\, \cos\theta\,, \qquad \Gamma^2{}_{12} = \Gamma^2{}_{21} = \cot\theta\,. \tag{4.118}$$

Calculating the Riemann tensor compoents from (4.65), then one finds the only non-vanishing ones are

$$R^1{}_{212} = -R^1{}_{221} = \sin^2\theta\,, \qquad R^2{}_{112} = -R^2{}_{121} = -1\,. \tag{4.119}$$

[9]Our principle focus in this course will be on four-dimensional metrics with signature $(-,+,+,+)$. But all of the tensor formalism that we have described so far is equally applicable in any dimension, and for any choice of metric signature. (Minor adjustments are needed in equations such as (4.60) for the divergence of a vector field, if the determinant of the metric is positive rather than negative.)

Lowering the upper index using the metric gives

$$R_{1212} = -R_{1221} = -R_{2112} = R_{2121} = a^2 \sin^2\theta\,. \tag{4.120}$$

It can be seen that these results are all consistent with the algebraic symmetries discussed earlier.

From (4.91) and (4.92) we find

$$R_{11} = 1\,,\quad R_{22} = \sin^2\theta\,,\quad R_{12} = R_{21} = 0\,,\qquad R = \frac{2}{a^2}\,. \tag{4.121}$$

Notice that we can write the Ricci tensor as

$$R_{\mu\nu} = \frac{1}{a^2}\, g_{\mu\nu}\,. \tag{4.122}$$

Metrics such as this, for which the Ricci tensor is a constant multiple of the metric tensor, are known as Einstein metrics.

5 Geodesics in General Relativity

Having introduced the basic elements of general-coordinate tensor analysis, we are now ready apply these ideas in the framework of general relativity. The essential idea in general relativity is that our four-dimensional spacetime is veiwed as a pseudo-Riemannian manifold, equipped with a smooth metric tensor $g_{\mu\nu}$ of signature $(-,+,+,+)$. In colloquial language, we may say that "spacetime tells matter how to move," and also that "matter tells spacetime how to curve."

The first half of the picture, the law governing how matter moves in spacetime, is a very natural generalisation of what we saw in chapter 3, when we studied the motion of a free particle in Minkowski spacetime, seen from the viewpoint of a non-inertial coordinate system. Locally, the description of free particle motion in a general curved spacetime is described by exactly the same *Geodesic Equation* (3.8) that described the motion of the particle in the Minkowski case. The only difference is that now $\Gamma^\mu{}_{\nu\rho}$ is the Christoffel connection (4.48) constructed from the metric tensor $g_{\mu\nu}$ of the spacetime. This chapter will be concerned with studying geodesic motion in general relativity in more detail.

The other half of the picture concerns the way in which matter tells spacetime how to curve. This is the stage where we will introduce the Einstein field equations, which are the analogue for gravity of the Maxwell field equations in electromagnetism. That will form the subject of the next chapter.

5.1 Geodesic motion in curved spacetime

In a local region of a curved spacetime, one can always choose coordinates where the metric looks approximately like the Minkowski metric. In fact, as we mentioned when proving the Bianchi idenetity for the Riemann tensor in the previous chapter, one can choose coordinates, which we shall call x'^μ, such that at an arbitrarily-chosen point $\bar{x}'^\mu$, one has

$$g'_{\mu\nu}(\bar{x}') = \eta_{\mu\nu}\,, \qquad \partial'_\mu g'_{\nu\rho}\Big|_{x'=\bar{x}'} = 0\,. \tag{5.1}$$

The latter equation implies $\Gamma'^\mu{}_{\nu\rho}(\bar{x}') = 0$ also, as can be seen from (4.48). Let us now prove that we can indeed choose coordinates such that the conditions in (5.1) hold at a point. That is to say, we make a coordinate transformation $x'^\mu = x'^\mu(x^\nu)$ and try to choose the functional dependences in such a way that (5.1) holds in the primed frame. Since we can always make "trivial" coordinate transformations in which we add appropriate constants to the coordinates, we may as well make life simple and consider the case where the chosen point is located at $x^\mu = 0$ and $x'^\mu = 0$. We can then expand the inverse coordinate transformation $x^\mu = x^\mu(x'^\nu)$ in a Taylor series around the origin:

$$x^\mu = a^\mu{}_\nu\, x'^\nu + \frac{1}{2!}\, a^\mu{}_{\nu\rho}\, x'^\nu\, x'^\rho + \frac{1}{3!}\, a^\mu{}_{\nu\rho\sigma}\, x'^\nu\, x'^\rho\, x'^\sigma + \cdots\,, \tag{5.2}$$

where $a^\mu{}_\nu$, $a^\mu{}_{\nu\rho}$, etc., are sets of constant coefficients. In the transformation rule of the metric components,

$$g'_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x'^\mu}\,\frac{\partial x^\sigma}{\partial x'^\nu}\, g_{\rho\sigma}(x) \tag{5.3}$$

we may also make a Taylor expansion of $g_{\rho\sigma}(x)$, in the form

$$g_{\mu\nu}(x) = g_{\mu\nu}(0) + \partial_\rho g_{\mu\nu}(0)\, x^\rho + \frac{1}{2!}\,\partial_\rho\partial_\sigma g_{\mu\nu}(0)\, x^\rho\, x^\sigma + \cdots\,. \tag{5.4}$$

Here, and subsequently, when we write expressions such as $\partial_\rho g_{\mu\nu}(0)$, we mean $\partial_\rho g_{\mu\nu}(x)$ with x subsequently set equal to zero.

Plugging the Taylor expansions into (5.3), we find

$$\begin{aligned} g'_{\mu\nu}(x') \;=\;& (a^\rho{}_\mu + a^\rho{}_{\mu\alpha}\, x'^\alpha + \tfrac12 a^\rho{}_{\mu\alpha_1\alpha_2} x'^{\alpha_1}\, x'^{\alpha_2} + \cdots)(a^\sigma{}_\nu + a^\sigma{}_{\nu\beta}\, x'^\beta + \tfrac12 a^\sigma{}_{\nu\beta_1\beta_2}\, x'^{\beta_1}\, x'^{\beta_2} + \cdots) \\ & \times\Big[g_{\rho\sigma}(0) + \partial_\gamma g_{\rho\sigma}(0)\,(a^\gamma{}_\delta\, x'^\delta + \tfrac12 a^\gamma{}_{\delta\tau}\, x'^\delta\, x'^\tau + \cdots) \\ & \quad + \tfrac12 \partial_\gamma\partial_\delta g_{\rho\sigma}(0)\,(a^\gamma{}_\theta\, x'^\theta + \cdots)(a^\delta{}_\eta\, x'^\eta + \cdots) + \cdots\Big]\,, \end{aligned} \tag{5.5}$$

First, we set $x'^\mu = 0$, which gives

$$g'_{\mu\nu}(0) = a^\rho{}_\mu\, a^\sigma{}_\nu\, g_{\rho\sigma}(0)\,. \tag{5.6}$$

There are $4 \times 4 = 16$ independent components $a^\rho{}_\mu$ that may be specified freely, and using 10 of these we can set the 10 independent components of $g'_{\mu\nu}(0)$ to be

$$g'_{\mu\nu}(0) = \eta_{\mu\nu}\,. \tag{5.7}$$

The $6 = 16 - 10$ remaining components of $a^\rho{}_\sigma$ are easily understood: they correspond to Lorentz transformations $\Lambda^\rho{}_\mu$ which will preserve the form of (5.7).

Next, we take the derivative ∂'_λ of (5.5) and then set $x'^\mu = 0$. This gives

$$\partial'_\lambda g'_{\mu\nu}(0) = a^\rho{}_\mu\, a^\sigma{}_\nu\, a^\gamma{}_\lambda\, \partial_\gamma g_{\rho\sigma}(0) + (a^\rho{}_{\mu\lambda}\, a^\sigma{}_\nu + a^\rho{}_\mu\, a^\sigma{}_{\nu\lambda})\, g_{\rho\sigma}(0)\,. \tag{5.8}$$

The $a^\rho{}_\mu$ cooeficients have already been fixed (modulo the Lorentaz transformations, which are not of interest here) in ensuring that (5.7) holds. But the $a^\rho{}_{\mu\lambda}$ coefficients are appearing linearly in the last two terms in (5.8), and by choosing these appropriately, we can in fact always make the right-hand side of (5.8) vanish. We can check this by counting how many parameters are available, and how many equations we wish to impose. The parameters $a^\rho{}_{\mu\lambda}$ are symmetric in μ and λ (since they are the coefficients of $x'^\mu\, x'^\lambda$ in the expansion of x'^ρ (see eqn (5.2)). Therefore, the number of independent $a^\rho{}_{\mu\lambda}$ is $(4 \times [(4 \times 5)/2])$, which equals 40. On the other hand, we would like to impose $\partial'_\lambda g'_{\mu\nu}(0) = 0$, and this is also $(4 \times [(4 \times 5)/2]) = 40$ independent equations (since $g'_{\mu\nu}$ is symmetric in μ and ν). Thus (5.8) amounts to 40 independent linear equations for the 40 independent unknowns in $a^\rho{}_{\mu\lambda}$, and so we can always find a unique solution.

The upshot of the above calculations is that we have proved that we can indeed always find a coordinate frame in which the conditions (5.1) hold at any given point.

It is instructive also to make sure that we are not able to prove "too much" by this method. Let us look now at the equations we shall obtain if we take two derivatives of (5.5) and then set $x'^\mu = 0$. We shall not labour all the details here, but it is easy to write down the result, and one will obtain something of the form

$$\partial'_{\lambda_1}\partial'_{\lambda_2} g'_{\mu\nu}(0) = a^\rho{}_\mu\, a^\sigma{}_\nu\, a^\gamma{}_{\lambda_1}\, a^\delta{}_{\lambda_2}\, \partial_\gamma\partial_\delta g_{\rho\sigma}(0) + (\text{ terms linear in } a^\rho{}_{\mu\alpha\beta}) + \text{more}\,. \tag{5.9}$$

The coefficients $a^\rho{}_{\mu\alpha\beta}$ are now available to us to try to set the second derivatives $\partial'_{\lambda_1}\partial'_{\lambda_2} g'_{\mu\nu}(0)$ to zero. But now, when we count equations and parameters, we find a problem. The $a^\rho{}_{\mu\alpha\beta}$ are symmetric in μ, α and β, so there are $4 \times [(4 \times 5 \times 6)/3!] = 80$ independent parameters. On the other hand, since $\partial'_{\lambda_1}\partial'_{\lambda_2} g'_{\mu\nu}(0)$ is symmetric in μ and ν, and also symmetric in λ_1 and λ_2, there are $[(4\times 5)/2] \times [(4\times 5)/2] = 100$ independent components. Thus we have only 80 parameters available to try to impose 100 independent conditions, so it cannot be done.

In fact we can impose 80 conditions on the 100 independent components in $\partial'_{\lambda_1}\partial'_{\lambda_2}g'_{\mu\nu}(0)$, but that leaves an irreducible core of 20 components that cannot be eliminated by means of coordinate transformations. We have seen this number before; it is the number of algebraically independent components of the Riemann tensor. This is no coincidence. The Riemann tensor is a general-coordinate covariant tensor constructed from second derivatives of the metric. What we have cofirmed above with our implementation of coordinate transformations is that there should indeed be 20 irreducible, coordinate-invariant, degrees of freedom associated with the second derivatives of the metric tensor, and these are precisely what are encoded in the Riemann tensor.

To summarise, we have seen that aside from effects due to curvature, the equation governing the motion of a free particle moving in a curved spacetime should be indistinguishable from the equation for a free particle moving in a flat spacetime described from a general non-inertial frame. We already constructed the equation for free-particle motion in a flat spacetime, viewed from an arbitrary non-inertial coordinates system, in chapter 3; it is the geodesic equation (3.8), which we reproduce here:

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu{}_{\nu\rho}\,\frac{dx^\nu}{d\tau}\,\frac{dx^\rho}{d\tau} = 0\,, \tag{5.10}$$

The only difference from before is that in chapter 3, the Christoffel connection $\Gamma^\mu{}_{\nu\rho}$ was the one calculated, using (4.48), from the metric (3.12) that was obtained by making a general coordinate transformation of the Minkowski spacetime. Now, instead, the metric $g_{\mu\nu}$ is, for the present, a completely arbitrary metric on the four-dimensional spacetime.

The geodesic equation (5.10) does not look manifestly covariant with respect to general coordinate transformations, but in fact it is. To see this, we first remark that the 4-velocity

$$U^\mu \equiv \frac{dx^\mu}{d\tau}\,, \tag{5.11}$$

is clearly a general-coordinate vector, since $d\tau = \sqrt{-ds^2}$ is a scalar and dx^μ transforms like a general-coordinate vector. If we consider the manifestly-covariant equation $U^\nu\nabla_\nu U^\mu = 0$, then using (4.31) and the chain rule we have

$$\begin{aligned} 0 = U^\nu\,\nabla_\nu\,U^\mu &= \frac{dx^\nu}{d\tau}\,\nabla_\nu\,\frac{dx^\mu}{d\tau} = \frac{dx^\nu}{d\tau}\,\partial_\nu\Big(\frac{dx^\mu}{d\tau}\Big) + \frac{dx^\nu}{d\tau}\,\Gamma^\mu{}_{\nu\rho}\,\frac{dx^\rho}{d\tau}\,,\\ &= \frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu{}_{\nu\rho}\,\frac{dx^\nu}{d\tau}\,\frac{dx^\rho}{d\tau}\,, \end{aligned} \tag{5.12}$$

which is precisely the geodesic equation (5.10).

Notice, looking back to our definition (4.102) for the parallel transport of a vector along a curve, that the geodesic equation

$$\frac{dx^\nu}{d\tau}\,\nabla_\nu\,\frac{dx^\mu}{d\tau} = 0\,, \tag{5.13}$$

which can also be written, following the notation in eqn (4.103), as

$$\frac{D}{D\tau}\Big(\frac{dx^\mu}{d\tau}\Big) = 0\,, \tag{5.14}$$

is in fact the equation for the parallel transport of the 4-velocity vector along its own integral curve. That is to say, the 4-velocity vector is parallel propagated along the direction in which it is pointing. It is in fact the nearest one could come, within the covariant framework of general relativity, to the notion of motion along a straight path.

We should add one further comment here, about the use of the proper time τ as the parameter along the path of the particle in geodesic motion. It is known as an *affine parameter*, and we can take the definition of an affine parameter to be one such that the geodesic equation takes the form (5.10). Suppose now we make a transformation to some other parameter σ, where $\sigma = \sigma(\tau)$. It would be sensible to choose the function $\sigma(\tau)$ to be such that σ, just like τ, increases monotonically along the path of the particle, that is to say, so that $d\sigma/d\tau > 0$ for all τ. What other restrictions on the choice of function arise, if we wish the geodesic equation to take the same form as (5.10) in terms of the parameter σ? Using the chain rule for differentiation, we see that

$$0 = \frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu{}_{\nu\rho}\,\frac{dx^\nu}{d\tau}\,\frac{dx^\rho}{d\tau} = \dot\sigma^2\,\Big[\frac{d^2x^\mu}{d\sigma^2} + \Gamma^\mu{}_{\nu\rho}\,\frac{dx^\nu}{d\sigma}\,\frac{dx^\rho}{d\sigma}\Big] + \ddot\sigma\,\frac{dx^\mu}{d\sigma}\,, \tag{5.15}$$

and so in general we have

$$\frac{d^2x^\mu}{d\sigma^2} + \Gamma^\mu{}_{\nu\rho}\,\frac{dx^\nu}{d\sigma}\,\frac{dx^\rho}{d\sigma} = -\frac{\ddot\sigma}{\dot\sigma^2}\,\frac{dx^\mu}{d\sigma}\,, \tag{5.16}$$

where $\dot\sigma \equiv d\sigma/d\tau$, etc. Thus the geodesic equation written in terms of the parameter σ takes the same form as (5.10) if and only if $\ddot\sigma = 0$, which means that σ must be related to τ by a so-called *affine transformation*, namely

$$\sigma = a + b\,\tau\,, \tag{5.17}$$

where a and b are constants. Any parameter for which the geodesic equation takes the standard form as in (5.10) is known as an *affine parameter*.

Note that if we write the geodesic equation in the more manifestly covariant way discussed above, then (5.16) can be written in the form

$$\frac{DV^\mu}{D\sigma} = f(\sigma)\,V^\mu\,, \qquad \text{where}\ \ V^\mu = \frac{dx^\mu}{d\sigma}\,. \tag{5.18}$$

Thus, in general, if we use a non-affine parameter the "acceleration" $DV^\mu/D\sigma$ is proportional to the "velocity" V^μ. The distinguishing feature that charactetrises an affine parameter is that the acceleration is zero along the path. Given a non-affine parameterisation of a geodesic, for which it satisfies the equation (5.18), one can always find a transformation to an affine parameterisation, by solving $\ddot{\sigma} = -\dot{\sigma}^2\, f(\sigma)$.

5.2 Geodesic deviation

We already mentioned that the local equation (5.10) for geodesic motion is the same whether the gravitational force is associated with "ponderable matter" or whether it is merely due to acceleration relative to a Minkowski spacetime inertial frame. In order to see the differences, one has to look at non-local effects, such as arise when comparing particle motions along two nearby geodesics. To do this, we can consider two nearby geodesics $x^\mu(\tau)$ and $x^\mu(\tau)+\delta x^\mu(\tau)$. If the separation is infinitesimal then δx^μ itself is a vector, and we shall write it as $Z^\mu \equiv \delta x^\mu$. One may think of it as defining the line joining the two infinitesimally-close particles. We can derive the equation for $\delta x^\mu(\tau)$ by making a variation of the geodesic equation (5.10), which gives

$$\frac{d^2 Z^\mu}{d\tau^2} + \partial_\sigma \Gamma^\mu_{\nu\rho}\, \frac{dx^\nu}{d\tau}\, \frac{dx^\rho}{d\tau}\, Z^\sigma + 2\Gamma^\mu{}_{\nu\rho}\, \frac{dx^\nu}{d\tau}\, \frac{dZ^\rho}{d\tau} = 0\,. \tag{5.19}$$

(The second term arises because $\Gamma^\mu{}_{\nu\rho}$ itself depends on the coordinates.) We would like to write the equation (5.19) for Z^μ in a covariant form.

Recalling the definition of the covariant directed derivative $D/D\lambda$ in (4.103), let us consider $D^2 Z^\mu/D\tau^2$. Expanding it out in terms of partial derivatives and connections, this is given by

$$\begin{aligned}
\frac{D^2 Z^\mu}{D\tau^2} &= \frac{d}{d\tau}\Big(\frac{DZ^\mu}{D\tau}\Big) + \frac{dx^\nu}{d\tau}\, \Gamma^\mu{}_{\nu\rho}\, \frac{DZ^\rho}{D\tau}\,, \\
&= \frac{d}{d\tau}\Big(\frac{dZ^\mu}{d\tau} + \frac{dx^\sigma}{d\tau}\, \Gamma^\mu{}_{\sigma\lambda}\, Z^\lambda\Big) + \frac{dx^\nu}{d\tau}\, \Gamma^\mu{}_{\nu\rho}\Big(\frac{dZ^\rho}{d\tau} + \frac{dx^\alpha}{d\tau}\, \Gamma^\rho{}_{\alpha\beta}\, Z^\beta\Big)\,, \\
&= \frac{d^2 Z^\mu}{d\tau^2} + \frac{d^2 x^\sigma}{d\tau^2}\, \Gamma^\mu{}_{\sigma\lambda}\, Z^\lambda + \partial_\alpha \Gamma^\mu{}_{\sigma\lambda}\, \frac{dx^\alpha}{d\tau}\, \frac{dx^\sigma}{d\tau}\, Z^\lambda + \frac{dx^\sigma}{d\tau}\, \Gamma^\mu{}_{\sigma\lambda}\, \frac{dZ^\lambda}{d\tau} \\
&\quad + \frac{dx^\nu}{d\tau}\, \Gamma^\mu{}_{\nu\rho}\, \frac{dZ^\rho}{d\tau} + \frac{dx^\nu}{d\tau}\, \Gamma^\mu{}_{\nu\rho}\, \frac{dx^\alpha}{d\tau}\, \Gamma^\rho{}_{\alpha\beta}\, Z^\beta\,.
\end{aligned} \tag{5.20}$$

We now use (5.19) to substitute for $d^2 Z^\mu/d\tau^2$ in the last line, and the geodesic equation (5.10) to substitute for $d^2 x^\sigma/d\tau^2$. We then find a variety of satisfying cancellations, including the fact that all the terms with single derivatives of Z cancel, and all the remaining $\partial\Gamma$ and $\Gamma\,\Gamma$ terms conspire to produce the Riemann tensor (see (4.65)). The upshot is that we obtain the elegant covariant equation

$$\frac{D^2 Z^\mu}{D\tau^2} = -R^\mu{}_{\rho\nu\sigma}\, \frac{dx^\rho}{d\tau}\, \frac{dx^\sigma}{d\tau}\, Z^\nu\,. \tag{5.21}$$

This is the equation of *Geodesic Deviation*. The left-hand side is a covariant expression for the 4-acceleration of one of the infinitesimally-separated particles relative to the other. If the spacetime is flat, with vanishing Riemann curvature, then there is no geodesic deviation. This is what a non-inertially moving observer in Minkowski spacetime would see. If, on the other hand, there is spacetime curvature (such as in the neighbourhood of the earth, the observer will see nearby geodesic accelerating relative to one another. (Such as would be seen by an observer in a freely-falling elevator, who watched two nearby particles in geodesic motion converging as they both accelerated towards the centre of the earth.) spacetime,

5.3 Geodesic equation from a Lagrangian

The geodesic equation (5.10) can be derived very easily from a Lagrangian. This also has the added bonus that it provides a very convenient and streamlined way of deriving the expressions for the Christoffel connection components $\Gamma^\mu{}_{\nu\rho}$ in a more efficient way than using the formula (4.48).

Consider the Lagrangian and action

$$L = \tfrac{1}{2} g_{\mu\nu}\, \dot{x}^\mu\, \dot{x}^\nu\,, \qquad I = \int L d\tau\,, \tag{5.22}$$

where $\dot{x}^\mu$ is a shorthand for $dx^\mu/d\tau$. Varying I with respect to the path $x^\mu(\tau)$ gives

$$\begin{aligned} \delta I &= \int d\tau \Big[\tfrac{1}{2}\partial_\rho g_{\mu\nu} \delta x^\rho\, \dot{x}^\mu\, \dot{x}^\nu + g_{\mu\nu}\, \dot{x}^m u\, \delta\dot{x}^\nu\Big]\,, \\ &= \int d\tau \Big[\tfrac{1}{2}\partial_\rho g_{\mu\nu} \delta x^\rho\, \dot{x}^\mu\, \dot{x}^\nu - \frac{d}{d\tau}(g_{\mu\nu}\, \dot{x}^\mu)\, \delta x^\nu\Big]\,, \\ &= \int d\tau \Big[\tfrac{1}{2}\partial_\nu g_{\mu\rho}\, \dot{x}^\mu\, \dot{x}^\rho - \partial_\rho g_{\mu\nu}\, \dot{x}^\rho\, \dot{x}^\mu - g_{\mu\nu}\, \ddot{x}^\mu\Big]\, \delta x^\nu\,. \end{aligned} \tag{5.23}$$

Thus the principle of stationary action $\delta I = 0$ gives

$$g_{\mu\nu}\, \ddot{x}^\mu + [\partial_\rho g_{\mu\nu} - \tfrac{1}{2}\partial_\nu g_{\mu\rho}]\, \dot{x}^\rho\, \dot{x}^\mu = 0\,. \tag{5.24}$$

This is, of course, a derivation of the Euler-Lagrange equations

$$\frac{d}{d\tau}\Big(\frac{\partial L}{\partial \dot{x}^\nu}\Big) - \frac{\partial L}{\partial x^\nu} = 0\,. \tag{5.25}$$

Multiplying by $g^{\sigma\nu}$, we therefore have

$$\ddot{x}^\sigma + \tfrac{1}{2} g^{\sigma\nu}\, (\partial_\rho g_{\mu\nu} + \partial_\mu g_{\rho\nu} - \partial_\nu g_{\mu\rho})\, \dot{x}^\rho\, \dot{x}^\mu = 0\,, \tag{5.26}$$

where we have used the symmetry of $\dot{x}^\rho\, \dot{x}^\mu$ to write $\partial_\rho g_{\mu\nu}\, \dot{x}^\rho\, \dot{x}^\mu$ as $\frac{1}{2}(\partial_\rho g_{\mu\nu} + \partial_\mu g_{\rho\nu})\dot{x}^\rho\, \dot{x}^\mu$. From the expression (4.48) for $\Gamma^\mu{}_{\nu\rho}$ we see that (5.26) is precisely the geodesic equation (5.10), i.e. (after index relabelling)

$$\ddot{x}^\mu + \Gamma^\mu{}_{\nu\rho}\, \dot{x}^\nu\, \dot{x}^\rho = 0\,. \tag{5.27}$$

Note also from the definition of the Lagrangian in (5.22) that along the geodesic path followed by the particle, one has

$$L = \tfrac{1}{2} g_{\mu\nu}\, \dot x^\mu\, \dot x^\nu = \tfrac{1}{2} g_{\mu\nu}\, \frac{dx^\mu}{d\tau}\, \frac{dx^\nu}{d\tau} = \tfrac{1}{2} \frac{g_{\mu\nu}\, dx^\mu\, dx^\nu}{d\tau^2} = -\tfrac{1}{2} \frac{d\tau^2}{d\tau^2} = -\tfrac{1}{2}\,. \qquad (5.28)$$

The fact that one derive the geodesic equation from the action given in (5.22) provides, as a bonus, a rather streamlined way of calculating the Christoffel connection for any metric. One uses the Euler-Lagrange equations (5.25) to derive the geodesic equation (5.27), and then simply reads off the commponents of the Christoffel connection. Consider as an example the 2-sphere metric (4.116). The Lagrangian L in (5.22) is therefore

$$L = \tfrac{1}{2} a^2\, \dot\theta^2 + \tfrac{1}{2} a^2\, \sin^2\theta\, \dot\varphi^2\,. \qquad (5.29)$$

(Because the metric signature is $(+,+)$ in ths example, we use proper distance s rather than proper time τ to parameterise the path of the geodesic, so $\dot x^\mu = dx^\mu/ds$ here.) The Euler-Lagrange equations (5.25) give

$$\ddot\theta - \sin\theta\, \cos\theta\, \dot\varphi^2 = 0\,, \qquad \ddot\varphi + 2\cot\theta\, \dot\theta\, \dot\varphi = 0\,. \qquad (5.30)$$

Taking $x^1 = \theta$ and $x^2 = \varphi$, we therefore immediately read off that the only non-zero components of $\Gamma^\mu{}_{\nu\rho}$ are[10]

$$\Gamma^1{}_{22} = -\sin\theta\, \cos\theta\,, \qquad \Gamma^2{}_{12} = \Gamma^2{}_{21} = \cot\theta\,. \qquad (5.31)$$

These can be seen to be in agreement with those that were found in (4.118) by using the formula (4.48). The great advantage (especially for a human) in using the method described above is that the results for all the $\Gamma^\mu{}_{\nu\rho}$ with a given value of the μ index come all at once, from a single equation. Thus one effectively only has to do n calculations for an n-dimensional metric. By contrast, if one uses the formula (4.48) one has to perform $\frac{1}{2}n^2(n+1)$ distinct calculations, one for each inequivalent choice of the index values for μ, ν and ρ. The saving may not be so impressive for $n = 2$, but for $n = 11$, say, the saving is considerable! A further point is that commonly, many of the components of $\Gamma^\mu{}_{\nu\rho}$ may in fact be zero, and a nice feature of the method described above is that these never appear in the calculation. By contrast, if one is grinding through the calculations, component by component, using (4.48), then one may be expending a lot of mental effort producing zero over and over again.

[10]Note that a common mistake is to fail to divide the coefficient of an off-diagonal term in $\dot x^\nu\, \dot x^\rho$ by two when reading off $\Gamma^\mu{}_{\nu\rho}$, such as in the $\dot\theta\dot\varphi$ term in the second equation in (5.30). The point is that both $\Gamma^2{}_{12}$ and $\Gamma^2{}_{21}$ contribute equally, and so each is equal to one half of the coefficient of $\dot\theta\dot\varphi$ in the geodesic equation in (5.30).

5.4 Null geodesics

A massless particle, such as a photon, follows a geodesic path, just as massive particles do. However, we can no longer use the proper time along the path of a photon, because the invariant proper-time interval between neighbouring points on the path given by $d\tau^2 = -g_{\mu\nu}dx^\mu dx^\nu$, is zero. Instead, we must choose some other parameter λ along the path of the photon. A possible choice would be to use the time coordinate t in a given coordinate frame, but we can leave things more general and just consider a parameter λ. We should choose a parameter that increases monotonically along the path (as the time coordinate t would), and also, we should, for convenience, choose an affine parameter.

The geodesic equation can be obtained by repeating the previous derivation for a massive particle, which started with the equation for the particle moving in Minkowski spacetime in an inertial frame. Instead of (3.2), we must now use a parameter λ that increases monotonically along the path of the null light ray, so that we have $d^2\tilde{x}^\mu/d\lambda^2 = 0$. Transforming to an arbitrary coordinate frame then gives (5.32), where the connection is given by (3.9). Finally, we generalise to an arbitrary background metric, and so the geodesic equation will still take the form (5.32), except that now the connection is the Christoffel connection given in terms of the spacetime metric by (4.48). Thus we find

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu{}_{\nu\rho}\frac{dx^\nu}{d\lambda}\frac{dx^\rho}{d\lambda} = 0\,. \tag{5.32}$$

This equation can be derived from the Lagrangian

$$L = \tfrac{1}{2}g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} \tag{5.33}$$

in the same way as in the massive case. One difference now, however, is that since $d\tau^2 = 0$ we have

$$L = 0 \tag{5.34}$$

on the path of the photon, rather than the previous result that $L = -\frac{1}{2}$ for the massive particle.

5.5 Geodesic motion in the Newtonian limit

The geodesic equation is the analogue in general relativity of Newton's second law applied to the case of a particle in a gravitational field. To see this, it is useful to consider the geodesic equation in the Newtonian limit, where the gravitational field is very weak and independent of time, and the particle is moving slowly. It will be convenient to split the spacetime coordinate index μ into $\mu = (0, i)$, where i ranges only over the spatial index values, $1 \le i \le 3$. Saying that the velocity is small (compared with the speed of light) means that

$$\Big|\frac{dx^i}{dt}\Big| << 1\,. \tag{5.35}$$

Since we are assuming weak gravitational fields here, we can assume that in a suitable coordinate system the metric is close to the Minkowski metric,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}\,, \tag{5.36}$$

where the deviations $h_{\mu\nu}$ are very small compared to 1. Since we are assuming time independence, this means that we may assume also that $\partial g_{\mu\nu}/\partial t = 0$.[11] Note that the inverse metric is of the form

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} + \mathcal{O}(h^2)\,, \tag{5.37}$$

where by definition $h^{\mu\nu} = \eta^{\mu\rho}\,\eta^{\nu\sigma}\,h_{\rho\sigma}$.

In the low-velocity limit, coordinate time t and proper time τ are essentially the same, and thus we also have

$$\frac{dx^0}{d\tau} \approx 1\,. \tag{5.38}$$

Consider now the spatial components of the geodesic equation (5.10). In this Newtonian limit, it therefore approximates to

$$\frac{d^2x^i}{dt^2} + \Gamma^i{}_{00} = 0\,. \tag{5.39}$$

From the expression (4.48) for the Christoffel connection, it follows from (5.36) and the assumption $\partial h_{\mu\nu}/\partial t = 0$ that

$$\Gamma^i{}_{00} \approx -\tfrac12 \partial_i h_{00}\,. \tag{5.40}$$

Thus the geodesic equation reduces in the Newtonian limit to

$$\frac{d^2x^i}{dt^2} = \tfrac12 \partial_i h_{00}\,. \tag{5.41}$$

We now compare this with the Newtonian equation for a particle moving in a gravitational field. If the Newtonian potential is Φ, then the equation of motion following from Newton's second law (assuming that the gravitational and inertial masses are equal!) is

$$\frac{d^2x^i}{dt^2} = -\partial_i\Phi\,. \tag{5.42}$$

[11]Of course one could always perversely then make a transformation to coordinates in which the metric components *did* depend on t. In this, as in many other examples, we cover ourselves by saying "there exists a choice of coordinates in which..."

Comparing with (5.41), we see that

$$h_{00} = -2\Phi\,. \tag{5.43}$$

(We can take the constant of integration to be zero, since at large distance, where the Newtonian potential vanishes, the metric should reduce to exactly the Minkowski metric.) Thus the spacetime metric in the weak-field Newtonian limit can be arranged to take the form[12]

$$ds^2 \approx -(1+2\Phi)\,dt^2 + (\delta_{ij} + h_{ij})\,dx^i dx^j\,. \tag{5.44}$$

Notice that in general relativity the equality of gravitational and inertial mass is built in from the outset; the geodesic equation (5.10) makes no reference to the mass of the particle.

Another important point is to note that in the geodesic equation (5.10), the Christoffel connection $\Gamma^\mu{}_{\nu\rho}$ is playing the rôle of the "gravitational force," since it is this term that describes the deviation from "linear motion" $d^2x^\mu/d\tau^2 = 0$. The fact that the gravitational force is described by a connection, and not by a tensor, is just as one would hope and expect. The point is that the "force of gravity" can come or go, depending on what system of coordinates one uses. In particular, if one chooses a free-fall frame, in which the metric at any given point can be taken to be the Minkowski metric, and the first derivatives can also be taken to vanish at the point, then the Christoffel connection vanishes at the point also. Thus indeed, we have the vanishing of gravity (weightlessness) in a local free-fall frame.

6 Einstein Equations, Schwarzschild Solution and Classic Tests

6.1 Derivation of the Einstein equations

So far, we have seen how matter responds to gravity, namely, according to the geodesic equation, which shows how matter moves under the influence of the gravitational field. The other side of the coin is to see how gravity is determined by matter. The equations which control this are the Einstein field equations. These are the analogue of the Newtonian equation

$$\nabla^2\,\Phi = 4\pi G\,\rho\,, \tag{6.1}$$

which governs the Newtonian gravitational potential Φ in the presence of a mass density ρ. Here G is Newton's constant.

[12]Here, we have enlarged the assumption of time independence to the stronger one that the metric is *static*. This amounts to saying that there exists a choice of coordinates where not only is $\partial g_{\mu\nu}/\partial t = 0$ but also that $g_{0i} = 0$, so there are no $dtdx^i$ cross-terms in the metric.

The required field equations in general relativity can be expected, like Newton's field equation, to be of order 2 in derivatives. Again we can proceed by considering first the Newtonian limit of general relativity. Since, as we have seen, the deviation h_{00} of the metric component g_{00} from its Minkowskian value -1 is equal to -2Φ in the Newtonian limit (see (5.43)), we are led to expect that the Einstein field equations should involve second derivatives of the metric. We also expect that the equation should be tensorial, since we would like it to have the same form in all coordinate frames. Luckily, there exist candidate tensors constructed from the metric, since, as we saw earlier, the Riemann tensor, and its contractions to the Ricci tensor and Ricci scalar, involve second derivatives of the metric. Some appropriate construct built from the curvature will therefore form the "left-hand side" of the Einstein equation.

There remains the question of what will sit on the right-hand side, generalising the mass density ρ. There is again a natural tensor generalisation, namely the *energy-momentum tensor*, or *stress tensor*, $T_{\mu\nu}$. This is a symmetric tensor that describes the distribution of mass (or energy) density, momentum flux density, and stresses in a matter system. We met some examples, in the context of special relativity, in section 2. Specifically, if we decompose the four-dimensional spacetime index μ as $\mu = (0, i)$ as before, then T_{00} describes the mass density (or energy density), T_{0i} describes the 3-momentum flux, and T_{ij} describes the stresses within the matter system.

A very important feature of the energy-momentum tensor for a closed system is that it is *conserved*, meaning that

$$\nabla^\nu T_{\mu\nu} = 0 \,. \tag{6.2}$$

This is analogous to the conservation law $\nabla^\mu J_\mu = 0$ for the 4-vector current density in electromagnetism. In that case, the conservation law ensures that charge is conserved, and by integrating J_0 over a closed spatial 3-volume and taking a time derivative, one shows that the rate of change of total charge within the 3-volume is balanced by the flux of electric current out of the 3-volume. Analogously, (6.2) ensures that the rate of change of total 4-momentum within a closed 3-volume is balanced by the 4-momentum flux out of the region.

If we are to build a field equation whose right-hand side is a constant multiple of $T_{\mu\nu}$, it follows, therefore, that the left-hand side must also satisfy a conservation condition. There is precisely one symmetric 2-index tensor built from the curvature that has this property, namely the *Einstein tensor*

$$G_{\mu\nu} \equiv R_{\mu\nu} - \tfrac{1}{2} R\, g_{\mu\nu} \,, \tag{6.3}$$

which we met in equation (4.95). Thus our candidate field equation is $G_{\mu\nu} = \lambda\, T_{\mu\nu}$, i.e.

$$R_{\mu\nu} - \tfrac{1}{2} R\, g_{\mu\nu} = \lambda\, T_{\mu\nu}\,, \tag{6.4}$$

for some universal constant λ, which we may determine by requiring that we obtain the correct weak-field Newtonian limit.

In a situation where the matter system has low velocities, its energy-momentum tensor will be dominated by the T_{00} component, which describes the mass density ρ. Thus to find the Newtonian limit of (6.4), we should examine the 00 component. To do this, it is useful first to take the trace of (6.4), by multiplying by $g^{\mu\nu}$. This gives

$$-R = \lambda\, g^{\mu\nu}\, T_{\mu\nu}\,. \tag{6.5}$$

Since $T_{\mu\nu}$ is dominated by $T_{00} = \rho$, and the metric is nearly the Minkowski metric (so $g^{00} \approx -1$), we see that

$$R \approx \lambda\, \rho \tag{6.6}$$

in the Newtonian limit. Thus, (6.4) reduces to

$$R_{00} \approx \tfrac{1}{2}\lambda\rho\,. \tag{6.7}$$

It is easily seen from the expression (4.65) for the Riemann tensor, and the definition (4.91) for the Ricci tensor, that from (5.40) the component R_{00} is dominated by

$$R_{00} \approx \partial_i \Gamma^i{}_{00} \approx -\tfrac{1}{2}\partial_i\partial^i\, h_{00}\,. \tag{6.8}$$

From (5.43) we therefore have that $R_{00} \approx \nabla^2\,\Phi$ in the Newtonian limit, and hence, from (6.7), we obtain the result

$$\nabla^2\,\Phi \approx \tfrac{1}{2}\lambda\,\rho\,. \tag{6.9}$$

It remains only to compare this with Newton's equation (6.1), thus determining that $\lambda = 8\pi G$.

In summary, we have arrived at the Einstein field equations[13]

$$R_{\mu\nu} - \tfrac{1}{2} R\, g_{\mu\nu} = 8\pi\, G\; T_{\mu\nu}\,, \tag{6.10}$$

[13]There is no universal agreement as to whether one should call (6.10) the Einstein field *equation*, or the Einstein field *equations*. On the one hand, eqn (6.10) comprises multiple differential equations (one for each value of μ and ν). On the other hand (6.10) is a single tensor equation, which could be written in a coodinate-free notation as $\mathrm{Ric} - \frac{1}{2} R\, \mathrm{met} = 8\pi T$, where $\mathrm{Ric} = R_{\mu\nu}\, dx^\mu \otimes dx^\nu$, etc. In practice, in these notes, I sometimes call them the Einstein equations and sometimes the Einstein equation.

and we have shown in particular that they have the proper Newtonian limit.

The Einstein equations could be viewed as the gravitational analogue of the Maxwell equations for electromagnetism. Thus, in electrodynamics we have the equation

$$\partial_\nu F^{\mu\nu} = 4\pi J^\mu \,. \tag{6.11}$$

(This equation is written in Minkowski spacetime here. We shall presently discuss the simple modification needed in order to write it in a general curved spacetime.) In each of (6.10) and (6.11) the left-hand side has terms involving derivatives of the field (gravitational or electromagnetic) of the theory. And each equation, on the right-hand side, has sources describing either the mass and momentum distribution, or the electric charge and current distribution, respectively. However, there is a very important qualitative difference between the two equations. The Maxwell equations are *linear* differential equations governing the electromagnetic field. By contrast, the Einstein equations are *non-linear* in the gravitational field. This is evident from the way that the Christoffel connection is constructed from the metric in (4.48), and the way that the Riemann tensor is then constructed from the connection, in (4.65). The reason for the non-linearity can easily be understood physically. The key point is that in general relativity, *all* systems with mass, energy and momentum tend to generate spacetime curvature. This includes the gravitational field itself, and hence the equations that govern the gravitational field must include the description of the gravitational field acting on itself. Hence the non-linearity. By contrast, the electromagnetic field is itself uncharged (the photon is neutral), and thus it does not act as a source for itself.[14]

6.2 The Schwarzschild solution

We now turn to our first example of the construction of a solution of the Einstein equations. This will be the gravitational analogue of the solution for a point charge in electromagnetism. It is also probably the most important of all the solutions in general relativity.

When one solves for the field of a point charge in electromagnetism one initially focuses on solving for the potential outside the origin, and so one simply takes the right-hand side of the Maxwell equations (6.11) to be zero. In the same vein, we shall begin our investigation of the gravitational field of speherically-symmetric system by focusing on an exterior region where we may assume that there is no matter at all, and so we take $T_{\mu\nu} = 0$ in (6.10).

[14] In the generalisation of electromagnetism to Yang-Mills theory, the Yang-Mills field *is* charged, and the associated Yang-Mills equations are consequently non-linear. In that case, the degree of non-linearity is much milder than for gravity.

The vacuum Einstein equations

$$R_{\mu\nu} - \tfrac{1}{2} R g_{\mu\upsilon} = 0 \tag{6.12}$$

can actually be reduced to the simpler condition of Ricci-flatness,

$$R_{\mu\nu} = 0\,. \tag{6.13}$$

Let us demonstrate this for an arbitrary spacetime dimensions n, which, as we shall see, must be greater than 2. Multiplying (6.12) by $g^{\mu\nu}$ gives

$$0 = R - \tfrac{1}{2} n R = -\tfrac{1}{2}(n-2) R\,. \tag{6.14}$$

Thus, provided that $n > 2$ we see that (6.12) implies $R = 0$, and plugging this back into (6.12) gives the Ricc-flat condition (6.13). Furthermore, $R_{\mu\nu} = 0$ implies $R = 0$, so the entire content of the vacuum Einstein equations is contained in the Ricci-flatnesss equation (6.13).

We shall assume that the solution we are looking for is spherically-symmetric, and also that it is static. It is not hard to see that the most general such metric can be conveniently written in the form

$$ds^2 = -B(r)\, dt^2 + A(r)\, dr^2 + r^2\, (d\theta^2 + \sin^2\theta\, d\varphi^2)\,, \tag{6.15}$$

where $A(r)$ and $B(r)$ are as-yet arbitrary functions of the radial variable r. That is to say, there exists a convenient choice of coordinate system in which it can be written as (6.15). We shall determine the functions $A(r)$ and $B(r)$ shortly, by requiring that the metric (6.15) satisfy (6.13). Note that if we had $A(r) = B(r) = 1$, then (6.15) would be just the Minkowski metric, but with the spatial part of the metric written in terms of spherical polar coordinates:

$$ds^2_{\text{Mink.}} = -dt^2 + dr^2 + r^2\, (d\theta^2 + \sin^2\theta\, d\varphi^2)\,. \tag{6.16}$$

Since we are expecting our solution to descrbe the gravitational field outside a spherically-symmetric static mass distribution, we can expect that the metric should approach (6.16) as r tends to infinity.

To proceed, we first calculate the Christoffel connection, which can be done either using (4.48), or, more efficiently, using the method we described earlier, in which one reads off the connection from the geodesic equation, derived from the Lagrangian in (5.22). Then, we calculate the Riemann tensor, using (4.65), taking the contraction to get the Ricci tensor, defined in (4.91). Taking the indexing of the coordinates to be

$$x^0 = t\,, \quad x^1 = r\,, \quad x^2 = \theta\,, \quad x^3 = \varphi\,, \tag{6.17}$$

it is not hard to see from (4.48) that the non-vanishing components of the Christoffel connection $\Gamma^\mu{}_{\nu\rho}$ are given by

$$\begin{aligned}
&\Gamma^0{}_{01} = \frac{B'}{2B}\,,\\
&\Gamma^1{}_{00} = \frac{B'}{2A}\,,\quad \Gamma^1{}_{11} = \frac{A'}{2A}\,,\quad \Gamma^1{}_{22} = -\frac{r}{A}\,,\quad \Gamma^1{}_{33} = -\frac{r\sin^2\theta}{A}\,,\\
&\Gamma^2{}_{12} = \frac{1}{r}\,,\quad \Gamma^2{}_{33} = -\sin\theta\cos\theta\,,\\
&\Gamma^3{}_{13} = \frac{1}{r}\,,\quad \Gamma^3{}_{23} = \cot\theta\,.
\end{aligned} \tag{6.18}$$

(Of course, as always the symmetry in the lower two indices is understood, so we do not need to list the further components that are implied by this.) The notation here is that $A' = dA/dr$ and $B' = dB/dr$. Plugging into the definition of the Rieman tensor, and then contracting to get the Ricci tensor, one then finds that the non-vanishing components are given by

$$\begin{aligned}
R_{00} &= \frac{B''}{2A} - \frac{B'}{4A}\Big(\frac{A'}{A} + \frac{B'}{B}\Big) + \frac{B'}{rA}\,,\\
R_{11} &= -\frac{B''}{2B} + \frac{B'}{4B}\Big(\frac{A'}{A} + \frac{B'}{B}\Big) + \frac{A'}{rA}\,,\\
R_{22} &= 1 + \frac{r}{2A}\Big(\frac{A'}{A} - \frac{B'}{B}\Big) - \frac{1}{A}\,,\\
R_{33} &= R_{22}\sin^2\theta\,.
\end{aligned} \tag{6.19}$$

Actually, it is worth remarking here that when one just wants to calculate the Ricci tensor, and does not want to know all the individual components of the Riemann tensor, it is more efficient to take the trace of (4.65) first, before starting the explicit calculations. Thus from (4.65) we have, after some index relabelling and using the symmetry of the Christoffel connection,

$$R_{\mu\nu} = \partial_\rho \Gamma^\rho{}_{\mu\nu} - \partial_\nu \Gamma^\rho{}_{\rho\mu} + \Gamma^\rho{}_{\rho\sigma}\,\Gamma^\sigma{}_{\mu\nu} - \Gamma^\rho{}_{\mu\sigma}\,\Gamma^\sigma{}_{\nu\rho}\,. \tag{6.20}$$

Now, in n dimensions, one only has to face doing $\frac{1}{2}n(n+1)$ calculations rather than the $\frac{1}{2}n^3(n-1)$ or so that one would do if one enumerated all the components of $R^\rho{}_{\sigma\mu\nu}$, where only the "obvious" antisymmetry in $\mu\nu$ would be immediately useful for reducing the labour.

To solve the Ricci-flatness condition (6.13) we first note from (6.19) that taking the combination $AR_{00} + BR_{11} = 0$ gives

$$\frac{1}{r}\Big(B' + \frac{A'\,B}{A}\Big) = 0\,, \tag{6.21}$$

which implies $(AB)' = 0$. Thus we have

$$AB = \text{constant}\,. \tag{6.22}$$

Now at large distance, we expect the metric to approach Minkowski spacetime, and so it should approach (6.16). This determines that $A(r)$ and $B(r)$ should both approach 1 at large distance, and hence we see that the constant in the solution (6.22) should be 1, and so $A = 1/B$.

From the condition $R_{22} = 0$, we then obtain the equation

$$1 - rB' - B = 0\,, \tag{6.23}$$

which can be written as

$$(rB)' = 1\,. \tag{6.24}$$

The solution to this, with the requirement that $B(r)$ approach 1 at large r, is given by

$$B = 1 + \frac{a}{r}\,, \tag{6.25}$$

where a is a constant. It is straightforward to verify that all the Einstein equations implied by $R_{\mu\nu} = 0$ are now satisfied.

Recalling that we showed previously that in the weak-field Newtonian limit, the metric $g_{\mu\nu}$ is approximately of the form $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $h_{00} = -2\Phi$, where Φ is the Newtonian gravitational potential (see equation (5.43)), it follows that the constant a in (6.25) can be determined, by considering the Newtonian limit. Thus we shall have $a = -2GM$, where G is Newton's constant. Usually, in general relativity we choose units where $G = 1$, and so we arrive at the Schwarzschild solution

$$ds^2 = -\Big(1 - \frac{2M}{r}\Big)\, dt^2 + \Big(1 - \frac{2M}{r}\Big)^{-1}\, dr^2 + r^2\, (d\theta^2 + \sin^2\theta\, d\phi^2)\,. \tag{6.26}$$

This describes the gravitational field outside a spherically-symmetric static mass M. The solution was first obtained by Karl Schwarzschild in 1916, less than a year after Einstein published his theory of general relativity.

As expected, the solution approaches Minkowski spacetime at large radius. It is clear that something rather drastic happens to the metric when r approaches $2M$. This radius, known as the *Schwarzschild Radius*, was thought for many years to correspond to some singularity of the solution. It was really only in the 1950's that it was first understood that the apparent singularity is merely a result of using a system of coordinates that becomes ill-behaved there. There is nothing singular about the solution as such. For example, the curvature is perfectly finite there, and in fact the only place where there is a curvature singularity is at $r = 0$.

We shall return to a more detailed discussion of the global structure of the Schwarzschild solution later on. For now, just to give a very simple example of the sort of things that can happen if one changes coordinate systems, consider the two-dimensional metric

$$ds^2 = \frac{du^2}{1-u^2} + (1-u^2)\, d\varphi^2\,. \tag{6.27}$$

This also exhibits rather singular-looking behaviour, at $u = \pm 1$, with the g_{uu} metric component blowing up there. However, a simple transformation of the u coordinate, by writing $u = \cos\theta$, puts the metric in the form

$$ds^2 = d\theta^2 + \sin^2\theta\, d\varphi^2\,, \tag{6.28}$$

which can now be recognised as the metric on a unit-radius 2-sphere (see (4.116)).

6.3 Classic tests of general relativity

Putting further discussion of the global structure to one side for now, we shall pass on to a discussion of some of the physical properties of the Schwarzschild solution, viewed as a description of the gravitational field outside a spherically-symmetric, static object such as a star. Note that if one puts in the numbers, and calculates the Schwarzschild radius for a spherically-symmetric object having the mass of the sun, one finds it is about 1 kilometre. This is tiny in comparison to the radius of the sun, and so in the exterior region outside the surface of the sun the $2M/r$ term in the function $(1-2M/r)$ that appears in the Schwarzschild solution is absolutely tiny compared to 1. Thus for the present purposes, we do not need to worry about the subtleties that arise when r goes down to the radius $2M$.

We shall now discuss the three "classic tests" of general relativity, namely the advance of the perihelion of a planet in its orbit around the sun; the bending of light that passes close to the sun; and the radar echo delay when a radio signal from earth is bounced off a planet on the far side of the sun, passing close to the sun's surface on the outward and return journey:

6.3.1 Orbits around a star or black hole

In section 5 we derived the geodesic equation (5.10), which describes how a test particle will move in an arbitrary gravitational field. We can now use this equation to study the orbits of particles moving in the Schwarzschild geometry. This allows us to study, for example, planetary orbits around the sun. In particular, we can then investigate the deviation from the usual Kepler laws of planetary orbits implied by general relativity. We can also consider orbits in the more extreme situation of a black hole.

As we saw earlier, the geodesic equation for a massive particle can be derived from the Lagrangian given in (5.22), which, for the case of the Schwarzschild metric (6.26), is given by

$$\mathcal{L} = -\tfrac{1}{2} B\, \dot{t}^2 + \tfrac{1}{2} B^{-1}\, \dot{r}^2 + \tfrac{1}{2} r^2 (\dot{\theta}^2 + \sin^2\theta\, \dot{\varphi}^2)\,, \tag{6.29}$$

where as before

$$B = 1 - \frac{2M}{r}\,. \tag{6.30}$$

As in any Lagrangian problem, if $\mathcal{L}$ does not depend on a particular coordinate q (i.e. it is what is called an "ignorable coordinate"), then one has an associated first integral, since its Euler-Lagrange equation

$$\frac{d}{d\tau}\Big(\frac{\partial \mathcal{L}}{\partial \dot{q}}\Big) - \frac{\partial \mathcal{L}}{\partial q} = 0 \tag{6.31}$$

reduces to

$$\frac{d}{d\tau}\Big(\frac{\partial \mathcal{L}}{\partial \dot{q}}\Big) = 0\,, \tag{6.32}$$

which can be integrated to give

$$\frac{\partial \mathcal{L}}{\partial \dot{q}} = \text{constant}\,. \tag{6.33}$$

In our case, t and φ are ignorable coordinates, and so we have the two first integrals

$$B\, \dot{t} = E\,, \qquad r^2 \sin^2\theta\, \dot{\varphi} = \ell\,, \tag{6.34}$$

for integration constants E and ℓ. The first of these is associated with energy conservation, and the second with angular-momentum conservation. We also have (5.28), which is like another first integral, giving

$$B\dot{t}^2 - B^{-1}\, \dot{r}^2 - r^2\dot{\theta}^2 - r^2\, \sin^2\, \dot{\varphi}^2 = 1\,. \tag{6.35}$$

Of course one can plug (6.34) into (6.35).

It is easy to see, because of the symmetries of the problem, that just as in Newtonian mechanics, planetary orbits will lie in a plane. Because of the symmetries, we can, without loss of generality, take this to be the equatorial plane, $\theta = \frac{1}{2}\pi$. (The test of the assertion that the motion lies in a plane is to verify that the Euler-Lagrange equation for θ implies that $\ddot{\theta} = 0$ if we set $\theta = \frac{1}{2}\pi$ and $\dot{\theta} = 0$. In other words, if one starts the particle off with motion in the equatorial plane, it stays in the equatorial plane. We leave this as an exercise.)

If we proceed by taking $\theta = \frac{1}{2}\pi$ we have three first integrals for the three coordinates t, φ and r, and so the Euler-Lagrange equation for r is superfluous (since we already know its first integral). From (6.34) and (6.35) we therefore have

$$\Big(1-\frac{2M}{r}\Big)\,\dot t = E\,,\qquad r^2\,\dot\varphi = \ell\,,\qquad \dot r^2 = E^2 - \Big(1+\frac{\ell^2}{r^2}\Big)\Big(1-\frac{2M}{r}\Big)\,. \tag{6.36}$$

Note that the third equation has been obtained by substituting the first two into (6.35), and using also (6.30).

If we rewrite the third equation in (6.36) as

$$\dot r^2 + V(r) = E^2\,, \tag{6.37}$$

where

$$V(r) = \Big(1+\frac{\ell^2}{r^2}\Big)\Big(1-\frac{2M}{r}\Big)\,, \tag{6.38}$$

then it can be recognised as the equation for the one-dimensional motion of a particle of mass $m = 2$ in the effective potential $V(r)$. It is worth remarking that if we were instead solving the problem of planetary orbits in Newtonian mechanics, we would have $V(r) = \ell^2/r^2 - 2M/r$. The extra term 1 in the general relativistic expression (6.38) is just a shift in the zero point of the total energy E^2, corresponding to the rest mass of the particle. The important difference in general relativity is the extra term $-2M\ell^2/r^3$ that comes from multiplying out the factors in (6.38). As we shall see, this term implies that the major axis of an elliptical planetary orbit will precess, rather than remaining fixed as it does in the Newtonian case. This is a testable prediction of general relativity, that has indeed been verified.

The nature of the orbits is determined by the shape of the effective potential $V(r)$ in equation (6.38). In particular, the crucial question is whether it has any critical points (where the derivative vanishes). From (6.38) we have

$$\frac{dV}{dr} = -\frac{2\ell^2}{r^3} + \frac{2M}{r^2} + \frac{6M\ell^2}{r^4}\,, \tag{6.39}$$

and so $dV/dr = 0$ if

$$r = \frac{\ell^2 \pm \ell\,\sqrt{\ell^2 - 12M^2}}{2M}\,. \tag{6.40}$$

If $\ell^2 < 12M^2$ there are therefore no critical points, and the effective potential just plunges from $V = 1$ at $r = \infty$ to $V = -\infty$ as r goes to zero. There are no orbits possible in this case.

If $\ell^2 > 12M^2$, the effective potential $V(r)$ has two critical points, at radii $r_\pm$ given by

$$r_\pm = \frac{\ell^2 \pm \ell\,\sqrt{\ell^2 - 12M^2}}{2M}\,. \tag{6.41}$$

The effective potential attains a maximum at $r = r_-$, and a local minimum at $r = r_+$. There is a potential well in the region $r_0 \le r \le \infty$, where $V(r_0) = 1$ and r_0 occurs at some value greater than r_- and less than r_+. If the integration constant E (related to the energy of the particle) is appropriately chosen, we can then obtain orbits in which r oscillates between turning points that lie within the region $r_0 \le r \le \infty$.

The simplest case to consider is a circular orbit, achieved when $r = r_+$ so that we are sitting at the local minimum at the bottom of the potential well. This will be achieved if

$$E^2 = V(r_+)\,, \tag{6.42}$$

since then, as can be seen from (6.37), we shall have $\dot{r} = 0$ and so $r = r_+$ for all τ.

To analyse the orbits in general, it is useful, as in the Newtonian case, to introduce a new variable u instead of r, defined by

$$u = \frac{M}{r}\,. \tag{6.43}$$

We also define a rescaled, dimensionless, angular momentum parameter $\tilde{\ell}$, defined by

$$\tilde{\ell} = \frac{\ell}{M}\,. \tag{6.44}$$

Since r and φ are both functions of τ it is then convenient to consider r, or the new variable u, as a function of φ. Elementary algebra shows that (6.37) gives rise to

$$\Big(\frac{du}{d\varphi}\Big)^2 + (1-2u)(u^2 + \tilde{\ell}^{-2}) = E^2\,\tilde{\ell}^{-2}\,. \tag{6.45}$$

In deriving this, we have used that $du/d\varphi = \dot{u}/\dot{\varphi}$, and we have substituted for $\dot{\varphi}$ from (6.36).

The circular orbit discussed above corresponds, of course, to $du/d\varphi = 0$, and so if we say this occurs at $u = u_0$, with energy given by E_0, we shall have

$$\tilde{\ell}^{-2} = u_0(1-3u_0)\,, \tag{6.46}$$

coming from the condition that $dV/dr = 0$ at $r = r_0 = M/u_0$, and also

$$(1-2u_0)(u_0^2 + \tilde{\ell}^{-2}) = E_0^2\,\tilde{\ell}^{-2}\,, \tag{6.47}$$

coming from (6.45) with $du/d\varphi = 0$. Plugging (6.46) into (6.47), we can rewrite (6.47) as

$$E_0^2 = \frac{(1-2u_0)^2}{1-3u_0}\,. \tag{6.48}$$

Thus we have $\tilde{\ell}$ and E_0 expressed in terms of the rescaled inverse radius u_0 of the circular orbit.

Having established the description of the circular orbit, we now consider an elliptical orbit. A convenient way to describe this is to think of keeping $\tilde{\ell}$ the same, and u_0 the same, but changing to a different energy E. Simple algebra shows that (6.45) can then be rewritten as

$$\Big(\frac{du}{d\varphi}\Big)^2 + (1-6u_0)(u-u_0)^2 - 2(u-u_0)^3 = (E^2 - E_0^2)\,\tilde{\ell}^{-2}\,. \tag{6.49}$$

Written in this way, it is manifest that we revert to the circular orbit with $u = u_0$ if we take the energy to be $E = E_0$.

The equation (6.49) is not easily solved analytically in terms of elementary functions. However, for our purposes it suffices to obtain an approximate solution. To do this we consider a slightly deformed orbit, in which we assume

$$u = u_0\,(1 + \epsilon\, \cos\omega\varphi)\,, \tag{6.50}$$

where $|\epsilon| << 1$. Plugging into (6.49), and working only up to order ϵ^2, we find

$$u_0^2\omega^2\epsilon^2 \sin^2\omega\varphi + (1-6u_0)u_0^2\epsilon^2\, \cos^2\omega\varphi = (E^2 - E_0^2)\tilde{\ell}^{-2}\,. \tag{6.51}$$

Thus our trial solution does indeed work, up to order ϵ^2, if we have

$$\omega^2 = 1 - 6u_0\,, \qquad E^2 = E_0^2 + \tilde{\ell}^2 u_0^2\,(1-6u_0)\,\epsilon^2\,. \tag{6.52}$$

The important equation here is the first one. From the form of the trial solution (6.50), we see that it is like the equation of an ellipse, which would be $u = u_0\,(1+\epsilon\,\cos\varphi)$, except that here to go from one perihelion (i.e. closest approach to the sun) to the next, the φ coordinate should advance through an angle $\Delta\varphi$, where

$$\omega\,\Delta\varphi = 2\pi\,. \tag{6.53}$$

Thus the azimuthal angle should advance by

$$\Delta\varphi = \frac{2\pi}{\sqrt{1-6u_0}}\,. \tag{6.54}$$

If $\Delta\varphi$ had been equal to 2π, the orbit would be a standard ellipse, returning to its perihelion after exactly a 2π rotation. Instead, we have the situation that $\Delta\varphi$ is bigger than 2π, and so the azimuthal angle must advance by a bit more than 2π before the next perihelion. Thus the perihelion advances by an angle $\delta\varphi$ per orbit, where

$$\delta\varphi = \Delta\varphi - 2\pi\,. \tag{6.55}$$

Now, we already noted that for a star such as the sun, the radius at its surface is hugely greater than the Schwarzschild radius for an object of the mass of the sun. Therefore since planetary orbits are certainly outside the sun (!), we have $r_0 >> M$, and so, from (6.43), we have $u_0 << 1$. We can therefore use a binomial approximation for $(1-6u_0)^{-1/2} = 1+3u_0+\cdots$ in (6.54), implying from (6.55) that the advance of the perihelion is approximated by

$$\delta\varphi \approx 6\pi u_0 = \frac{6\pi M}{r_0}\,. \tag{6.56}$$

Clearly the effect will be largest for the planet whose orbital radius r_0 is smallest. This can be understood intuitively since it is experiencing the greatest gravitational attraction (it is deepest in the sun's gravitational potential), and so it experiences the greatest deviation from Newtonian gravity. In our solar system, it is therefore the planet Mercury that will exhibit the largest perihelion advance.

We can easily restore the dimensionful constants G and c in any formula at any time, just by appealing to dimensional analysis, i.e. noting that Newton's constant and the speed of light have dimensions

$$[G] = M^{-1}\,L^3\,T^{-2}\,, \qquad [c] = LT^{-1}\,. \tag{6.57}$$

Thus equation (6.56) becomes

$$\delta\varphi \approx \frac{6\pi GM}{c^2 r_0}\,. \tag{6.58}$$

Putting in the numbers, this amounts to about 43 seconds of arc per century, for the advance of the perihelion of Mercury. Tiny though it is, this prediction has indeed been confirmed by observation, providing a striking vindication for Einstein's theory of general relativity.

6.3.2 Photon orbits, and bending of light by the sun

The motion of a light beam in the Schwarzschild metric is described by a null geodesic, for which we have

$$L = -\tfrac12 B\Big(\frac{dt}{d\lambda}\Big)^2 + \tfrac12 B^{-1}\Big(\frac{dr}{d\lambda}\Big)^2 + \tfrac12 r^2\Big(\frac{d\theta}{d\lambda}\Big)^2 + \tfrac12 r^2\,\sin^2\theta\,\Big(\frac{d\varphi}{d\lambda}\Big)^2\,. \tag{6.59}$$

As before, we can see from the Euler-Lagrange equation for θ that if the photon starts in the $\theta = \frac12\pi$ plane with $d\theta/d\lambda = 0$ initially, it remains in the $\theta = \frac12\pi$ plane for all time, so we can consider the reduced system for motion in the $\theta = \frac12\pi$ plane, described by the Lagrangian

$$L = -\tfrac12 B\Big(\frac{dt}{d\lambda}\Big)^2 + \tfrac12 B^{-1}\Big(\frac{dr}{d\lambda}\Big)^2 + \tfrac12 r^2\Big(\frac{d\varphi}{d\lambda}\Big)^2\,. \tag{6.60}$$

The Euler-Lagrange equations for t and φ, and the equation $L = 0$, then gives the equations

$$\begin{aligned} B\,\frac{dt}{d\lambda} &= E\,, \\ r^2\,\frac{d\varphi}{d\lambda} &= \ell\,, \\ B\Big(\frac{dt}{d\lambda}\Big)^2 - B^{-1}\Big(\frac{dr}{d\lambda}\Big)^2 - r^2\Big(\frac{d\varphi}{d\lambda}\Big)^2 &= 0\,, \end{aligned} \tag{6.61}$$

respectively, where E and ℓ are constants. Susbstituting the first two into the last equation then gives

$$\Big(\frac{dr}{d\lambda}\Big)^2 + \frac{\ell^2}{r^2}\Big(1 - \frac{2M}{r}\Big) = E^2\,. \tag{6.62}$$

The potential $V(r)$ for the one-dimensional problem $(dr/d\lambda)^2 + V(r) = E^2$ is now given by

$$V(r) = \frac{\ell^2}{r^2}\Big(1 - \frac{2M}{r}\Big)\,, \tag{6.63}$$

which can be compared with the potential given in (6.38) for the case of the massive particle. The potential (6.63) has a single stationary point, at

$$r = 3M\,, \tag{6.64}$$

and so this means that there exists a circular photon orbit at precisely this radius. Checking the second derivative there, we have $V''(3M) = -2\ell^2/(81M^4)$, which shows that the orbit is unstable.[15]

We now turn to another of the classic tests of general relativity, where a light beam from a distant star just grazes the surface of the sun, and then is observed here on earth. The apparent direction in which the distant star lies is then compared with where it would have been if the sun were not causing the path of the light beam to be deflected a little. The effect is a small one, so approximations can easily be made to make the problem tractable.

Defining

$$u = \frac{M}{r}\,, \qquad \tilde\ell = \frac{\ell}{M} \tag{6.65}$$

as we did when discussing the geodesics for massive particles, we obtain from the φ equation in (6.61) and from (6.62) that

$$\Big(\frac{du}{d\phi}\Big)^2 + u^2(1 - 2u) = \frac{E^2}{\tilde\ell^2}\,. \tag{6.66}$$

[15] As we already noted, if we are using the Schwarzschild metric to describe the gravitational field outside the sun then it is only valid for radii $r \ge R_{\rm sun}$, where $R_{\rm sun}$ is the radius of the sun. Since $R_{\rm sun} >> 2M$, the photon orbit at $r = 3M$ is not relevant when considering the sun, since it would be deep inside the sun where the Schwarzschild solution is not valid. If we were considering a black hole, on the other hand, then the photon orbit at $r = 3M$ is relevant, since it lies outside the event horizon at $r = 2M$.

Differentiating with respect to φ gives

$$\frac{d^2u}{d\varphi^2} + u = 3u^2\,. \tag{6.67}$$

Assuming that we are in the weak field regime, meaning that $M/r << 1$ and hence $u << 1$, we can treat the right-hand side of (6.61) as a small perturbation to the lowest-order approximation

$$\frac{d^2\bar{u}}{d\varphi^2} + \bar{u} = 0\,, \tag{6.68}$$

whose solution, with a suitable choice of origin for φ, is

$$\bar{u} = A\,\cos\varphi\,. \tag{6.69}$$

Here, the origin for φ has been chosen so that u is a maximum, and hence r is a minimum, at $\varphi = 0$. If we define the distance of closest approach for the light beam to be $r = b$, then it follows that $A = M/b$.

At the next order in a perturbative solution of (6.61) we can plug (6.69) with $A = M/b$ into the right-hand side, thus giving

$$\frac{d^2u}{d\varphi^2} + u = \frac{3M^2}{b^2}\,\cos^2\varphi\,. \tag{6.70}$$

This is easily solved, giving

$$u = \frac{M}{b}\,\cos\varphi + \frac{3M^2}{2b^2} - \frac{M^2}{2b^2}\,\cos 2\varphi\,. \tag{6.71}$$

The first term here is the zeroth-order approximation (6.69), and the remaining terms represent the first sub-leading order in a perturbative expansion for the solution. Since we are assuming the gravitational field is weak even at the point of closest approach, i.e. that $M/b << 1$, the approximate solution (6.71) is quite adequate for our purposes.

For all practical purposes, the light beam from the distant star starts out from $r = \infty$ (almost), heads in to a nearest approach to the sun at $r = b$, and then heads out again to $r = \infty$ (almost) where it is observed on earth. If it weren't for the effects of general relativity, the path of the light beam would just be described by the zeroth-order term in (6.71), i.e. $r(\varphi) = b/\cos\varphi$, with φ going from $\varphi = -\frac{1}{2}\pi$ at the start of the journey to $\varphi = +\frac{1}{2}\pi$ when the beam reaches the earth. This is the path the beam would follow if the sun were not there.

To find the effect of the deflection of light by the sun, we just need to solve the solution (6.71) for the two relevant values of φ for which $u = 0$ (and hence $r = \infty$). These will be at

$$\varphi_{\rm start} = -\tfrac{1}{2}\pi - \epsilon\,, \qquad \varphi_{\rm finish} = \tfrac{1}{2}\pi + \epsilon\,, \tag{6.72}$$

where ϵ is the (small) solution of

$$\frac{M}{b}\cos(\tfrac{1}{2}\pi+\epsilon)+\frac{3M^2}{2b^2}-\frac{M^2}{2b^2}\cos(\pi+2\epsilon)=0\,. \tag{6.73}$$

For small ϵ this gives at first non-trivial order

$$0\approx -\frac{M}{b}\,\epsilon+\frac{3M^2}{2b^2}+\frac{M^2}{2b^2}\,, \tag{6.74}$$

and hence to leading order we have

$$\epsilon=\frac{2M}{b}\,. \tag{6.75}$$

The total angle of deflection of the light beam, relative to when the sun is not there, is therefore given by

$$\delta=(\varphi_{\rm finish}-\varphi_{\rm start})-\pi\,, \tag{6.76}$$

and hence

$$\delta=\frac{4M}{b}\,. \tag{6.77}$$

The angular deflection δ in (6.77) is obviously maximised by taking b as small as possible. Thus, one wants to look at the apparent position in the sky of a star which is just peeking out from behind the sun, and compare its location, relative to stars that have a large angular separation from the sun and are thus much less deflected, with what the relative location is when the sun is not in the field of view. Putting in the numbers for the mass M and radius b of the sun, it turns out that

$$\delta\approx 1.75''\quad\text{(seconds of arc)}\,. \tag{6.78}$$

Of course, looking at stars that are immediately adjacent to the sun in the field of view is not easy! The one time when it can be done is during a total solar eclipse, and this was first attempted by Sir Arthur Eddington in May 1919, in an expedition to observe a total eclipse on an island off the coast of Africa. Within the limits of precision that could be achieved at the time, the observations confirmed the prediction of general relativity. This had a huge impact at the time, propelling Einstein to a level of pop-star recognition by the general public that has only been rivalled since then by one other scientist, Stephen Hawking.

6.3.3 Radar echo delay

From the t equation in (6.61) and the radial equation (6.62), we have

$$\Big(\frac{dr}{dt}\Big)^2=B^2(r)\,\Big[1-\frac{\ell^2\,B(r)}{E^2r^2}\Big]\,, \tag{6.79}$$

where $B(r) = 1 - 2M/r$. Suppose that the planet Mercury happens to be just emerging from behind the sun, as seen from earth, and that a radar pulse is sent from earth, it bounces off Mercury, and is received back on earth. Suppose that the point of nearest approach of the radar beam to the sun is at $r = r_0$. By definition, at this point $dr/dt = 0$, and so we have

$$\frac{\ell^2}{E^2 r_0^2} = \frac{1}{B(r_0)}\,. \tag{6.80}$$

Equation (6.79) can therefore be written as

$$\Big(\frac{dr}{dt}\Big)^2 = B^2(r)\,\Big[1 - \frac{r_0^2}{r^2}\,\frac{B(r)}{B(r_0)}\Big]\,. \tag{6.81}$$

Since we shall be assuming the gravitational field is weak along the entire path of the radar beam we have $M/r << 1$, and so (6.81) can be approximated by expandong (6.81) up to linear order in M, giving

$$\Big(\frac{dr}{dt}\Big)^2 = \Big(1 - \frac{r_0^2}{r^2}\Big)\Big[1 - \frac{4M}{r} - \frac{2M r_0}{(r+r_0)\,r}\Big]\,. \tag{6.82}$$

The time taken for the radar pulse to travel from r_0 to r is then given approximately by

$$\Delta t = \int dt \approx \int_{r_0}^{r} dr'\,\Big(1 - \frac{r_0^2}{r'^2}\Big)^{-1/2}\,\Big[1 + \frac{2M}{r'} + \frac{M r_0}{(r'+r_0)\,r'}\Big]\,. \tag{6.83}$$

The time for this journey if the sun were not there is, of course, just given by the same expression (6.83) but with M set to zero. Thus we find, performing the integrals, that

$$\Delta t = \sqrt{r^2 - r_0^2} + 2M\,\log\Big[\frac{r + \sqrt{r^2 - r_0^2}}{r_0}\Big] + M\,\sqrt{\frac{r - r_0}{r + r_0}}\,. \tag{6.84}$$

The first term is the result when the sun is not there, and the terms proportional to M are the leading-order corrections from general relativity.

If we consider the total round-trip time for the radar pulse, there will be two equal Δt contributions between the earth and the closest approach, and two equal Δt contributions between the closest approach and Mercury. If the earth and Mercury are at distances $r = R_e$ and $r = R_m$ from the sun respectively, we therefore have the total general-relativity induced correction to the total round-trip time of

$$\begin{aligned} \Delta T_{\rm delay} &= 4M\,\log\Big[\frac{R_e + \sqrt{R_e^2 - r_0^2}}{r_0}\Big] + 2M\,\sqrt{\frac{R_e - r_0}{R_e + r_0}}\,,\\ &\quad +4M\,\log\Big[\frac{R_m + \sqrt{R_m^2 - r_0^2}}{r_0}\Big] + 2M\,\sqrt{\frac{R_m - r_0}{R_m + r_0}}\,,\\ &\approx 4M\log\frac{2R_e}{r_0} + 2M + 4M\log\frac{2R_m}{r_0} + 2M\,,\\ &= 4M\,\Big[1 + \log\Big(\frac{4R_e R_m}{r_0^2}\Big)\Big]\,. \end{aligned} \tag{6.85}$$

Putting in the numbers, this gives

$$\Delta T_{\rm delay} \approx 240 \text{ microseconds}\,. \tag{6.86}$$

This is the extra time the round-trip journey for the radar pulse takes when it passes close to sun, as compared with the round-trip time for the same distance if the pulse does not pass close to the sun. Since light travels about 45 miles in 240 microseconds, this means that the orbital motions of the earth and Mercury must be known to within a few miles at any given time, so that a meaningful measurement can be extracted. Many other difficulties arise also, such as the fact that there is no radar reflector placed on Mercury, so the radar echo that is received is coming from a wide spread of surface locations at different distances from the earth. Apparently, nonetheless, the predicted time delay has been confirmed to a precision of order a few percent.

Much more accurate time delay data can now be obtained by using a distant spacecraft with a radio transponder. Experiments using the Cassini spacecraft, which was until recently orbiting Saturn, have achieved accuracies of order 0.002%.

7 Gravitational Action and Matter Couplings

7.1 Derivation of the Einstein equations from an action

It is often useful in physics to be able derive a system of field equations from an action principle. Familiar examples include the derivation of the equations of motion for a mechanical system of particles from an action, and the derivation of the Maxwell equations from an action. In this section, we show how the Einstein equations can also be derived from an action principle. We shall begin by discussing an action for the pure vacuum Einstein equations, and then, in the next section, we shall show how matter can be included too.

As we shall see, an action whose variation yields the pure vacuum Einstein equations is the following:

$$I_{\rm eh} = \frac{1}{16\pi G} \int d^4x \sqrt{-g}\, R\,, \tag{7.1}$$

where G is Newton's constant, g is the determinant of the metric $g_{\mu\nu}$, and R is the Ricci scalar. This is known as the Einstein-Hilbert action. Of course the overall constant round the front of the action is immaterial as far as the pure vacuum equations are concerned, but it will be important later when we couple matter to gravity.

The idea is that to obtain the vacuum Einstein equations, we make an infinitesimal variation of the metric in (7.1) around a solution, and we require that the variation of the action be zero. Recalling the definitions of the Ricci tensor (4.91) and Ricci scalar (4.92), we have

$$R_{\mu\nu} = R^\rho{}_{\mu\rho\nu}\,,\qquad R = g^{\mu\nu}\,R_{\mu\nu}\,,\tag{7.2}$$

where the Riemann tensor is given by (4.65)

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu\Gamma^\rho{}_{\nu\sigma} - \partial_\nu\Gamma^\rho{}_{\mu\sigma} + \Gamma^\rho{}_{\mu\lambda}\,\Gamma^\lambda{}_{\nu\sigma} - \Gamma^\rho{}_{\nu\lambda}\,\Gamma^\lambda{}_{\mu\sigma}\,,\tag{7.3}$$

and the Christoffel connection by (4.48)

$$\Gamma^\mu{}_{\nu\rho} = \tfrac{1}{2}g^{\mu\sigma}\,(\partial_\nu\,g_{\sigma\rho} + \partial_\rho\,g_{\sigma\nu} - \partial_\sigma\,g_{\nu\rho})\,.\tag{7.4}$$

Thus, to vary the metrics used in constructing R, we can go through a sequence of steps:

First, we note that when the metric is varied, the corresponding variation in the Christoffel connection, $\delta\Gamma^\mu{}_{\nu\rho}$, must be a tensor. This can be seen from the transformation rule (4.36) for the Christoffel connection; if we vary the metric so that Γ varies, the transformation rule implies

$$\delta\Gamma'^\nu{}_{\mu\alpha} = \frac{\partial x'^\nu}{\partial x^\sigma}\,\frac{\partial x^\rho}{\partial x'^\mu}\,\frac{\partial x^\lambda}{\partial x'^\alpha}\,\delta\Gamma^\sigma{}_{\rho\lambda}\,.\tag{7.5}$$

Crucially, the inhomogeneous second term in (4.36) has dropped out (because it does not change when the metric is varied), and so we are just left with the homogeneous transformation (7.5), which shows that $\delta\Gamma$ transformans as a general-coordinate $(1,2)$ tensor. (In fact, for the same reason, the *difference* between any two connections transforms as a tensor.)

Now, we look at the Riemann tensor. Making a variation of (7.3) with respect to the metric, we see that there will be two $\partial\delta\Gamma$ terms and four $\Gamma\delta\Gamma$ terms. It is a simple matter to check that the $\Gamma\delta\Gamma$ terms are precisely what is needed in order to covariantise the $\partial\delta\Gamma$ terms, and so in fact

$$\delta R^\rho{}_{\sigma\mu\nu} = \nabla_\mu\delta\Gamma^\rho{}_{\nu\sigma} - \nabla_\nu\delta\Gamma^\rho{}_{\mu\sigma}\,.\tag{7.6}$$

In fact we could see that this must be so, even without doing the calculation in detail. Since we already observed that $\delta\Gamma^\sigma{}_{\rho\lambda}$ is a tensor, it follows that in the expression δRiemann $=$ $\partial\delta\Gamma - \partial\delta\Gamma +$ four $\Gamma\delta\Gamma$ terms, there is no possible tensorial expression that it could give other than (7.6). This is an illustration of the power of tensor analysis; one can often use a "what else could it be" type of argument, based on invoking the known general covariance of an expression, to save a lot of calculation.

Next, we need an expression for $\delta\Gamma^\mu{}_{\nu\rho}$ in terms of variations of the metric. By varying (7.4), we see that there will be terms that are structurally of the form $\mathbf{g}^{-1}\,\partial\delta\mathbf{g}$ and terms of the structural form $(\delta\mathbf{g}^{-1})\,\partial\mathbf{g}$. We know that the resulting expression for $\delta\Gamma^\mu{}_{\nu\rho}$ must be a tensor, and so invoking general covariance, and recalling that $\partial\mathbf{g}$ terms can be written in terms of Γ, can see that the $(\delta\mathbf{g}^{-1})\,\partial\mathbf{g}$ terms must in fact covariantise the partial derivatives in the $\mathbf{g}^{-1}\,\partial\delta\mathbf{g}$ terms, and so the result must be

$$\delta\Gamma^\mu{}_{\nu\rho} = \tfrac{1}{2} g^{\mu\sigma}\left(\nabla_\nu\,\delta g_{\sigma\rho} + \nabla_\rho\,\delta g_{\sigma\nu} - \nabla_\sigma\,\delta g_{\nu\rho}\right). \tag{7.7}$$

It is a straightforward matter to do the pedestrian calculation of verifying this explicitly, and we leave this an an exercise.

Putting all this together, we have

$$\begin{aligned} \delta R &= \delta(g^{\mu\nu}\,R_{\mu\nu}) == (\delta g^{\mu\nu})\,R_{\mu\nu} + g^{\mu\nu}\,\delta R_{\mu\nu} = (\delta g^{\mu\nu})\,R_{\mu\nu} + g^{\mu\nu}\,\delta R^\rho{}_{\mu\rho\nu}\,,\\ &= R_{\mu\nu}\,\delta g^{\mu\nu} + g^{\mu\nu}\left(\nabla_\rho\delta\Gamma^\rho{}_{\nu\mu} - \nabla_\nu\delta\Gamma^\rho{}_{\rho\mu}\right),\\ &= R_{\mu\nu}\,\delta g^{\mu\nu} + \tfrac{1}{2} g^{\mu\nu}\,g^{\rho\sigma}\Big[\nabla_\rho(\nabla_\nu\delta g_{\sigma\mu} + \nabla_\mu\delta g_{\nu\sigma} - \nabla_\sigma\delta g_{\nu\mu})\\ &\qquad\qquad -\nabla_\nu(\nabla_\rho\delta g_{\sigma\mu} + \nabla_\mu\delta g_{\rho\sigma} - \nabla_\sigma\delta g_{\rho\mu})\Big]\,. \end{aligned} \tag{7.8}$$

Recall the matrix identity (4.56), which implies that $\delta g = g\,g^{\mu\nu}\,\delta g_{\mu\nu}$, where g is the determinant of $g_{\mu\nu}$. Note also that $\delta g_{\mu\nu} = -g_{\mu\rho}\,g_{\nu\sigma}\,\delta g^{\rho\sigma}$, which can be seen by varying $g_{\mu\nu}\,g^{\nu\rho} = \delta_\mu^\rho$, noting that the Kronecker delta does not change under the variation. After a little algebra, we then see from (7.8) that

$$\delta R = (R_{\mu\nu} - \nabla_\mu\nabla_\nu + g_{\mu\nu}\,\nabla^\rho\nabla_\rho)\,\delta g^{\mu\nu}\,, \tag{7.9}$$

and so, together with $\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}\,g_{\mu\nu}\,\delta g^{\mu\nu}$, we have

$$\delta(\sqrt{-g}\,R) = \sqrt{-g}\,\Big(R_{\mu\nu} - \tfrac{1}{2}R\,g_{\mu\nu} - \nabla_\mu\nabla_\nu + g_{\mu\nu}\,\nabla^\rho\nabla_\rho)\,\delta g^{\mu\nu}\,. \tag{7.10}$$

We are now nearly ready to prove that applying the principal of stationary action to the Einstein-Hilbert action (7.1) gives the vacuum Einstein equations.

First, we need to make an observation about the divergence theorem in Riemannian and pseudo-Riemannian geometry. If A^μ is a vector field, and if we integrate its divergence over a spacetime volume V whose boundary is S, then we shall have

$$\int_V \sqrt{-g}\,\nabla_\mu A^\mu\,d^4x = \int_V \partial_\mu(\sqrt{-g}\,A^\mu)\,d^4x = \int_S \sqrt{-g}\,A^\mu\,d\Sigma_\mu\,, \tag{7.11}$$

where, in the first equality we have used the result (4.60), which means that $\sqrt{-g}\,\nabla_\mu A^\mu = \partial_\mu(\sqrt{-g}\,A^\mu)$. The second equality then follows from a standard argument one uses to prove the divergence theorem in Cartesian analysis. $d\Sigma_\mu$ is the area element on the 3-dimensional boundary surface.

Considering now the variation of the Einstein-Hilbert action (7.1), we find

$$
\begin{aligned}
\delta I_{\rm eh} &= \frac{1}{16\pi G}\int \delta(\sqrt{-g}\,R)\,d^4x\\
&= \frac{1}{16\pi G}\int \sqrt{-g}\,\Big(R_{\mu\nu}-\tfrac12 R\,g_{\mu\nu}-\nabla_\mu\nabla_\nu+g_{\mu\nu}\,\nabla^\rho\nabla_\rho)\,\delta g^{\mu\nu}\,d^4x\,,\\
&= \frac{1}{16\pi G}\int \sqrt{-g}(R_{\mu\nu}-\tfrac12 R\,g_{\mu\nu})\delta g^{\mu\nu} d^4x\\
&\quad+\frac{1}{16\pi G}\int \sqrt{-g}\,\nabla_\mu(-\nabla_\nu\delta g^{\mu\nu}+g_{\rho\sigma}\,\nabla^\mu\,\delta g^{\rho\sigma})d^4x\,,\\
&= \frac{1}{16\pi G}\int \sqrt{-g}(R_{\mu\nu}-\tfrac12 R\,g_{\mu\nu})\delta g^{\mu\nu} d^4x\\
&\quad+\frac{1}{16\pi G}\int_S \sqrt{-g}\,(-\nabla_\nu\delta g^{\mu\nu}+g_{\rho\sigma}\,\nabla^\mu\,\delta g^{\rho\sigma})d\Sigma_\mu\,.
\end{aligned}
\tag{7.12}
$$

In the standard manner in a variational principle, we assume that the variations $\delta g^{\mu\nu}$ vanish on the boundary surface (at infinity, since the integration is over all of spacetime), and hence the surface integral gives zero. By the standard argument, we then conclude from the requirement of stationarity of the action for otherwise arbitrary $\delta g^{\mu\nu}$ that the cofactor of $\delta g^{\mu\nu}$ in the volume integal must vanish, i.e. that

$$
R_{\mu\nu}-\tfrac12 R\,g_{\mu\nu}=0\,. \tag{7.13}
$$

This is precisely the Einstein equation (6.10) in the case that the matter energy-momentum tensor $T_{\mu\nu}$ is assumed to be zero.

A small modification that one can make to the Einstein-Hilbert action is the inclusion of the cosmological constant. If we consider now the action

$$
I_{\rm ehc}=\frac{1}{16\pi G}\int \sqrt{-g}\,(R-2\Lambda)\,d^4x\,, \tag{7.14}
$$

where Λ is a constant, then using $\delta\sqrt{-g}=-\frac12\sqrt{-g}\,g_{\mu\nu}\,\delta g^{\mu\nu}$ we see that instead of (7.13) we now have

$$
R_{\mu\nu}-\tfrac12 R\,g_{\mu\nu}+\Lambda\,g_{\mu\nu}=0\,. \tag{7.15}
$$

Note that by taking the trace of this equation (i.e. contracting with $g^{\mu\nu}$) we get $-R+4\Lambda=0$, and plugging back into (7.15) then gives

$$
R_{\mu\nu}=\Lambda\,g_{\mu\nu}\,. \tag{7.16}
$$

As we had mentioned previously, metrics that satisfy this equation are known as Einstein metrics. As is well known, having introduced the cosmological constant Einstein later regretted it, calling it "the greatest blunder of my life." In retrospect, introducing it was actually a smart thing to do!

7.2 Coupling of the electromagnetic field to gravity

We reviewed the four-dimensional description of the Maxwell equations in special relativity earlier on. The equations in Minkowski spacetime are given in (2.62) and (2.63). Generalising these equations to an arbitrary curved spacetime background is very simple. We can follow the same technique we used earlier for deriving the parallel transport equation for a vector, and for deriving the geodesic equation. Namely, we first consider the Maxwell equations in Minkowski spacetime written in an arbitrary coordinate system. It is easy to see that the the partial derivative in the Maxwell field equation (2.62) becomes the covariant derivative, with the connection given by the the usual expression (3.9) that we derived in Minkowski spacetime. The extension to a general curved spacetime is then merely a matter of allowing the metric to be arbitrary, with the connection taken to be the Christoffel connection (4.48). The Bianchi identity (2.63) generalises even more easily. Writing it for Minkowski spacetime in an arbitrary coordinate system will cause the partial derivative in each of the three terms to be replaced by a covariant derivative, and again this immediately extends to the case of an arbitrary metric, as for the Maxwell field equation. But in fact, it is even simpler than this; one can easily verify that in fact all the connection terms cancel out in pairs, because the Christoffel connection is symmetric in its lower two indices. (We discussed the example of the curl of a co-vector earlier, in section 4.4, where we saw that $\nabla_{[\mu}V_{\nu]} = \partial_{[\mu}V_{\nu]}$. The same thing happens for the curl (i.e. totally antisymmetrised derivative) of any totally-antisymmetric $(0, q)$ tensor $W_{\mu_1\cdots\mu_q}$, i.e. $\nabla_{[\mu}W_{\nu_1\cdots\nu_q]} = \partial_{[\mu}W_{\nu_1\cdots\nu_q]}$.[16]) Thus in summary, the Maxwell equations in a general curved spacetime background are

$$\nabla_\mu F^{\mu\nu} = -4\pi J^\nu , \tag{7.17}$$

and

$$\partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} = 0 . \tag{7.18}$$

It should be remarked here that the process we have described for generalising Lorentz-covariant tensor equations in special relativity to generally-covariant equations in general relativity it a rather universal one. Essentially, we just replace all partial derivatives by covariant derivatives. (If it happens, as in the Bianchi identity, that the connection terms cancel out, then that is an added bonus.) In terms of the notation we introduced previously, where a partial derivative $\partial_\mu V_\nu$ was denote by a comma, $V_{\nu,\mu}$, and a covariant derivative $\nabla_\mu V_\nu$ by a semicolon, $V_{\nu;\mu}$, the rule for going from special to general relativity is sometimes

[16] Note that because of the antisymmetry of $F_{\mu\nu}$, the terms $\partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu}$ in the Bianchi identity can be written as $3\partial_{[\mu}F_{\nu\rho]}$.

known as the "comma goes to semicolon rule." To be more precise, the rule gives what is sometimes referred to as the "minimal coupling" of the theory (such as Maxwell electrodynamics) to gravity. One could imagine other more complicated covariantisations, in which, for example, higher-order terms involving the curvature arise too. We shall say it bit more about such possibilities later.

The Maxwell field equations (7.17) can be derived from a action principle, just as they can in Minkowski spacetime (see my E&M611 notes on my webpage). To do this, we first note that we can solve the Bianchi identity (7.18), just as in Minkowski spacetime, by writing $F_{\mu\nu}$ as the curl of a 4-vector potential:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \,. \tag{7.19}$$

Of course this itself is covariant, as we discussed earlier. We now consider the action

$$I_{\rm max} = -\frac{1}{16\pi} \int \sqrt{-g}\, F^{\mu\nu}\, F_{\mu\nu}\, d^4x \,, \tag{7.20}$$

where it is understood that A_μ is being treated as the fundamental field variable, with $F_{\mu\nu}$ then given by (7.19).

Varying with respect to A_ν gives

$$\begin{aligned} \delta I_{\rm max} &= -\frac{1}{8\pi} \int \sqrt{-g}\, F^{\mu\nu}\, \delta F_{\mu\nu}\, d^4x = -\frac{1}{8\pi} \int \sqrt{-g}\, F^{\mu\nu}\, (\partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu)\, d^4x \,, \\ &= -\frac{1}{4\pi} \int \sqrt{-g}\, F^{\mu\nu}\, \partial_\mu \delta A_\nu\, d^4x \,, \\ &= \frac{1}{4\pi} \int \Big[- \partial_\mu(\sqrt{-g}\, F^{\mu\nu}\, \delta A_\nu) + \partial_\mu(\sqrt{-g}\, F^{\mu\nu})\, \delta A_\nu \Big]\, d^4x \,. \end{aligned} \tag{7.21}$$

The first term on the last line can be turned into a surface integral using the divergence theorem. We take the original spacetime volume integral to be over a all of space, between an initial time t_i and a final time t_f. The surface integral therefore comprises a "cylinder" with endcaps at $t = t_i$ and $t = t_f$, on which by assumption δA_ν vanishes, and the sides of the cyclinder represent the "sphere at spatial infinity," and we assume the fields are zero there, by imposing appropriate fall-off conditions. Thus, as usual in a variational action principle we can drop the surface term. The remaining volume integral in the last line of (7.21) is assumed, under the variational principle, to vanish for all possible δA_ν, and hence we deduce

$$\partial_\mu(\sqrt{-g}\, F^{\mu\nu}) = 0 \,. \tag{7.22}$$

As we saw earlier when discussing the divergence operator (see eqn (4.60) and (4.61)), We can rewrite (7.22) in terms of the covariant derivative, as

$$\nabla_\mu F^{\mu\nu} = 0 \,. \tag{7.23}$$

This is precisely the Maxwell field equation (7.17) in the absence of any source terms. Sources, such as currents due to moving charges, could easily be added if desired.

This discussion of the Maxwell equations has up until now been in an unspecified gravitational background. We can now make the system of Maxwell fields in a gravitational background self-contained and dynamical, by allowing the Maxwell fields to become the source for gravity itself. We can achieve this by simply adding the Maxwell action $I_{\rm max}$ to the Einstein-Hilbert action $I_{\rm eh}$ for gravity (7.1), which we discussed earlier. Thus we consider the Einstein-Maxwell action

$$I = I_{\rm eh} + I_{\rm max} = \frac{1}{16\pi} \int \sqrt{-g}\,(R - F^2)\, d^4x\,, \tag{7.24}$$

where F^2 means $F^{\mu\nu}\,F_{\mu\nu}$. Note that here, and from now onwards unless specified to the contrary, we are choosing units for our measurements of mass and length such that Newton's constant G is set equal to 1.[17]

Varying the Einstein-Maxwell action with respect to A_ν and requiring $\delta I = 0$ continues to give the same source-free Maxwell equation (7.23) we obtained above, since A_ν does not appear in the Einstein-Hilbert term in the total action. Now consider what happens when we vary the Einstein-Maxwell action with respect to the metric. We already know the answer for the Einstein-Hilbert term; it is given in the first term in the last equality in eqn (7.12). Concentrating on the contribution from the Maxwell action, and remembering that

$$\delta\sqrt{-g} = -\tfrac{1}{2}\sqrt{-g}\, g_{\mu\nu}\,\delta g^{\mu\nu}\,, \tag{7.25}$$

we see that

$$\begin{aligned}
\delta I_{\rm max} &= -\frac{1}{16\pi}\int \delta(\sqrt{-g}\,F_{\mu\rho}\,F_{\nu\sigma}\,g^{\mu\nu}\,g^{\rho\sigma})\,d^4x\,,\\
&= -\frac{1}{16\pi}\int \sqrt{-g}\,(2F_{\mu\rho}\,F_{\nu\sigma}\,g^{\rho\sigma}\,\delta g^{\mu\nu} - \tfrac{1}{2}F^2\,g_{\mu\nu}\,\delta g^{\mu\nu})\,d^4x\,,\\
&= -\frac{1}{2}\int \sqrt{-g}\,T_{\mu\nu}\,\delta g^{\mu\nu}\,d^4x\,,
\end{aligned} \tag{7.26}$$

[17] As with all the dimensionful quantities like the speed of light, Newton's constant, Planck's constant, and so on, their common description as "fundamental constants of nature" is a bit of a misnomer. Seen from a different viewpoint they are merely the constants of proportionality that arise from our arbitrary choices of systems of units for time, length, mass, and so on. Indeed, even in the SI system there is no longer the concept of the speed of light as a fundamental constant of nature, since the metre is *defined* to be the distance travelled by light in 1/299,792,458 of a second. It is no longer meaningful, within the SI system, to "measure the speed of light." In the "natural units" that we are using, where $c = G = 1$, length, mass and time all have the same units.

where

$$T_{\mu\nu} = \frac{1}{4\pi}\,(F_{\mu\rho}\,F_\nu{}^\rho - \tfrac14 F^2\, g_{\mu\nu}) \tag{7.27}$$

is the energy-momentum tensor for the Maxwell field. (See eqn (2.86) for the energy-momentum tensor in the context of special relativity.) One can easily verify that (7.27) is covariantly conserved, $\nabla^\mu T_{\mu\nu} = 0$, by virtue of the source-free Maxwell equations (7.23).

Combining the contributions (7.12) and (7.26) to the variation of the Einstein-Maxwell action, we therefore arrive at the Einstein equations

$$R_{\mu\nu} - \tfrac12 R\, g_{\mu\nu} = 8\pi T_{\mu\nu} = 2(F_{\mu\rho}\,F_\nu{}^\rho - \tfrac14 F^2\, g_{\mu\nu}) \tag{7.28}$$

for the Einstein-Maxwell system. (Recall we have set $G = 1$ now.) Thus we have the source-free Maxwell equation (7.23), which incorporates the effects of the curved gravitational background on the Maxwell field. And we also have the Einstein equation (7.28), which incorporates the effects of the back-reaction of the Maxwell fields on the curvature of the spacetime in which they are propagating.

7.3 Tensor densities, and the invariant volume element

We may also consider more general couplings of other matter systems to gravity. Before doing so, it is useful to address a couple of more formal topics, which will be important for the discussion of matter couplings, and also more generally. The first topic concerns the definition of what are known as *tensor densities*. We already gave a discussion of general-coordinate tensors in section 4, with a (p,q) tensor transforming according to the rule (4.20). In particular, a $(0,0)$ tensor, i.e. a scalar field, has no $\partial x/\partial x'$ or $\partial x'/\partial x$ factors at all; it is invariant under general coordinate transformations. However, we have also met an object which, despite having no indices, is not in fact a scalar field but rather, it has a very specific transformation rule. This object is the g, the determinant of the metric tensor $g_{\mu\nu}$.

We know that $g_{\mu\nu}$ transforms according to

$$g'_{\mu\nu} = \frac{\partial x^\rho}{\partial x'^\mu}\,\frac{\partial x^\sigma}{\partial x'^\nu}\, g_{\rho\sigma}\,. \tag{7.29}$$

Taking the determinant of this equation therefore gives

$$g' = \Big|\frac{\partial x}{\partial x'}\Big|^2\, g\,, \qquad \text{where} \quad \Big|\frac{\partial x}{\partial x'}\Big| \equiv \det\Big(\frac{\partial x^\mu}{\partial x'^\nu}\Big)\,. \tag{7.30}$$

Here $|\partial x/\partial x'| = |\partial x'/\partial x|^{-1}$, where $|\partial x'/\partial x|$ is the Jacobian of the transformation from the unprimed to the primed coordinates. The quantity g is called a scalar density of weight

-2. More generally, and object H with components $H^{\mu_1\cdots\mu_p}{}_{\nu_1\cdots\nu_q}$ is called a (p,q) tensor density of weight w if it transforms according to the rule

$$H'^{\mu_1\cdots\mu_p}{}_{\nu_1\cdots\nu_q} = \Big|\frac{\partial x'}{\partial x}\Big|^w \frac{\partial x'^{\mu_1}}{\partial x^{\rho_1}} \cdots \frac{\partial x'^{\mu_p}}{\partial x^{\rho_p}} \frac{\partial x^{\sigma_1}}{\partial x'^{\nu_1}} \cdots \frac{\partial x^{\sigma_q}}{\partial x'^{\nu_q}} H^{\rho_1\cdots\rho_p}{}_{\sigma_1\cdots\sigma_q}\,. \tag{7.31}$$

In the previous subsections, when we wrote down the Einstein-Hilbert action (7.1) and the Maxwell action (7.20), we inserted a $\sqrt{-g}$ factor in the integrand. Beside the fact that it was needed in order to get the right equations of motion, it also served another very important role, which until now we have not commented upon. Namely, it ensured that the action itself was properly invariant under general coordinate transformations. To see this, we note that under a change of coordinates the "volume element" d^4x transforms in the standard way, namely with a Jacobian factor such that

$$d^4x' = \Big|\frac{\partial x'}{\partial x}\Big|\, d^4x\,. \tag{7.32}$$

Since g transforms according to (7.30), it follows that $\sqrt{-g}\, d^4x$ is invariant under general coordinate transformations,

$$\sqrt{-g'}\, d^4x' = \sqrt{-g}\, d^4x\,. \tag{7.33}$$

Since the Ricci scalar R is a scalar, and since $F^{\mu\nu}\, F_{\mu\nu}$ is a scalar, we see that in consequence the Einstein-Hilbert action and the Maxwell action are indeed genuine general-coordinate scalars. We should think of $\sqrt{-g}\, d^4x$ as being the invariant spacetime volume element.

An important tensor density is the alternating symbol $\varepsilon_{\mu\nu\rho\sigma}$, which is defined *in all coordinate frames* by the properties that

$$\begin{aligned} &(i) \qquad && \varepsilon_{\mu\nu\rho\sigma} = \varepsilon_{[\mu\nu\rho\sigma]}\,,\\ &(ii) \qquad && \varepsilon_{0123} = +1\,. \end{aligned} \tag{7.34}$$

(Note that we are using a script epsilon ε to denote this object. Shortly, we shall introduce another epsilon object, denoted by a non-script ϵ; it is important to distinguish the one from the other.) The first property states that $\varepsilon_{\mu\nu\rho\sigma}$ is totally antisymetric. This means that there is only one independent component, and this is then specified by property (ii). (Of course, other people may use the opposite convention, in which $\varepsilon_{0123} = -1$.) It is the natural four-dimensional generalisation of the 3-index epsilon tensor of three-dimensional Cartesian tensor analysis. The further generalisation to n dimensions is immediate. Using a basic result from linear algebra, that[18]

$$M_{\mu_1}{}^{\nu_1}\, M_{\mu_2}{}^{\nu_2}\, M_{\mu_3}{}^{\nu_3}\, M_{\mu_4}{}^{\nu_4}\, \varepsilon_{\nu_1\nu_2\nu_3\nu_4} = (\det M)\, \varepsilon_{\mu_1\mu_2\mu_3\mu_4}\,, \tag{7.35}$$

Thus we see that

$$\frac{\partial x^{\nu_1}}{\partial x'^{\mu_1}}\, \frac{\partial x^{\nu_2}}{\partial x'^{\mu_2}}\, \frac{\partial x^{\nu_3}}{\partial x'^{\mu_3}}\, \frac{\partial x^{\nu_4}}{\partial x'^{\mu_4}}\, \varepsilon_{\nu_1\nu_2\nu_3\nu_4} = \Big|\frac{\partial x'}{\partial x}\Big|^{-1}\, \varepsilon_{\mu_1\mu_2\mu_3\mu_4}\,, \tag{7.36}$$

which, comparing with (7.31), shows that $\varepsilon_{\mu_1\mu_2\mu_3\mu_4}$ as defined (in all frames) is an *invariant* tensor density of weight 1. It follows that we can then define the *Levi-Civita tensor*

$$\epsilon_{\mu\nu\rho\sigma} \equiv \sqrt{-g}\, \varepsilon_{\mu\nu\rho\sigma}\,, \tag{7.37}$$

which transforms as a genuine tensor. It is an invariant tensor, in the sense that $\epsilon'_{\mu\nu\rho\sigma} = \epsilon_{\mu\nu\rho\sigma}$.

7.4 Lie derivative and infinitesimal diffeomorphisms

We saw previously that the variation of the Einstein-Hilbert action with respect to the metric tensor produced the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu}$, which is conserved, $\nabla^\mu G_{\mu\nu} = 0$. We also saw that the variation of the Maxwell action with respect to the metric tensor produced the energy-monentum tensor $T_{\mu\nu}$ given by (7.27), which is also conserved, $\nabla^\mu T_{\mu\nu} = 0$. It is no coincidence that both of these variations produced conserved tensors. The underlying reason for it is related to the observation we made above, namely that in each case the action is a general-coordinate scalar. We can in fact give a nice general proof that if we vary *any* scalar action with respect to the metric, it will *always* give rise to a conserved tensor. In order to show this, we now need to introduce the notion of the *Lie derivative* of a tensor field.

To introduce the Lie derivative, we need to think a little carefully about what we mean by the general coordinate transformation properties of a field. We can start with a humble scalar field. When we say it is invariant under general coordinate transformations, and we write $\phi' = \phi$ (i.e. eqn (4.20) in the special case of a $(0,0)$ tensor), what we actually mean is that

$$\phi'(x') = \phi(x)\,. \tag{7.38}$$

[18]This can be proved rather mechanically, by first noting that the left-hand side is obviously totally antisymmetric in μ_1, μ_2, μ_3 and μ_4, which means that only one non-vanishing special case needs to be checked, and then taking, for example, $\mu_1 = 0$, $\mu_2 = 1$, $\mu_3 = 2$ and $\mu_4 = 3$ in order to verify the identity. It is instructive, and simpler, to check the analogous, simpler, examples of $n = 2$ and $n = 3$ dimensions first.

(Of course here, when we write x it is standing for all of the coordinates x^μ, and likewise for x'.) General-coordnate transformations are also sometimes called *diffeomorphisms*. Consider now an infinitesimal diffeomorphism, with

$$x'^\mu = x^\mu - \xi^\mu(x)\,. \tag{7.39}$$

We may now calculate the infinitesimal change $\delta\phi(x)$, which is *by definition*

$$\delta\phi(x) \equiv \phi'(x) - \phi(x)\,. \tag{7.40}$$

Now from (7.39) and using Taylor's theorem, we have

$$\begin{aligned} \phi'(x') &= \phi'(x) - \xi^\nu\,\partial_\nu\phi'(x) + \cdots\,, \\ &= \phi'(x) - \xi^\nu\,\partial_\nu\phi(x) + \cdots\,, \end{aligned} \tag{7.41}$$

where in getting to the second line we can drop the prime on $\phi'(x)$ in the second term, since $\phi'(x)$ and $\phi(x)$ differ only infinitesimally, and the prefactor ξ^ν in that term is already infinitesimal. Thus from the epxression in the second line, together with (7.38), we see from (7.40) that

$$\delta\phi(x) = \xi^\nu\,\partial_\nu\phi(x)\,. \tag{7.42}$$

Now consider the analogous calculation for the infinitesimal diffeomorphism of a vector field, whose general-coordinate transformation is

$$V'^\mu(x') = \frac{\partial x'^\mu}{\partial x^\nu}\,V^\nu(x)\,. \tag{7.43}$$

From (7.39) we have

$$\frac{\partial x'^\mu}{\partial x^\nu} = \delta^\mu_\nu - \partial_\nu\xi^\mu\,, \tag{7.44}$$

and so (7.43) implies

$$\begin{aligned} V'^\mu(x') &= (\delta^\mu_\nu - \partial_\nu\xi^\mu)\,V^\nu(x)\,, \\ &= V^\mu(x) - (\partial_\nu\xi^\mu)\,V^\nu(x)\,. \end{aligned} \tag{7.45}$$

As in the scalar case, we now use Taylor's theorem and (7.39) to relate $V'^\mu(x')$ to $V'^\mu(x)$:

$$\begin{aligned} V'^\mu(x') &= V'^\mu(x) - \xi^\nu\,\partial_\nu V'^\mu(x) + \cdots\,, \\ &= V'^\mu(x) - \xi^\nu\,\partial_\nu V^\mu(x) + \cdots\,, \end{aligned} \tag{7.46}$$

Thus we find that the infinitesimal variation defined by

$$\delta V^\mu(x) \equiv V'^\mu(x) - V^\mu(x) \tag{7.47}$$

is given by

$$\delta V^\mu = \xi^\nu\, \partial_\nu V^\mu - V^\nu\, \partial_\nu \xi^\mu\,. \tag{7.48}$$

We define the right-hand side here to be the *Lie derivative* of the vector V with resepct to the vector ξ. It is written as $\delta V^\mu = \mathcal{L}_\xi\, V^\mu$, where

$$\mathcal{L}_\xi V^\mu = \xi^\nu\, \partial_\nu V^\mu - V^\nu\, \partial_\nu \xi^\mu\,. \tag{7.49}$$

Note that the Lie derivative of the vector field V with respect to the vector field ξ is in fact expressible simply as the commutator of the vector fields:

$$\mathcal{L}_\xi\, V = [\xi, V]\,. \tag{7.50}$$

In other words, we have

$$\begin{aligned} \mathcal{L}_\xi\, V &= \mathcal{L}_\xi\, V^\mu\, \partial_\mu \\ &= \xi^\nu \partial_\nu V^\mu\, \partial_\mu - V^\nu \partial_\nu \xi^\mu\, \partial_\mu \\ &= [\xi^\mu\, \partial_\mu, V^\nu\, \partial_\nu]\,, \end{aligned} \tag{7.51}$$

which indeed implies (7.50).

The result we derived for the infinitesimal diffeomorphism of the scalar field ϕ in (7.42 can also be written as $\delta\phi = \mathcal{L}_\xi\, \phi$, where the Lie derivative of ϕ with respect to ξ is simply given by

$$\mathcal{L}_\xi\, \phi = \xi^\nu \partial_\nu\, \phi\,. \tag{7.52}$$

Finally, if we carry out the analogous calculation for a co-vector field U_μ, whose transformation rule is

$$U'_\mu(x') = \frac{\partial x^\nu}{\partial x'^\mu}\, U_\nu(x)\,, \tag{7.53}$$

for which we need to observe from (7.44) that up to first order in ξ we shall have

$$\frac{\partial x^\nu}{\partial x'^\mu} = \delta^\nu_\mu + \partial_\mu \xi^\nu\,, \tag{7.54}$$

then the outcome will be that $\delta U_\mu(x) \equiv U'_\mu(x) - U_\mu(x)$ is given by $\delta U_\mu = \mathcal{L}_\xi\, U_\mu$, where the Lie derivative of a co-vector with respect to the vector ξ is given by

$$\mathcal{L}_\xi\, U_\mu = \xi^\nu\, \partial_\nu U_\mu + U_\nu\, \partial_\mu \xi^\nu\,. \tag{7.55}$$

The calculation is now easily extended to an arbitrary (p,q) tensor T. Under the infinitesimal diffeomorphism one finds $\delta T^{\mu_1\cdots\mu_p}{}_{\nu_1\cdots\nu_q} = \mathcal{L}_\xi\, T^{\mu_1\cdots\mu_p}{}_{\nu_1\cdots\nu_q}$, where the Lie derivative is defined by

$$\begin{aligned} \mathcal{L}_\xi\, T^{\mu_1\cdots\mu_p}{}_{\nu_1\cdots\nu_q} &= \xi^\rho\, \partial_\rho T^{\mu_1\cdots\mu_p}{}_{\nu_1\cdots\nu_q} - T^{\rho\mu_2\cdots\mu_p}{}_{\nu_1\cdots\nu_q}\, \partial_\rho \xi^{\mu_1} - \cdots T^{\mu_1\mu_2\cdots\rho}{}_{\nu_1\cdots\nu_q}\, \partial_\rho \xi^{\mu_p} \\ &\quad + T^{\mu_1\cdots\mu_p}{}_{\rho\nu_2\cdots\nu_q}\, \partial_{\nu_1} \xi^\rho + \cdots T^{\mu_1\cdots\mu_p}{}_{\nu_1\nu_2\cdots\rho}\, \partial_{\nu_q} \xi^\rho\,. \end{aligned} \tag{7.56}$$

The first term, sometimes called the "transport term," is present for any (p, q) tensor, even a scalar field. There is then a term of the form of the second term in (7.49) for each upstairs index, and a term of the form of the second term in (7.55) for each downstairs index.

Note that although we introduced the notion of the Lie derivative as the differential operator that describes the variation of a tensor field under an infinitesimal general coordinate transformation, it in fact has a much wider applicability. Another point to notice is that although it does not look manifestly covariant in (7.49), (7.55) or (7.56), it *is* in fact covariant with respect to general coodinate transformations. Thus the right-hand side in (7.56) is in fact a (p, q) general-coordinate tensor. One can check this by replacing all the partial derivatives by covariant derivatives, thus giving an expression that *is* manifestly a (p, q) tensor, and then verifying that all the Christoffel connection terms in fact cancel out. We leave this as an exercise for the reader.[19]

An important example of an infinitesimal diffeorophism, which we shall need shortly, is the transformation of the metric tensor. Specialising (7.56) to this case, we therefore have

$$\delta g_{\mu\nu} = \xi^\rho\, \partial_\rho g_{\mu\nu} + g_{\rho\nu}\, \partial_\mu \xi^\rho + g_{\mu\rho}\, \partial_\nu \xi^\rho\,. \tag{7.57}$$

As we remarked above, it is easy to verify that we can replace the partial derivatives by covariant derivatives, and so

$$\begin{aligned} \delta g_{\mu\nu} &= \xi^\rho\, \nabla_\rho g_{\mu\nu} + g_{\rho\nu}\, \nabla_\mu \xi^\rho + g_{\mu\rho}\, \nabla_\nu \xi^\rho\,, \\ &= \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu\,. \end{aligned} \tag{7.58}$$

where, in getting to the second line, we have used the fact that $g_{\mu\nu}$ is covariantly constant.

7.5 General matter action, and conservation of $T_{\mu\nu}$

Now let us consider a matter field, or more generally a system of matter fields, described by an action $I_{\rm mat}$. The action will be required to be a general-coordinate scalar, and it may be written schematically as

$$I_{\rm mat} = \int \mathbf{L}(g_{\mu\nu}, \Phi)\,, \tag{7.59}$$

Here, Φ represents the matter field (or fields). In the example we already considered, of the Maxwell field, we had

$$\mathbf{L}(g_{\mu\nu}, A_\mu) = -\frac{1}{16\pi}\, \sqrt{-g}\, F_{\mu\nu}\, F_{\rho\sigma}\, g^{\mu\rho}\, g^{\nu\sigma}\, d^4x\,, \tag{7.60}$$

[19] It was in fact guaranteed from the way we constructed the Lie derivative that it *must* map a tensor to another tensor, but it is sometimes good to check things like this explicitly.

where $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$. In the electromagnetic example we saw that under a variation of the action with respect to the metric we had

$$\delta_g I_{\rm max} = -\tfrac{1}{2} \int \sqrt{-g}\, T_{\mu\nu}\, \delta g^{\mu\nu}\, d^4x = \tfrac{1}{2} \int \sqrt{-g}\, T^{\mu\nu}\, \delta g_{\mu\nu}\, d^4x\,, \tag{7.61}$$

where $T_{\mu\nu}$ is the energy-momentum tensor, given by (7.27) in the Maxwell example. (The symbol δ_g here denotes that a variation is made just with respect to the metric $g_{\mu\nu}$.) For an arbitrary matter system we define its energy-momentum tensor by the analogous variational formula:[20]

$$\delta_g I_{\rm mat} = -\tfrac{1}{2} \int \sqrt{-g}\, T_{\mu\nu}\, \delta g^{\mu\nu}\, d^4x = \tfrac{1}{2} \int \sqrt{-g}\, T^{\mu\nu}\, \delta g_{\mu\nu}\, d^4x\,, \tag{7.62}$$

We now consider making an infinitesimal diffeomorphism, parameterised by the vector field ξ^μ. Since $I_{\rm mat}$ is a scalar, and furthermore it is independent of x (since the coordinates have been integrated out), it must be that $\delta I_{\rm mat} = 0$, where δ here denotes a variation of all the fields (metric and matter) under the infinitesimal diffeomorphism. Thus from (7.59) we have

$$0 = \delta I_{\rm mat} = \int \frac{\delta \mathbf{L}}{\delta g_{\mu\nu}}\, \delta g_{\mu\nu} + \int \frac{\delta \mathbf{L}}{\delta \Phi}\, \delta\Phi\,. \tag{7.63}$$

Note that Φ can represent any kind of matter field, or a set of matter fields. We use an implicit summation convention over all the fields, and over whatever spacetime indices the fields may carry, when we write $\frac{\delta \mathbf{L}}{\delta\Phi}\, \delta\Phi$.

Now, the crucial point is that the second term on the right-hand side will vanish by virtue of the field equations that the matter field(s) satisfy. Also, in view of (7.62) the first term can be written in terms of $T_{\mu\nu}$, so we shall have

$$0 = \tfrac{1}{2} \int \sqrt{-g}\, T^{\mu\nu}\, \delta g_{\mu\nu}\, d^4x\,. \tag{7.64}$$

It must be emphasised that here $\delta g_{\mu\nu}$ is specifically an infinitesimal diffeomorphism transformation as given by (7.58); it is *not* an arbitrary variation of the metric. Substituting (7.58) into this, we therefore find

$$\int \sqrt{-g}\, T^{\mu\nu}\, \nabla_\mu \xi_\nu\, d^4x = 0\,. \tag{7.65}$$

Integrating by parts by using the divergence theorem, and under the assumption that the surface term drops out because the fields are assumed to vanish at infinity, we therefore have

$$\int \sqrt{-g}\, (\nabla_\mu T^{\mu\nu})\, \xi_\nu\, d^4x = 0\,. \tag{7.66}$$

[20]Recall that since $g_{\mu\nu}\, g^{\nu\rho} = \delta^\rho_\mu$, it follows by varying this that we shall have $\delta g_{\mu\nu} = -g_{\mu\rho}\, g_{\nu\sigma}\, \delta g^{\rho\sigma}$.

Since this is true for an arbitrary diffeomorphism parameter ξ_ν, it therefore follows that

$$\nabla_\mu T^{\mu\nu} = 0\,. \tag{7.67}$$

Thus we have concluded that the energy-momentum tensor for an arbitrary matter system that is derived from a diffeomorphism-invariant action is covariantly conserved. The conservation holds by virtue of the fact that the matter field(s) satisfy their equations of motion. We saw this explictly earlier, in the example of the electromagnetic field. Another simple example of a matter action is to consider a scalar field of mass m, satisfying the Klein-Gordon equation

$$-\Box\phi + m^2\phi = 0\,,\qquad \text{where }\ \Box\phi \equiv \nabla^\mu\nabla_\mu\phi \tag{7.68}$$

This can be derived from the matter action

$$I_{\rm mat} = \frac{1}{16\pi}\int \sqrt{-g}\,\Big[-\tfrac12(\partial\phi)^2 - \tfrac12 m^2\phi^2\Big]\,d^4x\,, \text{where }\ (\partial\phi)^2 \equiv g^{\mu\nu}\,\partial_\mu\phi\,\partial_\nu\phi\,. \tag{7.69}$$

Varying with respect to ϕ, and dropping the boundary term in the necessary integration by parts in the usual way, we have:

$$\begin{aligned}\delta I_{\rm mat} &= \frac{1}{16\pi}\int \sqrt{-g}\,\Big[-\partial^\mu\phi\,\partial_\mu\delta\phi - m^2\phi\,\delta\phi\Big]\,d^4x\,,\\ &= \frac{1}{16\pi}\int \sqrt{-g}\,\Big[\nabla^\mu\partial_\mu\phi - m^2\phi\Big]\,\delta\phi\,d^4x\,,\end{aligned} \tag{7.70}$$

and so requiring $\delta I_{\rm mat}=0$ for all possible $\delta\phi$ then indeed implies the Klein-Gordon equation (7.68).

Now, we calculate the energy-momentum tensor for the scalar field by varying the action with respect to the metric and using (7.62). Thus we have[21]

$$\begin{aligned}\delta I_{\rm mat} &= \frac{1}{16\pi}\int \Big(\sqrt{-g}\,[-\tfrac12\delta g^{\mu\nu}\,\partial_\mu\phi\,\partial_\nu\phi] + (\delta\sqrt{-g})[-\tfrac12(\partial\phi)^2 - \tfrac12 m^2\phi^2]\Big)\,d^4x\,,\\ &= \frac{1}{16\pi}\int \sqrt{-g}\Big[-\tfrac12\partial_\mu\phi\,\partial_\nu\phi - \tfrac12[-\tfrac12(\partial\phi)^2 - \tfrac12 m^2\phi^2]\,g_{\mu\nu}\Big]\,\delta g^{\mu\nu}\,d^4x\,,\end{aligned} \tag{7.71}$$

from which it follows, using (7.62), that

$$T_{\mu\nu} = \frac{1}{16\pi}\,\Big[\partial_\mu\phi\,\partial_\nu\phi - \tfrac12(\partial\phi)^2\,g_{\mu\nu} - \tfrac12 m^2\phi^2\,g_{\mu\nu}\Big]\,. \tag{7.72}$$

One can easily verify that this is indeed covariantly conserved, i.e. $\nabla^\mu T_{\mu\nu}=0$, by virtue of the fact that ϕ satisfies the Klein-Gordon equation (7.68).

[21] It should always be clear from the context what one is varying an action with respect to. Previously, in (7.70), we varied $I_{\rm mat}$ with respect to ϕ. Here, instead, we are varying it with respect to $g^{\mu\nu}$. In an earlier discussion, we considered the variation of an action with respect to a diffeomorphism.

7.6 Killing vectors

We saw earlier that under an infinitesimal diffeomorphism $x^\mu \to x'^\mu = x^\mu - \xi^\mu(x)$, the metric tensor transforms as

$$\delta g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu \,. \tag{7.73}$$

We may define a *Killing vector*[22] K^μ as the generator of a diffeomorphism that leaves the metric invariant, i.e.

$$\nabla_\mu K_\nu + \nabla_\nu K_\mu = 0 \,, \tag{7.74}$$

and so if $\xi^\mu = \epsilon\, K^\mu$, where ϵ is an infinitesimal constant parameter, we then have $\delta g_{\mu\nu} = 0$.

Let us consider the Schwarzschild metric as an example;

$$ds^2 = -\Big(1 - \frac{2M}{r}\Big)\, dt^2 + \Big(1 - \frac{2M}{r}\Big)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2) \,. \tag{7.75}$$

It is clear that if we consider the diffeomorphism

$$x^\mu \to x'^\mu = x^\mu - \xi^\mu \qquad \text{with} \quad \xi^0 = \epsilon \,, \quad \xi^1 = \xi^2 = \xi^3 = 0 \,, \tag{7.76}$$

that is to say, the pure time translation $t \to t' = t - \epsilon$, where ϵ is a constant, then it will leave the metric unchanged, that is to say

$$g'_{\mu\nu}(x') = g'_{\mu\nu}(x) = g_{\mu\nu}(x) \,, \tag{7.77}$$

and hence $\delta g_{\mu\nu}(x) \equiv g'_{\mu\nu}(x) - g_{\mu\nu}(x) = 0$. In other words, the vector field

$$K = \frac{\partial}{\partial t} \tag{7.78}$$

is a Killing vector in the Schwarzschild metric. One can explicitly verify that it does indeed obey the Killing vector equation (7.74).

In fact one can easily see that whenever the components of a metric tensor are all independent of a particular coordinate, say z, then there correspondingly exists a Killing vector

$$K = \frac{\partial}{\partial z} \,. \tag{7.79}$$

(In the language of classical mechanics, one could say that z is an *ignorable coordinate*.) Thus we see that there is another obvious Killing vector in the Schwarzschild metric (7.75), namely

$$L = \frac{\partial}{\partial \varphi} \,. \tag{7.80}$$

[22]Named after the German mathematician Wilhelm Killing.

This Killing vector is the generator of infinitesimal rotations around the azimuthal axis of the 2-sphere.

Not every Killing vector corresponds to an ignorable coordinate in the metric. Taking Schwarzschild as an example again, it has two further Killing vectors that describe the further rotational symmetries of the 2-sphere. Unlike translations of the azimuthal coordinate φ, these further symmetry transformations involve θ-dependent translations of both the φ and θ coordinates of the sphere. In fact they take the forms

$$L_x = -\sin\varphi\,\frac{\partial}{\partial\theta} - \cot\theta\cos\varphi\,\frac{\partial}{\partial\varphi}\,, \qquad L_y = \cos\varphi\,\frac{\partial}{\partial\theta} - \cot\theta\sin\varphi\,\frac{\partial}{\partial\varphi}\,. \tag{7.81}$$

Together with $L_z = \partial/\partial\varphi$ which we met already, these three Killing vectors are the generators of infinitesimal rotations around the x, y and z axes respectively, if we view the unit 2-sphere as embedded in Cartesian 3-space via the standard relations

$$x = \sin\theta\cos\varphi\,, \qquad y = \sin\theta\sin\varphi\,, \qquad z = \cos\theta\,. \tag{7.82}$$

It is a straightforward matter to verify that the vector fields L_x and L_y indeed satisfy the Killing equation (7.74) in the Schwarzschild metric.

In the example of the Schwarzschild metric, one can show that the four Killing vectors we have enumerated above, namely the time translation Killing vector (7.78) and the three rotational Killing vectors L_x, L_y and L_z on the 2-sphere, exhaust the complete set of independent Killing vectors. The latter three generate the rotation group $SO(3)$ of three dimensional Euclidean space, and in fact they obey the commutator algebra

$$[L_x, L_y] = -L_z\,, \qquad [L_y, L_z] = -L_x\,, \qquad [L_z, L_x] = -L_y\,. \tag{7.83}$$

The full symmetry group of the Schwarzschild metric is therefore $\mathbb{R} \times SO(3)$, where $\mathbb{R}$ indicates translations along the real line in the time direction. This group of symmeries is known as the *isometry group* of the Schwarzschild metric.

8 Further Solutions of the Einstein Equations

In this chapter, we discuss some further important examples of solutions of the Einstein equations, both with and without matter sources.

8.1 Reissner-Nordström solution

The Reissner-Nordström metric is a static, spherically symmetric solution in the Einstein-Maxwell theory, for which the field equations were derived in section 7.2:

$$\begin{aligned} R_{\mu\nu} - \tfrac{1}{2} R\, g_{\mu\nu} &= 2(F_{\mu\rho}\, F_\nu{}^\rho - \tfrac{1}{4} F^2\, g_{\mu\nu})\,, \\ \nabla_\mu\, F^{\mu\nu} &= 0\,. \end{aligned} \tag{8.1}$$

Note that by taking the trace of the Einstein equation (and noting also that the energy-momentum tensor for the Maxwell field is tracefree in four dimensions), we obtain $R = 0$ and hence the equation can be written in the simpler form

$$R_{\mu\nu} = 2(F_{\mu\rho}\, F_\nu{}^\rho - \tfrac{1}{4} F^2\, g_{\mu\nu})\,. \tag{8.2}$$

To construct the static, speherically-symmetric solution we can take the metric to have the same general form (6.15) as in the derivation of the Schwarzschild solution. For the Maxwell field, we can choose a gauge where the potential A_μ is given by

$$A_0 = -\phi(r)\,, \qquad A_1 = A_2 = A_3 = 0\,. \tag{8.3}$$

Thus the field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ just has the non-vanishing components

$$F_{01} = -F_{10} = \phi'\,. \tag{8.4}$$

From this, it is easily seen that the right-hand side of (8.2) is diagonal, with

$$2(F_{\mu\rho}\, F_\nu{}^\rho - \tfrac{1}{4} F^2\, g_{\mu\nu}) = \text{diag}\Big(\frac{\phi'^2}{A}, -\frac{\phi'^2}{B}, \frac{r^2\,\phi'^2}{AB}, \frac{r^2\,\phi'^2\,\sin^2\theta}{AB}\Big)\,. \tag{8.5}$$

From this, and the expressions (6.19) for the Ricci tensor for the metric (6.15), we see that $AR_{00} + BR_{11} = 0$ and so $(AB)' = 0$, just as in Schwarzschild. Thus we again have

$$A = \frac{1}{B}\,, \tag{8.6}$$

and hence the 22 component of the Einstein equations implies

$$(rB)' = 1 - \phi'^2\, r^2\,. \tag{8.7}$$

The Maxwell equation $\nabla_\mu F^{\mu\nu} = 0$ can be written as $\partial_\mu(\sqrt{-g}\, F^{\mu\nu}) = 0$, which, with $F_{\mu\nu}$ given by (8.4) implies

$$(r^2\,\phi')' = 0\,. \tag{8.8}$$

Integrating once gives $r^2\,\phi' = -q$ (an arbitrary integration constant), and integrating again gives

$$\phi = \frac{q}{r}\,. \tag{8.9}$$

Here, we have dropped the second constant of integration, since it is just the trivial additive constant that we can remove by requiring the electric potential to satisfy $\phi = 0$ at infinity. Plugging this expression for ϕ into (8.7), we can solve for B, obtaining

$$B = 1 - \frac{2M}{r} + \frac{q^2}{r^2}\,. \tag{8.10}$$

Thus, in summary, the solution, known as the Reissner-Nordström solution, is given by

$$ds^2 = -B\,dt^2 + \frac{dr^2}{B} + r^2\,(d\theta^2 + \sin^2\theta\,d\varphi^2)\,, \qquad \phi = \frac{q}{r}\,. \tag{8.11}$$

It reduces, obviously, to the Schwarzschild solution if $q = 0$. When q is non-zero, it describes the fields outside a spherically-symmetric static object with mass M and electric charge q. As in the case of Schwarzschild, the the Reissner-Nordström metric can also be taken to describe the solution for a black hole, for which it is a solution for all $r > 0$. We shall discuss some of its properties in greater detail later.

For now, recall that in the Schwarzschild solution there is a single radius $r = 2M$ at which $B(r)$ vanishes and $A(r)$ goes to infinity. This signals the fact that the light cones (the paths followed by null rays (light rays) in spacetime) tip over such that not even light can escape. This radius $r = 2M$ in Schwarzschild is the radius of the *event horizon* of the black hole. By contrast, in the Reissner-Nordström solution it can be seen that there are two values of r at which $B(r)$ vanishes and $A(r)$ diverges, namely at $r = r_\pm$, where

$$r_\pm = M \pm \sqrt{M^2 - q^2}\,. \tag{8.12}$$

These are the radii of the *outer horizon* (at $r = r_+$) and the *inner horizon* (at $r = r_-$). As in Schwarzschild, there is a genuine curvature singularity at $r = 0$, and so as long as

$$|q| \le M\,, \tag{8.13}$$

the singularity is hidden from external view behind the outer horizon. If $|q|$ exceeds M, then $B(r)$ has no real roots and so the singularity at $r = 0$ is no longer hidden behind an horizon. It is then known as a *naked singularity.*

The case when $|q| = M$ is called the *extremal* Reissner-Nordström solution. In this case, the outer and inner horizons coalesce, at $r_+ = r_- = M$. The extremal case is of considerable theoretical interest, but it is not one that is likely to be encountered observationally. If one restores all the constants in order to express things in SI units, it will be seen that an extremal Reissner-Nordström black hole of a typical mass that is seen at the centre of a galaxy would have to carry a huge and totally unrealistic amount of charge in order to be extremal. (The infalling matter that forms the black hole is predominantly electrically neutral.)

8.2 Kerr and Kerr-Newman solutions

8.2.1 Kerr solution

Another solution of very great importance is the Kerr solution in pure Einstein gravity, which describes the metric outside a rotating black hole. Einstein was surprised when Schwarzschild found his solution in 1916, one year after the formulation of the theory. He died eight years before Roy Kerr found the exact solution for the rotating black hole, in 1963. Had he lived, he would probably have been completely astonished that an exact solution could be obtained for this hugely more complicated situation, of a black hole with rotation.

We shall not present a derivation of the Kerr solution here, but merely give the result. If the reader has the strength to perform the calculations,[23] it is in principle straightforward, although tedious, to confirm that this metric solves the vacuum Einstein equations:

$$ds^2 = -\frac{\Delta}{\rho^2}\,(dt - a\sin^2\theta\, d\varphi)^2 + \rho^2\Big(\frac{dr^2}{\Delta} + d\theta^2\Big) + \frac{\sin^2\theta}{\rho^2}\,[(r^2+a^2)\,d\varphi - a\,dt]^2\,, \qquad (8.14)$$

where

$$\rho^2 \equiv r^2 + a^2\,\cos^2\theta\,, \qquad \Delta \equiv r^2 - 2M\,r + a^2\,. \qquad (8.15)$$

It describes a rotating black hole with mass M and angular momentum $J = aM$. There is a curvature singularity at $\rho = 0$. Although, from the definition of ρ, one might think this means $r = 0$ and $\theta = \frac{1}{2}\pi$, in fact the curvature singularity is actually a ring, occurring at imaginary values of the r coordinate such that $r^2 = -a^2\,\cos^2\theta$. To see this, one needs to carry out a more careful analysis, recognising that the coordinate r is not a good one in the vicinity of the singularity.

The Kerr metric is asymptotically flat, approaching the Minkowski metric (written in a spheroidal coordinate system) at large r. It reduces to the Schwarzschild solution if the rotation parameter a is set to zero.

[23] In fact, if one wants to check that this is indeed Ricci flat, it is well worthwhile writing a little routine in Mathematica to perform the calculation of the Christoffel connection and then the curvature. The calculations would be very tedious to perform by hand, but are a complete triviality for a computer.

As in the case of the Reissner-Nordström black hole, it can be seen that the Kerr black hole has an inner and an outer horizon, at radii $r = r_\pm$ given by the roots of $\Delta = 0$ in this case:

$$r_\pm = M \pm \sqrt{M^2 - a^2}\,. \tag{8.16}$$

There is again an extremal special case, where $|a| = M$, at which the two horizons coalesce, with $r_+ = r_- = M$. Since the angular momentum is $J = aM$, it follows that $|J| = M^2$ in the extremal limit. If $|a|$ exceeds M then $\Delta = 0$ has no real roots, and there is a naked curvature singularity with no horizon to clothe it.

The Kerr solution is of enormous physical importance, since almost every galaxy in the universe is believed to have a supermassive black hole at its centre. Typically, since the black hole forms and expands by the accretion of stars and other matter that is swirling around outside, the angular momentum will be considerable. In fact, a typical black hole at a galactic center is well described by the Kerr solution that is fairly close to the extremal limit $|a| = M$. This is because the black hole typically forms from the infalling of matter that is spiralling around it, carrying a large amount of orbital angular momentum.

8.2.2 Kerr-Newman solution

There also exists a charged generalisation, which is a solution of the Einstein-Maxwell equations, with the metric and vector potential given by

$$\begin{aligned} ds^2 &= -\frac{\Delta}{\rho^2}\,(dt - a\sin^2\theta\, d\varphi)^2 + \rho^2\Big(\frac{dr^2}{\Delta} + d\theta^2\Big) + \frac{\sin^2\theta}{\rho^2}\,[(r^2+a^2)\,d\varphi - a\,dt]^2\,, \\ A_\mu\, dx^\mu &= -\frac{q\,r\,(r^2+a^2)}{\Sigma}\,dt + \frac{aq\,r\,\sin^2\theta}{\rho^2}\,(d\varphi - f\,dt)\,, \end{aligned} \tag{8.17}$$

where

$$\begin{aligned} \rho^2 &= r^2 + a^2\,\sin^2\theta\,, \qquad \Delta = r^2 - 2Mr + a^2 + q^2\,, \\ \Sigma &= (r^2+a^2)^2 - a^2\,\Delta\,\sin^2\theta\,, \quad f = \frac{a\,(2Mr - q^2)}{\Sigma}\,. \end{aligned} \tag{8.18}$$

The solution, known as the Kerr-Newman solution, describes a rotating black hole with mass M, angular mmomentum $J = aM$ and electric charge q. It reduces to the Kerr solution if $q = 0$, and it reduces to the Reissner-Nordström solution if instead $a = 0$. Verifying this solution by hand would be considerably more challenging even than the case of the Kerr solution. Again, though, it is very easy to verify it using Mathematica.

8.3 Asymptotically anti-de Sitter spacetimes

The solutions we have discussed so far, that is the Schwarzschild, Kerr and Kerr-Newman solutions, have all *asymptotically flat*, meaning that at large distances the metric approaches the Minkowski metric. Solutions that have different asymptotic behaviour can also be found, and an especially important case is solutions that are asymptotic to de Sitter spacetime or anti-de Sitter spacetime.

8.3.1 Anti-de Sitter and de Sitter spacetimes

We can first construct the de Sitter and anti-de Sitter metrics themselves. These are solutions of the vacuum Einstein equations with a cosmological constant, satisfying (7.16). These metrics are maximally symmetric, and they are defined analogously to the way one defines an n-dimensional sphere as a constant-radius surface embedded in a Euclidean space of dimension $(n+1)$. The difference is that instead one defines a hyperbolic "constant-radius" surface in an $(n+1)$-dimensional spacetime with an appropriate indefinite signature.

To be concrete, let us consider the case of four-dimensional anti-de Sitter spacetime. This is defined as the surface

$$-(X^0)^2 + (X^1)^2 + (X^2)^2 + (X^3)^2 - (X^4)^2 = -\ell^2\,, \tag{8.19}$$

where ℓ is a constant, in the five-dimensional flat spacetime with coordinates $(X^0, X^1, X^2, X^3, X^4)$ and metric

$$ds_5^2 = -(dX^0)^2 + (dX^1)^2 + (dX^2)^2 + (dX^3)^2 - (dX^4)^2\,. \tag{8.20}$$

The constraint (8.19) can be solved by writing

$$\begin{aligned} X^0 &= \sqrt{r^2+\ell^2}\,\sin\frac{t}{\ell}\,, \qquad X^4 = \sqrt{r^2+\ell^2}\,\cos\frac{t}{\ell}\,, \\ X^1 &= r\,\sin\theta\,\cos\varphi\,, \qquad X^2 = r\,\sin\theta\,\sin\varphi\,, \qquad X^3 = r\,\cos\theta\,. \end{aligned} \tag{8.21}$$

Substituting into (8.20) gives the four-dimensional induced metric

$$ds^2 = -\Big(1+\frac{r^2}{\ell^2}\Big)\,dt^2 + (1+\frac{r^2}{\ell^2}\Big)^{-1}\,dr^2 + r^2\,(d\theta^2 + \sin^2\theta\,d\varphi^2)\,. \tag{8.22}$$

This is the four-dimensional metric on anti-de Sitter (AdS) spacetime. It is easy to verify (for example, from the expressions for the Ricci tensor given in (6.19)), that it satisfies (7.16) with cosmological constant given by

$$\Lambda = -\frac{3}{\ell^2}\,. \tag{8.23}$$

Thus, we can write the anti-de Sitter metric (8.22) as

$$ds^2 = -\Big(1 - \tfrac{1}{3}\Lambda\, r^2\Big)\, dt^2 + (1 - \tfrac{1}{3}\Lambda\, r^2\Big)^{-1}\, dr^2 + r^2\,(d\theta^2 + \sin^2\theta\, d\varphi^2)\,. \tag{8.24}$$

Since the AdS metric was defined via the constraint (8.19) and the 5-metric (8.20), both of which are invariant under the 5-dimensional (pseudo) rotation group $SO(3,2)$, it follows that this is also the symmetry group of the metric (8.24).

The metric (8.24) describes four-dimensional anti-de Sitter spacetime if the cosmological constant Λ is negative. If instead Λ is positive, it becomes the de Sitter metric. One can straightforwardly show, by a construction analogous to the one given above, that it can be described in terms of the surface

$$-(X^0)^2 + (X^1)^2 + (X^2)^2 + (X^3)^2 + (X^4)^2 = \ell^2\,, \tag{8.25}$$

embedded in a five-dimensional spacetime with $(-,+,+,+)$ signature and the metric

$$ds_5^2 = -(dX^0)^2 + (dX^1)^2 + (dX^2)^2 + (dX^3)^2 + (dX^4)^2\,. \tag{8.26}$$

The de Sitter metric has the symmetry group $SO(4,1)$.

The generalisation to n-dimensional AdS spacetime is straightforward. One now defines it via an embedding in an $(n{+}1)$-dimensional spacetime with signature $(-,+,+,+,\cdots+,+,-)$ (i.e. two minus, the rest plus), with

$$-(X^0)^2 + (X^1)^2 + (X^2)^2 + \cdots + (X^{n-2})^2 + (X^{n-1})^2 - (X^n)^2 = -\ell^2\,, \tag{8.27}$$

$$ds_n^2 = -(dX^0)^2 + (dX^1)^2 + (dX^2)^2 + \cdots + (dX^{n-2})^2 + (dX^{n-1})^2 - (dX^n)^2\,. \tag{8.28}$$

One can show that this metric, which has $SO(n-1,2)$ symmetry, satisfies the vacuum Einstein equation (7.16) with $\Lambda = -(n-1)\,\ell^{-2}$. The construction of n-dimensional de Sitter spacetime similarly generalises the four-dimensional de Sitter construction discussed above.

8.4 Schwarzschild-AdS solution

Anti-de Sitter or de Sitter spacetime can be viewed as the natural generalisation of the maximally-symmetric $\Lambda = 0$ Minkowski background to the case of Λ being negative or positive, respectively. The symmetry group of Minkowski spacetime is the Poincaré group, which as we discussed earlier, has 10 parameters (6 for the Lorentz transformations plus 4 for the translations). Likewise, the $SO(3,2)$ or $SO(4,1)$ symmetry groups of the anti-de

Sitter and de Sitter metrics each have 10 parameters. This is the maximal possible number of parameters in four dimensions, hence the term "maximal symmetry."

It is straightforward to generalise the Schwarzschild solution, which is the static, spherically symmetric, solution of the vacuum Einstein equations with $\Lambda = 0$ to the case when $\Lambda \neq 0$, satisfying (7.16). This can be done along the same lines as in the steps followed earlier in the course when deriving the Schwarzschild metric. In particular, the results (6.19) for the components of the Ricci tensor for the most general static, spherically symmetric, metric (6.15) can be employed. One finds (we leave this as an exercise for the reader), that the solution to (7.16) is given by

$$ds^2 = -\Big(1 - \frac{2M}{r} - \tfrac{1}{3}\Lambda\, r^2\Big)\, dt^2 + \Big(1 - \frac{2M}{r} - \tfrac{1}{3}\Lambda\, r^2\Big)^{-1} dr^2 + r^2\,(d\theta^2 + \sin^2\theta\, d\phi^2)\,. \quad (8.29)$$

As can be seen, at large r this approaches the anti-de Sitter metric (8.24). The solution (8.29) is usually called the Schwarzschild-anti-de Sitter metric (or Schwarzschild-AdS) when Λ is negative, and the Schwarzschild-de Sitter metric when Λ is positive.

8.5 Interior solution for a static, spherically-symmetric star

We saw earlier that the Schwarzschild solution describes the spacetime geometry outside a static, spherically-symmetric, massive object. If the object in question is a star, then the Schwarzschild solution, for which we assumed there was no matter source, is valid only outside the radius of the star. On the other hand, the solution can also be viewed as being valid for any radius $r > 0$ in the case where the object itself has collapsed down to form a black hole. We shall discuss the black hole geometry in greater detail later.

In this subsection, we shall consider the case where the gravitating object is a non-collapsed star. We shall show how the Schwarzschild solution, valid for radii greater than the radius of the star, can be matched on to an appropriate interior solution. We shall assume that the entire system is static and spherically symmetric. This, of course, is an idealisation, but it will nonetheless provide useful insights.

To address this question, we must make some assumption about the nature of the matter of which the star is composed. For these purposes, it will be appropriate to treat the matter as a perfect fluid, whose energy-momentum tensor, as discussed previously, takes the general form

$$T_{\mu\nu} = (\rho + P)\, U_\mu\, U_\nu + P\, g_{\mu\nu}\,, \quad (8.30)$$

where ρ is the energy density, P is the pressure, and U^μ is the 4-velocity field in the fluid.

We shall assume the same static, spherically-symmetric, metric ansatz as before:

$$ds^2 = -B(r)\, dt^2 + A(r)\, dr^2 + r^2\, (d\theta^2 + \sin^2\theta\, d\varphi^2)\,, \tag{8.31}$$

Similarly, the energy density ρ and the pressure P will be functions only of r. Since we are assuming everything is static, the 3-velocity of the fluid must vanish, and so U^μ will have only a non-vanishing 0 component. Since the 4-velocity must satisfy $g_{\mu\nu}\, U^\mu\, U^\nu = -1$, it therefore follows that

$$U^0 = B^{-1/2}\,, \qquad U_0 = -B^{1/2}\,, \tag{8.32}$$

with all other components vanishing. It then follows from (8.30) that the energy-momentum tensor is diagonal, with the non-vanishing components being

$$T_{00} = \rho\, B\,, \qquad T_{11} = P\, A\,, \qquad T_{22} = P\, r^2\,, \qquad T_{33} = P\, r^2\, \sin^2\theta\,. \tag{8.33}$$

From the expressions (6.19) for the components of the Ricci tensor for the metric (8.31), it can be seen that the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$ is also diagonal with the non-vanishing components

$$\begin{aligned} G_{00} &= B\Big[\frac{A'}{rA^2} - \frac{1}{r^2 A} + \frac{1}{r^2}\Big]\,,\\ G_{11} &= \frac{B'}{rB} - \frac{A}{r^2} + \frac{1}{r^2}\,,\\ G_{22} &= \frac{r^2}{A}\Big[\frac{B''}{2B} - \frac{B'}{4B}\Big(\frac{A'}{A} + \frac{B'}{B}\Big) - \frac{A'}{2rA} + \frac{B'}{2rB}\Big]\,,\\ G_{33} &= \sin^2\theta\, G_{22}\,. \end{aligned} \tag{8.34}$$

The 00 component of the Einstein equations $G_{\mu\nu} = 8\pi T_{\mu\nu}$ implies

$$8\pi\rho = \frac{A'}{rA^2} - \frac{1}{r^2 A} + \frac{1}{r^2}\,. \tag{8.35}$$

This is an equation involving only the metric function A, but not B. It can be written as

$$8\pi\rho = \frac{1}{r^2}\, \frac{d}{dr}\, [r\, (1 - A^{-1})]\,. \tag{8.36}$$

Being mindful of the form of the function A in the Schwarzschild solution, it is natural to express $A(r)$ in terms of a function $m(r)$, with

$$A(r) = \Big[1 - \frac{2m(r)}{r}\Big]^{-1}\,, \tag{8.37}$$

so that (8.36) becomes

$$8\pi\rho(r) = \frac{2}{r^2}\, \frac{dm(r)}{dr}\,. \tag{8.38}$$

Thus we can solve for $m(r)$, giving

$$m(r) = 4\pi \int_0^r \rho(r')\, r'^2\, dr' + a\,, \tag{8.39}$$

where a is a constant of integration. As r goes to zero it must be that $A(r)$ approaches 1, since otherwise there would be a conical singularity, and so in fact we must have $a = 0$. (There would in fact be a power-law divergence in the Ricci tensor as r went to zero, if a were non-zero, and this would be in conflict with other components of the Einstein equations, for non-singular matter sources.) Thus we have

$$m(r) = 4\pi \int_0^r \rho(r')\, r'^2\, dr'\,. \tag{8.40}$$

For the solution to be static we must certainly have $g_{11} > 0$, and so we see from (8.37) that we must have

$$2m(r) < r \tag{8.41}$$

for all values of r. The interior solution must match onto the exterior Schwarzschild solution (6.26) at the surface of the star (at $r = r_0$, say) and so in particular we must have

$$m(r_0) = M\,. \tag{8.42}$$

The 11 component of the Einstein equations implies

$$8\pi P = \frac{B'}{rAB} + \frac{1}{r^2 A} - \frac{1}{r^2}\,, \tag{8.43}$$

which, in view of (8.37), can be written as

$$\frac{B'(r)}{B(r)} = \frac{2[m(r) + 4\pi r^3\, P(r)]}{r\,[r - 2m(r)]}\,. \tag{8.44}$$

We also know that the energy-momentum tensor must be conserved. It is straightforward to calculate $\nabla_\mu T^{\mu\nu}$, and one finds that only the $\nu = 1$ component is not trivially zero; it implies

$$\frac{dP(r)}{dr} = -\tfrac{1}{2}[\rho(r) + P(r)]\, \frac{B'(r)}{B(r)}\,. \tag{8.45}$$

Using (8.44), we find

$$\frac{dP(r)}{dr} = -[\rho(r) + P(r)]\, \frac{m(r) + 4\pi r^3\, P(r)}{r\,[r - 2m(r)]}\,. \tag{8.46}$$

This is known as the *Tolman-Oppenheimer-Volkov* (TOV) equation of hydrostatic equilibrium. In the Newtonian limit, where $m(r) << r$ and $P(r) << \rho(r)$, it becomes the Newtonian hydrostatic equation

$$\frac{dP(r)}{dr} = -\frac{\rho(r)\, m(r)}{r^2}\,. \tag{8.47}$$

To summarise, we have seen that the interior solution for a static, spherically-symmetric star composed of a perfect fluid is given by

$$ds^2 = -B(r)\,dt^2 + \Big(1 - \frac{2m(r)}{r}\Big)^{-1}\,dr^2 + r^2 d\theta^2 + r^2\,\sin^2\theta\,d\varphi^2\,, \tag{8.48}$$

where $m(r)$ is given by (8.40) and $B(r)$ is obtained by solving (8.44). To make further progress, one can specify an *equation of state* for the perfect fluid, i.e. specify P as a function of ρ. Having specified $P(\rho)$, one can in principle then specify a value for ρ at the centre of the star, $\rho(0) = \rho_c$. This then implies that the pressure at the centre will be $P_c = P(\rho_c)$. One then integrates outwards from $r = 0$, using (8.40) and (8.46). The surface of the star, at $r = r_0$, will be, by definition, where $P(r)$ and $\rho(r)$ become zero. One then integrates out equation (8.44) to solve for the metric function $B(r)$. These results must then match onto the Schwarzschild solution at the surface of the star, at $r = r_0$.

An alternative approach, rather than specifying an equation of state, is to specify the energy density ρ as a function of r inside the star. A simple choice is to consider the case where the perfect fluid is *incompressible*, meaning that ρ is a constant. Thus we may take:

$$\begin{aligned} \rho(r) &= \rho_0 \quad \text{for } 0 \le r \le r_0\,, \\ \rho(r) &= 0 \quad \text{for } r > r_0\,, \end{aligned} \tag{8.49}$$

where ρ_0 is a constant. Equation (8.40) then gives

$$m(r) = \tfrac{4}{3}\pi\, r^3\,\rho_0\,, \quad \text{for } 0 \le r \le r_0\,. \tag{8.50}$$

The solution matches onto the Schwarzschild solution (6.26) at $r = r_0$, so we shall have

$$M = \tfrac{4}{3}\pi\, r_0^3\,\rho_0\,. \tag{8.51}$$

The TOV equation (8.46) can then be solved, giving

$$P(r) = \rho_0\,\Big[\frac{(1-2M/r_0)^{1/2} - (1-2Mr^2/r_0^3)^{1/2}}{(1-2Mr^2/r_0^3)^{1/2} - 3(1-2M/r_0)^{1/2}}\Big]\,. \tag{8.52}$$

The pressure at the centre of the star, i.e. $r = 0$, is given by

$$P_c = P(0) = \rho_0\,\Big[\frac{1-(1-2M/r_0)^{1/2}}{3(1-2M/r_0)^{1/2} - 1}\Big]\,. \tag{8.53}$$

This becomes infinite if

$$r_0 = \tfrac{9}{4} M\,, \tag{8.54}$$

mean that a star composed of an incompressible perfect fluid can only exist if its radius satisfies

$$r_0 > \tfrac{9}{4} M\,. \tag{8.55}$$

In view of (8.51), this bound can alternatively be expressed as the statement that for a given uniform energy density ρ_0, there is an upper bound on the possible mass of the star:

$$M \le \frac{4}{9\sqrt{3\pi}}\,\frac{1}{\sqrt{\rho_0}}\,. \tag{8.56}$$

No such bound would arise in Newtonian physics, of course: One could in principle assemble an arbitrarily large quantity of incompressible fluid with density ρ_0, and build a star of arbitrarily high mass.

A general observation that one can make, based on the TOV equation (8.46), is that the right-hand side is always more negative (assuming the pressure is positive), for a given energy density function $\rho(r)$, than in the Newtonian case given in (8.47), regardless of the details of the equation of state. This is immediately evident from the fact that the numerator and the denominator factors in (8.46) satisfy

$$\begin{aligned} [\rho(r)+P(r)]\,[m(r)+4\pi r^3\,P(r)] &\ \ge\ \rho(r)\,m(r)\,,\\ r\,[r-2m(r)] &\ \le\ r^2\,, \end{aligned} \tag{8.57}$$

and so

$$[\rho(r)+P(r)]\,\frac{m(r)+4\pi r^3\,P(r)}{r\,[r-2m(r)]} \ge \frac{\rho(r)\,m(r)}{r^2}\,. \tag{8.58}$$

This has the consequence that the pressure $P(0)$ at the centre of the star will always be greater, for a given $\rho(r)$, in general relativity than in the Newtonian case. This means that it is harder to maintain an equilibrium in general relativity. This was very clear in the example considered above, where a constant energy density ρ_0 inside the star was assumed. It then turned out that it was not possible to have any equilibrium at all, in general relativity, if the mass was too large for a given energy density ρ_0.

9 Gravitational Waves

Another important class of solutions in general relativity is gravitational waves, which are the gravitational analogue of the electromagnetic waves of Maxwell's electrodynamics.

9.1 Plane gravitational waves

The simplest situation to consider, and the one that is most relevant in practice, is the case of a gravitational wave propagating in a flat Minkowski spacetime background. Thus we may choose a coordinate system in which the metric is just perturbed slightly away ffrom the Minkowski metric:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}\,, \tag{9.1}$$

where each component of $h_{\mu\nu}$ can be assumed to be small; $|h_{\mu\nu}| << 1$. It is then straightforward to see that up to the first order in powers of h, the inverse metric is given by

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} + \cdots \,. \tag{9.2}$$

where here, and in the equations that follow, it is assumed that indices on h and other small quantities are raised and lowered using the Minkowski background metric. Thus

$$h^{\mu\nu} \equiv \eta^{\mu\rho}\, \eta^{\nu\sigma}\, h_{\rho\sigma} \,. \tag{9.3}$$

Linearising the Christoffel connection

$$\Gamma^{\mu}{}_{\nu\rho} = \tfrac{1}{2} g^{\mu\sigma}\, (\partial_\nu g_{\sigma\rho} + \partial_\rho g_{\nu\sigma} - \partial_\sigma g_{\nu\rho}) \tag{9.4}$$

gives

$$\Gamma^{\mu}_{\text{lin.}\,\nu\rho} = \tfrac{1}{2} \eta^{\mu\sigma}\, (\partial_\nu h_{\sigma\rho} + \partial_\rho h_{\nu\sigma} - \partial_\sigma h_{\nu\rho}) \,. \tag{9.5}$$

Since the Christoffel connection has no zeroth-order term, it follows that up to linear order the Riemann tensor, which has the structural form $\partial\Gamma - \partial\Gamma + \Gamma\Gamma - \Gamma\Gamma$, will receive contribitions only from the $\partial\Gamma$ terms, and likewise for the Ricci tensor. Thus we shall have

$$\begin{aligned} R^{\text{lin.}}_{\mu\nu} &= \partial_\rho \Gamma^{\rho}_{\text{lin.}\,\mu\nu} - \partial_\nu \Gamma^{\rho}_{\text{lin.}\,\rho\mu} \,, \\ &= \tfrac{1}{2} \eta^{\rho\sigma}\, (\partial_\rho\partial_\nu h_{\sigma\mu} + \partial_\rho\partial_\mu h_{\sigma\nu} - \partial_\rho\partial_\sigma h_{\nu\mu} - \partial_\nu\partial_\rho h_{\sigma\mu} - \partial_\nu\partial_\mu h_{\rho\sigma} + \partial_\nu\partial_\sigma h_{\rho\mu}) \,, \\ &= \tfrac{1}{2}(-\Box h_{\mu\nu} + \partial_\mu\partial_\sigma h^{\sigma}{}_{\nu} + \partial_\nu\partial_\sigma h^{\sigma}{}_{\mu} - \partial_\mu\partial_\nu h) \,, \end{aligned} \tag{9.6}$$

where we have defined

$$\Box \equiv \eta^{\mu\nu}\, \partial_\mu\partial_\nu \,, \qquad h \equiv \eta^{\mu\nu}\, h_{\mu\nu} \,. \tag{9.7}$$

Note that another simple way to derive the Riemann tensor, and hence Ricci tensor, in this case is to use the exact expression for $R_{\mu\nu\rho\sigma}$ given in eqn (4.71). Since the Christoffel connection is linear in $h_{\mu\nu}$ the $\Gamma\Gamma$ terms can be neglected in the linear approximation to which we are working, and the $\partial\partial g$ terms will just give $\partial\partial h$, so

$$R^{\text{lin.}}_{\mu\nu\rho\sigma} = \tfrac{1}{2}(\partial_\mu\partial_\sigma h_{\nu\rho} - \partial_\mu\partial_\rho h_{\nu\sigma} + \partial_\nu\partial_\rho h_{\mu\sigma} - \partial_\nu\partial_\sigma h_{\mu\rho}) \,. \tag{9.8}$$

Linearised gravitational waves propagating in the Minkowski spacetime background will obey $R^{\text{lin.}}_{\mu\nu} = 0$, and hence

$$\Box h_{\mu\nu} - \partial_\mu\partial_\sigma h^{\sigma}{}_{\nu} - \partial_\nu\partial_\sigma h^{\sigma}{}_{\mu} + \partial_\mu\partial_\nu h = 0 \,. \tag{9.9}$$

The analysis that follows will be closely analogous to the way one studies electromagnetic waves in electrodynamics.[24] We can simplify the equation (9.9) by making a judicious coordinate transformation. Recall from (7.57) that if one makes an infinitesimal diffeomorphism of the form

$$\delta x^\mu = x'^\mu - x^\mu = -\xi^\mu\,, \tag{9.10}$$

then the components of the metric tensor change according to

$$\delta g_{\mu\nu} = \xi^\rho\,\partial_\rho g_{\mu\nu} + g_{\rho\nu}\,\partial_\mu\xi^\rho + g_{\mu\rho}\,\partial_\nu\xi^\rho\,. \tag{9.11}$$

Now, with $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $h_{\mu\nu}$ itself is small, then the leading terms in the transformation of $h_{\mu\nu}$ will be given by

$$\delta h_{\mu\nu} = h'_{\mu\nu} - h_{\mu\nu} = \partial_\mu\xi_\nu + \partial_\nu\xi_\mu\,. \tag{9.12}$$

Note that the linearised Ricci tensor we obtained in (9.6) must be invariant under this transformation, and one can easily check that this is indeed the case. (These transformations are the gravitational analogue of the $\delta A_\mu = \partial_\mu\Lambda$ infinitesimal gauge transformations in electrodynamics, which, of course, leave $F_{\mu\nu}$ invariant.)

We can use the four parameters ξ^μ of the infinitesimal diffeomorphism to impose four conditions on the linearised metric fluctions $h_{\mu\nu}$. The most convenient choice is to impose what is known as the *de Donder gauge* condition

$$\partial_\mu h^\mu{}_\nu - \tfrac{1}{2}\partial_\nu h = 0\,. \tag{9.13}$$

Note that this is a set of four equations, and so we can indeed expect to be able to use the four parameters ξ^μ to achieve this. The de Donder gauge is sometimes called the *harmonic gauge*, for the following reason: The covariant d'Alembertian on a scalar field ϕ is given by

$$\nabla^\mu\nabla_\mu\phi = g^{\mu\nu}\nabla_\mu\partial_\nu\phi = g^{\mu\nu}\partial_\mu\partial_\nu\phi - g^{\mu\nu}\,\Gamma^\rho{}_{\mu\nu}\,\partial_\rho\phi\,. \tag{9.14}$$

If we act with this operator on the coordinates x^σ, and impose the *harmonic condition* $\nabla^\mu\nabla_\mu x^\sigma = 0$ then this gives

$$g^{\mu\nu}\,\Gamma^\sigma{}_{\mu\nu} = 0\,, \tag{9.15}$$

[24] In electrodynamics, the equations are already linear, and so, writing $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, the source-free field equation $\partial^\mu F_{\mu\nu} = 0$ implies $\Box A_\mu - \partial_\mu\partial^\nu A_\nu = 0$, which is the electromagnetic analogue of (9.9). One then simplifies this equation by using the gauge transformations ($\delta A_\mu = \partial_\mu\Lambda$ at the infinitesimal level) to impose the Lorenz gauge $\partial^\mu A_\mu = 0$, thus leading to $\Box A_\mu = 0$.

since $\partial_\mu \partial_\nu \, x^\sigma = 0$. For our situation, where the linearised Christoffel connection is given by (9.5), we see that up to first order in the small quantities $h_{\mu\nu}$, the harmonic condition (9.15) gives

$$\eta^{\mu\nu} \, \Gamma^\sigma_{\text{lin.}\,\mu\nu} = 0 \,, \tag{9.16}$$

which leads precisely to the de Donder gauge condition (9.13).

The convenience of the de Donder gauge choice (9.13) can be appreciated when we substitute it into the expression (9.9) for the gravitational waves; it reduces the equation simply to

$$\Box h_{\mu\nu} = 0 \,. \tag{9.17}$$

We can then look for plane-wave solutions, in which we write

$$h_{\mu\nu} = \epsilon_{\mu\nu} \, e^{\text{i}k\cdot x} \,, \tag{9.18}$$

where $\epsilon_{\mu\nu}$ is a constant symmetric *polarisation tensor*, k_μ is the constant wave-vector, and we adopt the notation

$$k \cdot x \equiv k_\mu \, x^\mu \,. \tag{9.19}$$

The wave equation (9.17) implies

$$0 = \Box h_{\mu\nu} = (\text{i}k^\rho) \, (\text{i}k_\rho) \, \epsilon_{\mu\nu} \, e^{\text{i}k\cdot x} \,, \tag{9.20}$$

and the de Donder gauge condition (9.13) implies

$$0 = \text{i}k_\mu \, \epsilon^\mu{}_\nu \, e^{\text{i}k\cdot x} - \tfrac{\text{i}}{2} k_\nu \, \epsilon^\mu{}_\mu \, e^{\text{i}k\cdot x} \,. \tag{9.21}$$

Thus, in all we see that the polarisation and wave vectors must satisfy the conditions

$$k^2 \equiv k^\mu \, k_\mu = 0 \,, \tag{9.22}$$

$$k_\mu \, \epsilon^\mu{}_\nu - \tfrac{1}{2} k_\nu \, \epsilon^\mu{}_\mu = 0 \,. \tag{9.23}$$

We can make a counting of degrees of freedom at this point. The polarisation tensor $\epsilon_{\mu\nu}$ is symmetric, and so it has $(4 \times 5)/2 = 10$ independent components. The de Donder gauge imposes the four conditions (9.23), and so this leaves $10 - 4 = 6$ free independent components of the polarisation tensor. But, we are not finished yet; in the words of Peter van Nieuwenhuizen, one of the discoverers of supergravity, "the gauge shoots twice." We can actualy still squeeze more juice out of the freedom to make gauge conditions. We used the infinitesimal diffeomorphisms (9.10) to impose the de Donder gauge (9.13). Suppose now we ask if we can make a *further* diffeomorphism, with the requirement that it must preserve the already-established de Donder gauge. Therefore, we consider a diffeomorphism parameter ξ^μ such that its associated transformation of $h_{\mu\nu}$, given by (9.12), leaves the de Donder gauge condition unchanged;

$$\partial^\mu(\partial_\mu\xi_\nu + \partial_\nu\xi_\mu) - \tfrac{1}{2}\partial_\nu[\eta^{\rho\sigma}\,(\partial_\rho\xi_\sigma + \partial_\sigma\xi_\rho)] = 0\,. \tag{9.24}$$

In other words, the diffeomrphism must satisfy

$$\Box\xi_\mu = 0\,. \tag{9.25}$$

We are thus led to consider a diffeomorphism with

$$\xi_\mu = \mathrm{i}\epsilon_\mu\, e^{\mathrm{i}k\cdot x}\,, \tag{9.26}$$

where ϵ_μ is a constant vector, and the i factor is put in for convenience (it could of course be absorbed into ϵ_μ, but is is nicer to keep it as an explicit factor). Note that we could have chosen any null vector as the wave vector, but we have specifically chosen the same wave vector that appears in our plane wave solution (9.18). The reason for choosing this will become clear shortly.

From (9.12), the change in $h_{\mu\nu}$ under this further diffeomorphism is given by

$$h'_{\mu\nu} = h_{\mu\nu} - (k_\mu\epsilon_\nu + k_\nu\epsilon_\mu)\, e^{\mathrm{i}k\cdot x} = [\epsilon_{\mu\nu} - (k_\mu\epsilon_\nu + k_\nu\epsilon_\mu)]\, e^{\mathrm{i}k\cdot x}\,, \tag{9.27}$$

and hence we see that the polarisation tensor $\epsilon_{\mu\nu}$ in the plane wave (9.18) changes according to

$$\epsilon'_{\mu\nu} = \epsilon_{\mu\nu} - (k_\mu\epsilon_\nu + k_\nu\epsilon_\mu)\,. \tag{9.28}$$

(Note that the $e^{\mathrm{i}k\cdot x}$ factors have cancelled out.) There are thus four parameters ϵ_μ available, which can be used to impose four further conditions on the previously-remaining six independent components of $\epsilon_{\mu\nu}$. Thus the gauge has indeed shot for a second time, and the final counting is that there are $10 - 4 - 4 = 2$ independent polarisation states in the gravitational wave.

9.2 Spin of the gravitational waves

It is useful at this stage to consider an explicit example of a gravitational plane wave. Let us supose that it is traveling in the z direction, and so the null vector k^μ can be taken to be

$$k^\mu = (k, 0, 0, k)\,, \qquad k > 0\,. \tag{9.29}$$

The wave (9.18) has the coordinate dependence $e^{{\rm i}k\cdot x} = e^{-{\rm i}k(t-z)}$, so for $k > 0$ it is a positive-frequency wave propagating at the speed of light along the positive z direction.

The de Donder conditions (9.23) for $\nu = 0, 1, 2, 3$ imply, respectively,

$$\begin{aligned} \epsilon_{00} + \epsilon_{30} + \tfrac{1}{2}(-\epsilon_{00} + \epsilon_{11} + \epsilon_{22} + \epsilon_{33}) &= 0\,, \\ \epsilon_{01} + \epsilon_{31} &= 0\,, \\ \epsilon_{02} + \epsilon_{32} &= 0\,, \\ \epsilon_{03} + \epsilon_{33} - \tfrac{1}{2}(-\epsilon_{00} + \epsilon_{11} + \epsilon_{22} + \epsilon_{33}) &= 0\,. \end{aligned} \qquad (9.30)$$

Thus we find the four conditions

$$\epsilon_{01} = -\epsilon_{31}\,, \qquad \epsilon_{02} = -\epsilon_{32}\,, \qquad \epsilon_{03} = -\tfrac{1}{2}(\epsilon_{00} + \epsilon_{33})\,, \qquad \epsilon_{22} = -\epsilon_{11}\,. \qquad (9.31)$$

Making the further gauge transformations (9.28) then gives

$$\begin{aligned} \epsilon'_{12} &= \epsilon_{12}\,, \qquad \epsilon'_{13} = \epsilon_{13} - k\,\epsilon_1\,, \qquad \epsilon'_{23} = \epsilon_{23} - k\,\epsilon_2\,, \\ \epsilon'_{00} &= \epsilon_{00} + 2k\,\epsilon_0\,, \qquad \epsilon'_{11} = \epsilon_{11}\,, \qquad \epsilon'_{33} = \epsilon_{33} - 2k\,\epsilon_3\,. \end{aligned} \qquad (9.32)$$

If we choose the components of the vector ϵ_μ so that

$$\epsilon_0 = -\frac{1}{2k}\,\epsilon_{00}\,, \qquad \epsilon_1 = \frac{1}{k}\,\epsilon_{13}\,, \qquad \epsilon_2 = \frac{1}{k}\,\epsilon_{23}\,, \qquad \epsilon_3 = \frac{1}{2k}\,\epsilon_{33}\,, \qquad (9.33)$$

then we see that the only non-vanishing components of the transformed polarisation tensor $\epsilon'_{\mu\nu}$ will be

$$\epsilon'_{11} = -\epsilon'_{22}\,, \quad \text{and} \quad \epsilon'_{12}\,. \qquad (9.34)$$

From now on, we shall assume that this gauge choice has been made, and we shall drop the primes.

The spin, or more properly the helicity, of the states can determined by looking at how the components of the polarisation tensor transform under the so-called *little group*, which is the rotation subgroup of the Lorentz transformations that leaves the null wave-vector k^μ invariant. This will therefore correspond to a Lorentz transformation matrix $\Lambda_\mu{}^\nu = S_\mu{}^\nu$, given by

$$S_\mu{}^\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\,. \qquad (9.35)$$

Note that the little goup is just $SO(2)$ transformations, comprising, in this case, rotations in the (x, y) plane. It is helpful to group the remaining polarisation states (9.34) (now with the primes dropped) into the complex combinations

$$\epsilon_\pm \equiv \epsilon_{11} \mp \mathrm{i}\epsilon_{12} \,. \tag{9.36}$$

It is also instructive to make the complex combinations

$$\alpha_\pm \equiv \epsilon_{31} \mp \mathrm{i}\epsilon_{32} \tag{9.37}$$

from components that we actually chose to set to zero by means of the diffeomorphism gauge transformations. After a little simple algebra, we then find that after acting with the rotation (9.35) according to the standard Lorentz transformation rule

$$\tilde\epsilon_{\mu\nu} = S_\mu{}^\rho\, S_\nu{}^\sigma\, \epsilon_{\rho\sigma} \,, \tag{9.38}$$

that the various components transform as

$$\begin{aligned}
\epsilon_\pm &\longrightarrow \tilde\epsilon_\pm = e^{\pm 2\mathrm{i}\theta}\,\epsilon_\pm \,,\\
\alpha_\pm &\longrightarrow \tilde\alpha_\pm = e^{\pm \mathrm{i}\theta}\,\alpha_\pm \,,\\
\epsilon_{33} &\longrightarrow \tilde\epsilon_{33} = \epsilon_{33} \,, \qquad \epsilon_{00} \longrightarrow \tilde\epsilon_{00} = \epsilon_{00} \,.
\end{aligned} \tag{9.39}$$

These equations show that $\epsilon_\pm$ transform as states of helicity ± 2, while the states $\alpha_\pm$ have helicity ± 1 and the states ϵ_{00} and ϵ_{33} have helicity 0. When the gauge "shot for the second time," it led to the removal of the helicity-1 and helicity-0 components of the gravitational wave. In other words, the true physical degrees of freedom in the wave are just the helicity $+2$ and helicity -2 states. These are the polarisations of the massless spin-2 graviton. (This is closely analogous to the situation for electromagnetism, where the gauge-independent physical states in a plane wave are purely spin-1, with states of helicity $+1$ and -1 only.)

9.3 Observable effects of gravitational waves

Gravitational waves are generally very weak, and actually detecting them has been a tremendous technical challenge. Finally, in 2015, advances in detector technology allowed the first observation of gravitational waves. The general principles of how a gravity-wave detector works can be seen from the following calculation.

We saw in chapter 5 that if two particles follow nearby geodesic paths, then their separation vector Z^μ will obey the equation of geodesic deviation (5.21)

$$\frac{D^2 Z^\mu}{D\tau^2} = -R^\mu{}_{\rho\nu\sigma}\,\frac{dx^\rho}{d\tau}\,\frac{dx^\sigma}{d\tau}\,Z^\nu \,. \tag{9.40}$$

In a nearly-Minkowski spacetime, in the case that the 3-velocities of the particles are small, we shall have $\tau \approx t$, and $dx^\mu/d\tau$ will be approximately given by $dx^\mu/d\tau \approx (1,0,0,0)$. Thus the spatial components of the separation vector Z^μ will approximately satisfy

$$\frac{d^2 Z^i}{dt^2} \approx -R^i{}_{0j0}\, Z^j\,. \tag{9.41}$$

Furthermore, with the Christoffel connection being assumed to be small (given approximately by (9.5)), it follows from (4.65) that

$$R^i{}_{0j0} \approx \partial_j \Gamma^i{}_{00} - \partial_0 \Gamma^i{}_{j0}\,. \tag{9.42}$$

If we consider the gravitational wave (9.18) with

$$\epsilon_{11} = -\epsilon_{22} = \epsilon\,, \qquad k^\mu = (k,0,0,k)\,, \tag{9.43}$$

with all other $\epsilon_{\mu\nu} = 0$, so that the physical wave can be taken to be

$$h_{11} = -h_{22} = \epsilon\, \sin k(t-z)\,, \tag{9.44}$$

with all other $h_{\mu\nu} = 0$, then

$$\begin{aligned} \Gamma^i{}_{00} &\approx \tfrac{1}{2}\eta^{ik}\,(\partial_0 h_{k0} + \partial_0 h_{0k} - \partial_k h_{00}) = 0\,,\\ \Gamma^i{}_{j0} &\approx \tfrac{1}{2}\eta^{ik}\,(\partial_j h_{k0} + \partial_0 h_{jk} - \partial_k h_{j0}) = \tfrac{1}{2}\partial_0 h_{ij}\,, \end{aligned} \tag{9.45}$$

and so

$$R^i{}_{0j0} \approx -\tfrac{1}{2}\frac{\partial^2 h_{ij}}{\partial t^2}\,. \tag{9.46}$$

Thus we shall have

$$R^1{}_{010} \approx -R^2{}_{020} \approx \tfrac{1}{2}\epsilon\, k^2\, \sin k(t-z)\,. \tag{9.47}$$

(Note that here k^2 means the square of the constant k in (9.43), and not $k^\mu k_\mu$ as it did earlier!)

If we consider a ring of freely-falling particles in the XY plane, centered on the origin, then equation (9.41) implies that

$$\frac{d^2 X}{dt^2} = -\tfrac{1}{2} X\, \epsilon\, k^2\, \sin k(t-z)\,, \qquad \frac{d^2 Y}{dt^2} = \tfrac{1}{2} Y\, \epsilon\, k^2\, \sin k(t-z)\,, \tag{9.48}$$

where $X = Z^1$ and $Y = Z^2$. The ring of particles will oscillate to become a stretched or squashed ellipse in a periodic fashion. A solid object will tend to undergo periodic distortions of a similar nature.

9.4 Generation of gravitational waves

Until now, our discussion of gravity waves has been concerned with how they propagte in spacetime, and how they might be detected. For these purposes, it was sufficient to consider the source-free Einstein equations. Here, we shall examine how they might actually be generated, and for this it is necessary to consider the details of the matter sources that could give rise to gravitational waves. Thus, we consider the Einstein equation

$$R_{\mu\nu} - \tfrac{1}{2} R g_{\mu\nu} = 8\pi T_{\mu\nu} \,. \tag{9.49}$$

We may continue with the assumption of a weak field for which the metric is given by (9.1), and again we shall impose the de Donder gauge condition (9.13), so that we shall have

$$R_{\mu\nu} \approx -\tfrac{1}{2} \Box h_{\mu\nu} \,. \tag{9.50}$$

The linearisation of the Einstein equation (9.49) then gives

$$\Box \psi_{\mu\nu} = -16\pi T_{\mu\nu} \,, \tag{9.51}$$

where

$$\psi_{\mu\nu} \equiv h_{\mu\nu} - \tfrac{1}{2} h \eta_{\mu\nu} \,, \tag{9.52}$$

and as before, $h = \eta^{\mu\nu}\, h_{\mu\nu}$. $T_{\mu\nu}$ is now understood to be just the energy-momentum tensor in the Minkowski background, and which therefore satisfies the conservation equation

$$\partial_\mu T^{\mu\nu} = 0 \tag{9.53}$$

in the Minkowski background metric.

The field equation (9.51) can be solved in terms of a retarded potential, in exactly the same way as one solves the equation $\Box A_\mu = -4\pi J_\mu$ in electrodynamics (see, for example, my EM611 lectures online). Thus we shall have

$$\psi_{\mu\nu}(x) = 4 \int \frac{T_{\mu\nu}(t - |\vec r - \vec r\,'|, \vec r\,')}{|\vec r - \vec r\,'|}\, d^3\vec r\,' \,, \tag{9.54}$$

where $x^\mu = (t, \vec r)$, etc. We shall assume a compact matter source near to the origin of the coordinate system, and we then consider the case where the observation point $\vec r$ is at a very large distance in comparison to the size of the matter source. Thus $R = |\vec r|$ will be very large in comparison to $|\vec r\,'|$ for all points $\vec r\,'$ within the source, and so we may approximate (9.54) by

$$\psi_{\mu\nu} = \frac{4}{R} \int T_{\mu\nu}\, dV \,. \tag{9.55}$$

This approximation corresponds to considering the far-field *radiation zone*. Since we are using R to denote the distance to the point of observation we can, without risk of confusion, switch to using unprimed variables for the integration on the right-hand side. Thus dV, the integration volume element, is now written as $d^3\vec{r}$, and the arguments of $T_{\mu\nu}$ are $T_{\mu\nu}(t - R, \vec{r})$. If we consider the spatial components of $\psi_{\mu\nu}$, we have

$$\begin{aligned}\int T^{ij}\, dV &= \int \Big[\partial_k(T^{kj}\, x^i) - (\partial_k T^{kj})\, x^i\Big]\, dV\,,\\ &= \partial_0 \int T^{0j}\, x^i\, dV\,,\\ &= \tfrac{1}{2}\partial_0 \int (T^{0j}\, x^i + T^{0i}\, x^j)\, dV\,,\\ &= \tfrac{1}{2}\partial_0 \int \Big[\partial_k(T^{0k}\, x^i x^j) - (\partial_k T^{0k})\, x^i x^j\Big]\, dV\,,\\ &= \tfrac{1}{2}\partial_0^2 \int T^{00}\, x^i x^j\, dV\,, \end{aligned} \tag{9.56}$$

where we have made use of the conservation equation $0 = \partial_\mu T^{\mu\nu} = \partial_0 T^{0\mu} + \partial_k T^{k\mu}$ and the symmetry of $T^{\mu\nu}$, and we have dropped boundary terms arising when using the divergence theorem. Thus, since $T^{00} = \rho$, the energy density, we have

$$\psi_{ij} = \frac{2}{R}\,\frac{\partial^2}{\partial t^2} \int \rho x^i x^j\, dV\,. \tag{9.57}$$

The equation (9.52) defining $\psi_{\mu\nu}$ in terms of $h_{\mu\nu}$ can be inverted (by taking the $\eta_{\mu\nu}$ trace and substituting back in for h) to give

$$h_{\mu\nu} = \psi_{\mu\nu} - \tfrac{1}{2}\psi\, \eta_{\mu\nu}\,, \tag{9.58}$$

where $\psi = \eta^{\mu\nu}\, \psi_{\mu\nu} = -h$. Using the additional gauge transformations we discussed earlier, with $\delta h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$ and $\Box \xi_\mu = 0$, thus preserving the de Donder gauge, one may choose to set $h_{ii} = 0$ (summed over the three spatial directions). In fact the gauge choices made in the previous example we discussed had the consequence that $h_{ii} = 0$ (see equation (9.34)).Thus from (9.58) we have $\psi_{ii} - \frac{3}{2}\psi = 0$, and hence

$$h_{ij} = \psi_{ij} - \tfrac{1}{3}\psi_{kk}\, \delta_{ij}\,, \tag{9.59}$$

leading to

$$h_{ij} = \frac{2}{R}\,\frac{\partial^2}{\partial t^2} \int \rho(x^i x^j - \tfrac{1}{3} r^2 \delta_{ij})\, dV\,, \tag{9.60}$$

where $r^2 = x^i x^i$. Thus we see that the gravitational wave is generated at leading order by the time-dependent quadrupole moment of the matter source.

It is instructive to compare the above with what happens in electromagnetism. In that case (see, for example, my EM611 lecture notes), electromagnetic waves are generated at leading order by the time-dependent electric dipole moment. It is not possible to have an isolated time-dependent electric monopole source, because charge is conserved. Thus the leading-order possibility for a time-dependent source is at the dipole order; positive and negative charges can oscillate back and forth, while keeping the total charge conserved.

In the case of gravity, not only can the mass of the isolated source system not change in time, but also its dipole moment cannot change in time. This is because unlike electric charges, which can be positive or negative, masses can only be positive. Thus the leading order at which the isolated system can have a time-dependent moment is at the quadrupole order.

10 Global Structure of Schwarzschild Black holes

In this section, we shall discuss the global structure of the Schwarzschild black hole solution, in particular studying its structure at infinity, on the event horizon, and at the curvature singularity.

The Schwarzschild solution can be thought of as a kind of gravitational analogue of the point charge solution in classical electrodynamics. Of course the non-linear nature of the Einstein equations means that the solution is more complicated, and much more subtle, than the humble point charge. Also, the very essence of general relativity is that one is using a description that is covariant with respect to arbitrary changes of coordinate system. This means that one has to be very careful to distinguish between genuine physics on the one hand, and mere artefacts of particular coordinate systems on the other. This is the beauty and the subtlety of the subject. As Sidney Coleman has remarked, "In General Relativity you don't know where you are, and you don't know what time it is." The profundity of this observation should become apparent as we proceed.

For convenience, we reproduce here the Schwarzschild metric, which was obtained in eqn (6.26) in section 6:

$$ds^2 = -\Big(1 - \frac{2M}{r}\Big)\, dt^2 + \Big(1 - \frac{2M}{r}\Big)^{-1}\, dr^2 + r^2\, (d\theta^2 + \sin^2\theta\, d\phi^2)\,. \tag{10.1}$$

As remarked previously, the apparently singular behaviour of the metric at $r = 2M$ is in fact merely an artefact of a breakdown of the coordinate system, and does not actually indicate any true physical singularity at that location in the spacetime. Studying this in detail will form a large part of the discussion in this section.

By contrast, there is a genuine curvature singularity at $r = 0$, as may be seen by calculating a suitable scalar built from the Riemann tensor. The Ricci scalar is too special for demonstrating this singularity, since by construction it vanishes, as a consequence of the Ricci-flatness $R_{\mu\nu} = 0$ of the Schwarzschild solution. For the same reason, the scalar invariant $R^{\mu\nu}\, R_{\mu\nu}$ is of no use to us either, since it too vanishes by construction. The curvature singularity can be seen, however, if we calculate the scalar formed by squaring the Riemann tensor,

$$|\text{Riem}|^2 \equiv R^{\mu\nu\rho\sigma}\, R_{\mu\nu\rho\sigma}\,. \tag{10.2}$$

A somewhat lengthy, but entirely straightforward, calculation shows that this is given by

$$|\text{Riem}|^2 = \frac{48M^2}{r^6} \tag{10.3}$$

for the Schwarzschild metric. We see that this diverges like $1/r^6$ as r goes to zero. Since it is a scalar quantity, it will take the same form in all coordinate frames, and so no amount of changing from one coordinate system to another can get rid of this true singularity in spacetime

So far, we have been concerned here only with *local* considerations; writing down the metric ansatz (6.15), calculating the curvature, and then solving the vacuum Einstein equations to obtain (10.1). Now, the time has come to study the global structure of the Schwarzschild solution.

We already noted that at large distance, the Schwarzschild solution approaches Minkowski spacetime, and in fact in that large-r region it nicely approaches a Newtonian limit in which $g_{00} \to -1-2\Phi$, where $\Phi = -M/r$ is the Newtonian gravitational potential for a spherically-symmetric object of mass M.

Of much greater interest to us here is to take the Schwarzschild metric seriously even at small values of r, to see where that leads us. The first thing one notices about (10.1) is that it becomes singular at $r = 2M$. This is in some sense unexpected, since when we started out we looked for a spherically-symmetric solution that would be expected to describe the geometry outside a "point mass" located at $r = 0$. There is indeed a singularity at $r = 0$, of a rather severe nature. We saw that the metric becomes singular also at $r = 0$, but, as we shall see below, one cannot judge a solution in general relativity just by looking at singularities in the metric, because these can change drastically in different coordinate systems. There is, however, a reliable indicator as to when there is a genuine singularity in the spacetime, namely by looking at scalar invariants built from the Riemann tensor. The point about looking at scalar invariants is that they are, by definition, invariant under changes of coordinate system, and so they provide a coordinate-independent indication of whether or not there are genuine singularities. As we saw in (10.3), the scalar built from the square of the Riemann tensor indeed diverges at $r = 0$, showing that there is a genuine curvature singularity there. By contrast, the square of the Riemann tensor is perfectly finite at $r = 2M$.

Note that we were somewhat fortunate here in finding that $|\text{Riem}|^2$ was divergent at $r = 0$; this means that we can be sure that there is a genuine spacetime singularity. The converse is not necessarily true; one can encounter circumstances where the curvature is actually divergent, but $|\text{Riem}|^2$ is not. In the Schwarzschild example, $|\text{Riem}|^2$ in (10.3) is a sum of squares with positive coefficients, because there are always an even number of "0" indices on the non-vanishing components of the Riemann tensor. In more general cases, there might be components with an odd number of "0" components, and the squares of these would enter with minus signs in the calculation of $|\text{Riem}|^2$, because of the indefinite metric signature. Thus one could encounter circumstances where singular behaviour cancelled out between different components of the Riemann tensor.

Let us now turn our attention to the singular behaviour of the Schwarzschild metric (10.1) at $r = 2M$. It was decades after the original discovery of the Schwarzschild solution before this was properly understood, in in the early days people would speak of the "Schwarzschild singularity" at $r = 2M$ as if it were a genuine singularity in the spacetime. In fact, as we shall see, there is physically nothing singular at $r = 2M$; the apparent singularity in (10.1) is simply a consequence of the fact that the (t, r, θ, φ) coordinate system breaks down there. There are many physically interesting phenomena associated with this region in the spacetime, but there is no singularity. It is known, for reasons that will become clear, as an "event horizon."

The notion of a coordinate system breaking down at an otherwise perfectly regular point or region in a space is a perfectly familiar one. We can consider polar coordinates on the plane as an example, where the metric is

$$ds^2 = dr^2 + r^2\, d\theta^2 \ . \tag{10.4}$$

This metric is singular at the origin; the metric component $g_{\theta\theta}$ vanishes there, and the determinant of the metric vanishes too. But, as we well know, a transformation to Cartesian coordinates (x, y), related to (r, θ) by $x = r \cos\theta$ and $y = r \sin\theta$, puts the metric (10.4) into the standard Cartesian form $ds^2 = dx^2 + dy^2$, and now we see that indeed $r = 0$, which is now described by $x = y = 0$, is perfectly regular.

10.1 A toy example

It is worth making a little detour to consider a toy example that will perhaps help to illustrate some of the concepts that we shall encounter below when studying the global properties of the Schwarzschild black hole. Let us consider the two-dimensional spacetime metric

$$ds^2 = -dt^2 + e^{2z}\, dz^2\,. \tag{10.5}$$

Secretly, we can see that this is nothing but Minkowski spacetime with metric

$$ds^2 = -dt^2 + dx^2\,, \tag{10.6}$$

as is revealed by making the coordinate redefinition $z = \log x$. But suppose we haven't yet noticed this, and so we are studying the spacetime using the original coordinates (t, z) of (10.5). The metric (10.5) looks nonsingular for all t and all z, i.e. $-\infty \le t \le \infty$ and $\infty \le z \le \infty$, except that g_{zz} goes to zero at $z = -\infty$ and to infinity at $z = +\infty$.

We can gain further insights into the structure of the spacetime by looking at the behaviour of its geodesics. These are described, for massive geodesics, by

$$L = -\tfrac{1}{2}\dot{t}^2 + e^{2z}\,\dot{z}^2\,, \qquad L = -\tfrac{1}{2} \quad \text{on shell}\,, \tag{10.7}$$

where a dot means $d/d\tau$. The Euler-Lagrange equation $d(\partial L/\partial\dot{t})/d\tau - \partial L/\partial t = 0$ gives the first integral

$$\dot{t} = c \tag{10.8}$$

where c is a constant, and so the on-shell constraint gives

$$\dot{z}\, e^z = \pm\sqrt{c^2 - 1}\,. \tag{10.9}$$

Integrating this, we learn that, making a convenient choice of sign and origin for τ,

$$e^z = -\sqrt{c^2 - 1}\,\tau\,. \tag{10.10}$$

Thus as τ increases from some initial negative value τ_0, the particle moves in the direction of decreasing z from its initial point z_0 until it reaches $z = -\infty$ at $\tau = 0$. The crucial point is that the particle has reached $z = -\infty$ in a *finite proper time*. That is to say, a physical traveller can actually reach the "edge of the world" after a finite travel time. In such a circumstance the spacetime as originally described by the (t, z) coordinates with, in particular, $-\infty \le z \le \infty$ is said to be *geodesically incomplete*.[25]

[25]By contrast, the traveller would take an infinite proper time to get from the initial point z_0 to the other "end of the world" at $z = \infty$. Thus this does not signal any geodesic incompleteness at $z = \infty$, since no one could ever actually get there.

When one finds that a spacetime is geodesically incomplete, it is a giving a strong hint that there is something defective about the coordinate system one is using in that region. Of course we know how to remedy the situation in this case; we should define a new coordinate x by setting

$$z = \log x \,, \tag{10.11}$$

and then the metric becomes $ds^2 = -dt^2 + dx^2$ which is perfectly geodesically complete with $-\infty \le t \le \infty$ and $-\infty \le x \le \infty$. It is very revealing now to look at our solution (10.10) for the geodesic motion in terms of the new x coordinate; we have $e^z = e^{\log x} = x$, and so the solution is simply

$$x = -\sqrt{c^2 - 1}\,\tau \,. \tag{10.12}$$

This now makes perfect sense. As τ increases from the initial negative value τ_0 nothing weird happens when τ reaches 0. We don't encounter any "edge of the world" there. Instead, the x coordinate is simply falling from the (positive) starting value $x_0 = e^{z_0}$ and reaching zero at $\tau = 0$. As τ increases further, the particle (or oberver) smoothly carries on to negative values of x.

Notice, however, that negative x means that the old z coordinate becomes complex: when $x < 0$ we have

$$z = \log x = \log(-|x|) = \mathrm{i}\pi + \log(|x|) = \mathrm{i}\pi + \log(-x) \,. \tag{10.13}$$

(We have made a specific choice of branch cut here.) So when the clock in the traveller's spacecraft reaches $\tau = 0$ and then beyond to positive proper times he doesn't hit a brick wall or drop of the edge of the world. he simply discovers that the spacetime was bigger than he thought, and that his old (t, z) coordinates were not able to describe the part that he has now reached.

By changing to the (t, x) coordinates we have constructed an *analytic extension* of the original spacetime that was defined by (t, z) with $-\infty \le z \le \infty$. In fact what we have constructed, namely Minkowski spacetime, is the *maximal analytic extension* of the original one. That is to say, there is no need for any further extension and it cannot be extended any further; it is now geodesically complete.

10.2 Radial geodesics in Schwarzschild

Before getting down to a detailed study of the global structure of the Schwarzschild metric, let us pause to make sure that the discussion is not going to be purely academic. If it were the case that an observer out at large distance could never reach the region $r = 2M$, then one might question why it would be so important to study the global structure there. On the other hand, if an observer can reach it in a finite time, then it is clearly of great importance (especially to the observer!) to understand what he will find there. This is actually already a slightly subtle issue because, as we shall see, an observer who stays safely out near infinity will never see the infalling observer pass through the event horizon at $r = 2M$. However, the infalling observer himself will fall through the horizon in a finite time interval, as measured in his own frame.

Let us, therefore, calculate the motion of radially-infalling geodesics in the Schwarzschild metric. (We could consider more general geodesic motion with angular dependence too, which would be relevant for considering planetary orbits, *etc.* From the point of view of testing whether an observer crosses the event horizon, however, any non-radial component to the motion would merely be a "time-wasting" manoeuvre, counter-productive from the point of view of getting there as quickly as possible.) For radial motion, the Lagrangian (5.22) that gives the geodesic equations is

$$L = -\tfrac{1}{2}\Big(1 - \frac{2M}{r}\Big)\, \dot t^2 + \tfrac{1}{2}\Big(1 - \frac{2M}{r}\Big)^{-1}\, \dot r^2\,. \tag{10.14}$$

The Euler-Lagrange equation for t gives

$$\Big(1 - \frac{2M}{r}\Big)\, \dot t = E\,, \tag{10.15}$$

where E is a constant. The constant of the motion $L = -1/2$ then gives us the equation for infalling radial motion:

$$\dot r = -\Big(E^2 - 1 + \frac{2M}{r}\Big)^{1/2}\,, \tag{10.16}$$

where the choice of sign is determined by the fact that we are looking for the *ingoing* solution. Note that for a particle coming in from infinity the constant E must be such that $E^2 > 1$.

Suppose that at proper time τ_0 the particle is at radius $r_0 > 2M$. It follows, by integrating (10.16), that the further elapse of proper time for it to reach $r = 2M$ is given by

$$\begin{aligned} \tau_{2M} - \tau_0 &= \int d\tau = \int_{r_0}^{2M} \frac{dr}{\dot r}\,, \\ &= \int_{2M}^{r_0} \frac{dr}{\sqrt{E^2 - 1 + \frac{2M}{r}}}\,. \end{aligned} \tag{10.17}$$

This is perfectly finite, and so the ingoing particle does indeed fall through the event horizon in a finite proper time.

Notice, however, that an observer who watches from infinity will never see the particle reach the horizon. Such an observer measures time using the coordinate t itself, and so his calculation of the elapsed time will be

$$\begin{aligned} t_{2M} - t_0 &= \int dt = \int_{r_0}^{2M} \frac{\dot{t}\, dr}{\dot{r}} \ , \\ &= \int_{2M}^{r_0} \frac{E\, dr}{\left(1 - \frac{2M}{r}\right)\sqrt{E^2 - 1 + \frac{2M}{r}}} \ , \end{aligned} \tag{10.18}$$

which diverges logarithmically. In fact as the particle gets nearer and nearer the horizon the time measured in the t cooordinate gets more and more "stretched out," and radiation, or signals, from the particle get more and more red-shifted, but it is never seen to reach, or cross, the horizon. Seen from infinity, infalling observers, like old soldiers, never die; they just fade away.

10.3 The event horizon

In order to test the suspicion that $r = 2M$ is non-singular, and just not well-described by the (t, r, θ, φ) coordinate system, let us try changing variables to a different coordinate system. Of course it is not the (θ, φ) part that is at issue here, and in fact we can effectively suppress this in all of the subsequent discussion. We really need only concern ourselves with what is happening in the (t, r) plane, with the understanding that each point in this plane really represents a 2-sphere of radius r in the original spacetime. To abbreviate the writing, we can define the metric $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\varphi^2$ on the unit-radius 2-sphere. To establish notation, let us denote by $\mathbf{g}$ the original Schwarzschild metric (10.1), and denote by $\mathcal{M}$ the manifold on which it is valid, namely,

$$\mathcal{M}: \qquad r > 2M \ . \tag{10.19}$$

(Actually, there are two disjoint regions where the metric is valid, namely $0 < r < 2M$, and $r > 2M$. Since we want to include the description of the asymptotic external region far from the mass, it is natural to choose $\mathcal{M}$ as the $r > 2M$ region.) Together, we may refer to the pair $(\mathcal{M}, \mathbf{g})$ as the original Schwarzschild spacetime.

The best starting point for the sequence of coordinate transformations that we shall be using is to consider a *null* ingoing geodesic, rather than the timelike ones followed by massive particles that we considered previously. A null geodesic has the property that

$$g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = 0 \tag{10.20}$$

where λ parameterises points along its path, $x^\mu = x^\mu(\lambda)$. Note that we can't use the proper time τ as the parameter now, since $d\tau = 0$ along the path of a null geodesic (such as a light beam), and so we choose some other parameterisation in terms of λ instead. From the Schwarzschild metric (10.1) we can see that a radial null geodesic (for which $ds^2 = 0$) must satisfy

$$dt^2 = \frac{dr^2}{\left(1 - \frac{2M}{r}\right)^2} \; . \tag{10.21}$$

It is natural to introduce a new radial coordinate r^*, defined by

$$r^* \equiv \int^r \frac{dr}{1 - \frac{2M}{r}} = r + 2M \, \log\left(\frac{r - 2M}{2M}\right) \; . \tag{10.22}$$

This is known as the Regger-Wheeler radial coordinate, and it has the effect of stretching out the distance to horizon, pushing it to infinity. Sometimes r^* is called the "tortoise coordinate," although this is a bit of a misnomer since the fabled tortoise gets there in the end.

We now define advanced and retarded *null* coordinates v and u, known as "Eddington-Finkelstein coordinates:"

$$v = t + r^* \; , \qquad -\infty < v < \infty \; , \tag{10.23}$$

$$u = t - r^* \; , \qquad -\infty < v < \infty \; . \tag{10.24}$$

Radially-infalling null geodesics are described by $v =$constant, while radially-outgoing null geodesics are described by $u =$constant. If we plot the lines of constant u and constant v in the (t, r) plane, we can begin to see what is going on. (See Figure 1.) Out near infinity, we have $v \approx t + r$ and $u \approx t - r$, and the lines $v =$constant and $u =$ constant just asymptote to 45-degree straight lines of gradient -1 and $+1$ respectively. Light-cones look normal out near infinity, with 45-degree edges defined by $v =$ constant and $u =$ constant. As we get nearer the horizon, these light cones become more and more acute-angled, until on the horizon itself they have become squeezed into cones of zero vertex-angle. Inside the horizon they have tipped over, and lie on their sides.

Note that because of the way we have defined r^* in (10.22), it becomes complex when $r < 2M$, with

$$r^* = r + 2\mathrm{i}\,\pi\, M + 2M \log\left(\frac{2M - r}{2M}\right) . \tag{10.25}$$

(We have made a specific choice for the location of the branch cut of the logarithm here.) This might seem disturbing but recall that we saw something very similar in our toy example of two-dimensional Minkowski spacetime with the metric (10.5). For the present, we

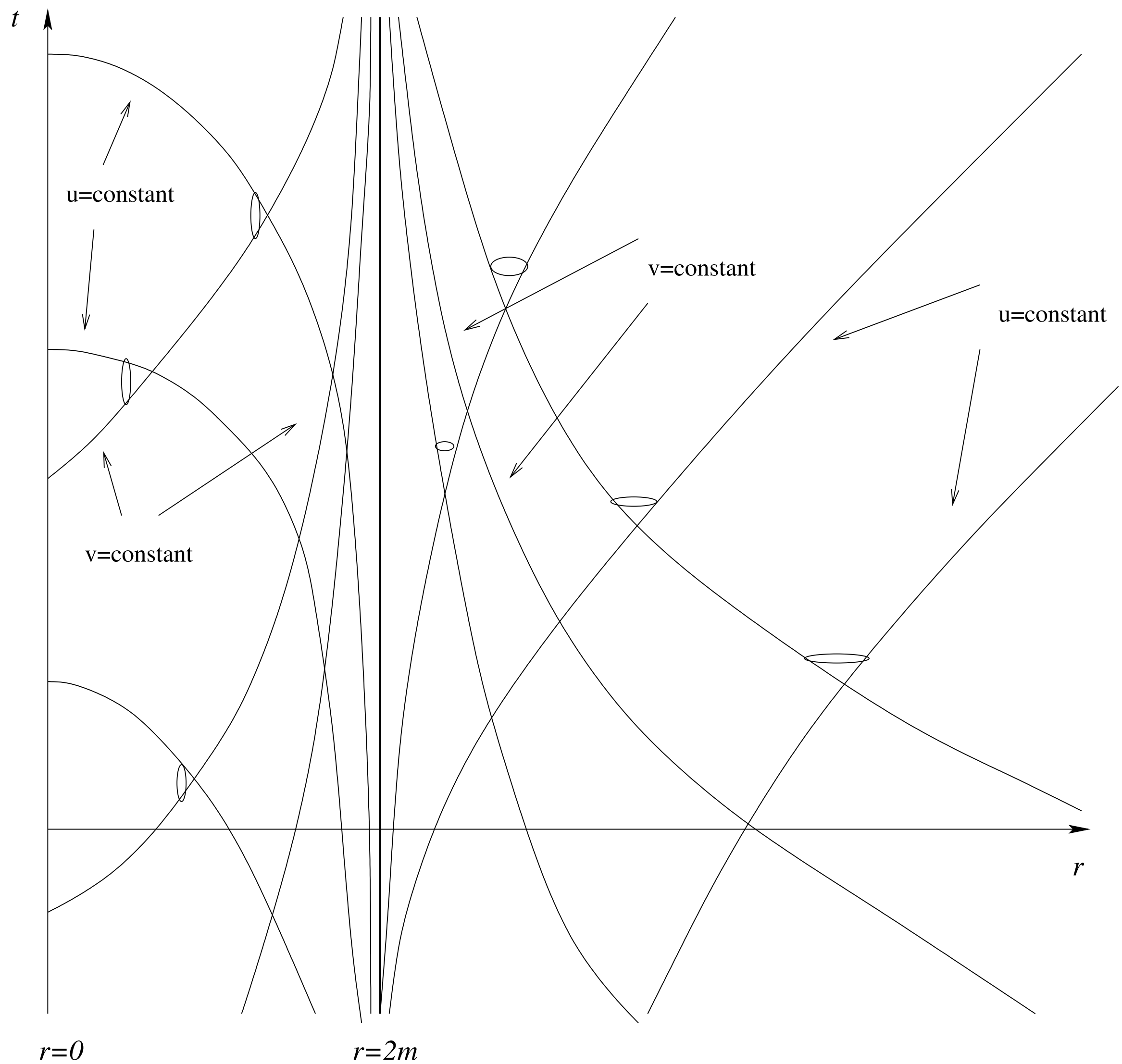

Figure 1: Schwarzschild spacetime $(\mathcal{M}, \mathbf{g})$.

can sidestep needing to worry about the additive imaginary constant in (10.25) by simply thinking of the lines $u =$constant and $v =$constant as being lines along which $du = 0$ or $dv = 0$, and then we won't ever see the additive $2\mathrm{i}\,\pi\,M$ term anyway. In other words, the two sets of curves are characterised by

$$du = dt - \frac{dr}{1 - \frac{2M}{r}} = 0\,, \qquad \text{or} \quad dv = dr + \frac{dr}{1 - \frac{2M}{r}} = 0 \tag{10.26}$$

respectively. Later, we shall see that the $2\mathrm{i}\,\pi\,M$ plays an important role, however.

The light cones are getting squeezed like this because we are trying to describe things near the horizon using the time coordinate t which is really appropriate only for an observer out at large distances. We have already seen that the use of the coordinate t to describe an infalling particle leads to the misleading impression that it never actually reaches $r = 2M$, let alone passes through it.

Guided by the behaviour of the light-cones, we are therefore led to try replacing the coordinate t in the original Schwarzschild metric (10.1) by v, using (10.23) to set $dt = dv - dr^* = dv - (1 - 2M/r)^{-1}\, dr$. Thus we find that the metric becomes

$$ds^2 = -\Big(1 - \frac{2M}{r}\Big)\, dv^2 + 2dr\, dv + r^2\, d\Omega^2\ . \tag{10.27}$$

This now has no divergence at $r = 2M$, and, because of the constant cross-term $2dr\, dv$, its inverse is perfectly finite there too; in other words, the metric is non-singular at $r = 2M$, and in fact it is well defined for all $r > 0$ and for all v with $-\infty \le v \le \infty$. We can now plot another spacetime diagram, where we use v and r as the coordinates on the plane. Since we know that out near infinity the $v =$constant lines are well thought-of as being at 45-degrees with slope -1, it is natural to choose this as our plotting scheme everywhere. This can be achieved by introducing a time-like coordinate t', defined by

$$t' \equiv v - r\,, \tag{10.28}$$

and using this as the coordinate on the vertical axis of the spacetime diagram. This gives us the picture shown in Figure 2. We see now that the light-cones do not degenerate on the horizon. They do, however, tilt over more and more as one approaches the horizon, until at $r = 2M$ itself they have tipped so that the future light-cone lies entirely within the direction of decreasing r. In fact $r = 2M$ is a null surface, and the spacetime is not time symmetric. The surface $r = 2M$ acts as a one-way membrane; future-directed timelike and null paths can cross only in one direction, from $r > 2M$ to $r < 2M$. They reach the singularity at $r = 0$ in a finite proper time or affine distance. Past-directed timelike or null curves in the region $0 < r < 2M$, on the other hand, cannot reach the singularity at $r = 0$. In other words a future-directed null ray has only one way to go; inwards. The fate of a massive particle, whose path must lie inside the null cone, is the same.

Let us denote by $\mathbf{g}'$ the metric (10.27). Since there is no metric singularity at $r = 2M$, we see that the range of the radial coordinate r, which was restricted to the region $r > 2M$ in the original spacetime $(\mathcal{M}, \mathbf{g})$ with metric $\mathbf{g}$ given by (10.1), can now be extended to cover the entire region $r > 0$. Thus we have an analyic extension $(\mathcal{M}', \mathbf{g}')$ of the Schwarzschild spacetime, where

$$\mathcal{M}' : \qquad r > 0\ . \tag{10.29}$$

There is an alternative analytic extension of $(\mathcal{M}, \mathbf{g})$ that we can consider, where we substitute for the time coordinate using the retarded Eddington-Finkelstein coordinate u

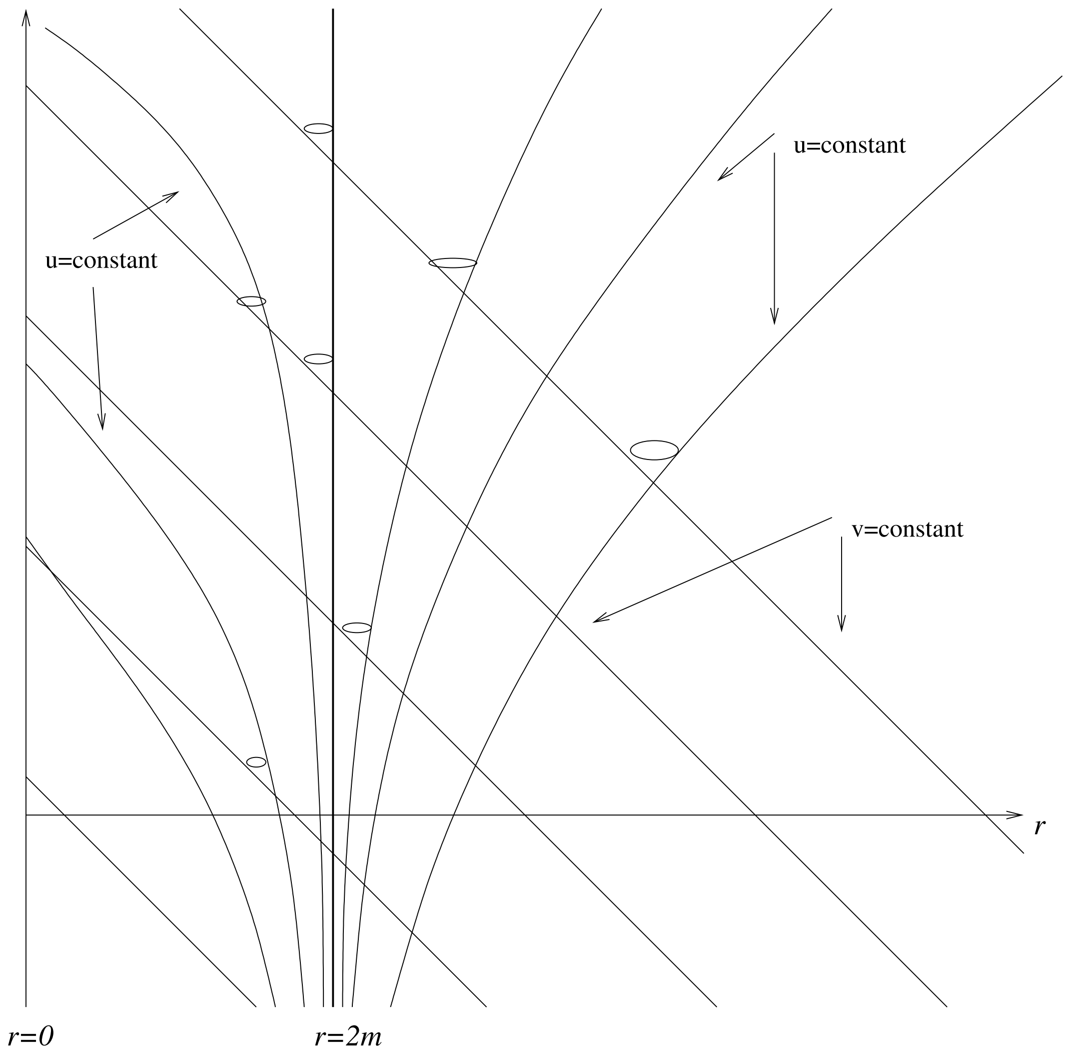

Figure 2: Schwarzschild spacetime $(\mathcal{M}', \mathbf{g}')$. The vertical axis is $t' = v - r$ here.

defined in (10.24), rather than the advanced coordinate v. This gives another form for the Schwarzschild metric, which we shall call $\mathbf{g}''$:

$$ds^2 = -\Big(1 - \frac{2M}{r}\Big)\,du^2 - 2du\,dr + r^2\,d\Omega^2 \ . \tag{10.30}$$

This is again nonsingular at $r = 2M$, and is analytic on a manifold $\mathcal{M}''$ with

$$\mathcal{M}'' : \qquad r > 0 \ . \tag{10.31}$$

However, although the region of analyticity here is the same as for the extension $\mathcal{M}'$, the two analytic extensions $\mathcal{M}'$ and $\mathcal{M}''$ are quite different. The time asymmetry in the $\mathcal{M}''$ manifold is the opposite of that in $\mathcal{M}'$. The surface $r = 2M$ is again null, but this time it is a one-way membrane acting in the opposite direction; it is now only past-directed timelike or null curves that can cross from $r > 2M$ to $r < 2M$. With the vertical axis now being a new time-like coordinate $t^{''}$, defined now by

$$t'' \equiv u + r\,, \tag{10.32}$$

this is depicted in Figure 3.

It is clear that neither of the analytic extensions $(\mathcal{M}', \mathbf{g}')$ or $(\mathcal{M}'', \mathbf{g}'')$ by itself captures the entire structure of the full Schwarzschild geometry. We can, however, go one stage further and construct a larger extension of the spacetime by using both the v and u coordinates, in place of t and r. Thus from (10.1), (10.23) and (10.24) we obtain the metric

$$ds^2 = -\Big(1 - \frac{2M}{r}\Big)\,dv\,du + r^2\,d\Omega^2 \ . \tag{10.33}$$

Here, we are now using r simply as a shorthand symbol for the quantity defined by

$$\tfrac{1}{2}(v-u) = r + 2M\,\log\Big(\frac{r-2M}{2M}\Big) \ . \tag{10.34}$$

Now define new coordinates V and U, known as Kruskal coordinates, by

$$V = e^{\frac{v}{4M}}\ , \qquad U = -e^{-\frac{u}{4M}} \ . \tag{10.35}$$

At this stage, we see that we must have

$$V > 0\,, \qquad U < 0\,, \tag{10.36}$$

in order for u and v to be real. The quantity r is now defined implicitly through the equation

$$UV = -e^{\frac{v-u}{4M}} = -e^{\frac{r^*}{4M}} = -e^{\frac{r}{2M}}\,\Big(\frac{r-2M}{2M}\Big)\,. \tag{10.37}$$

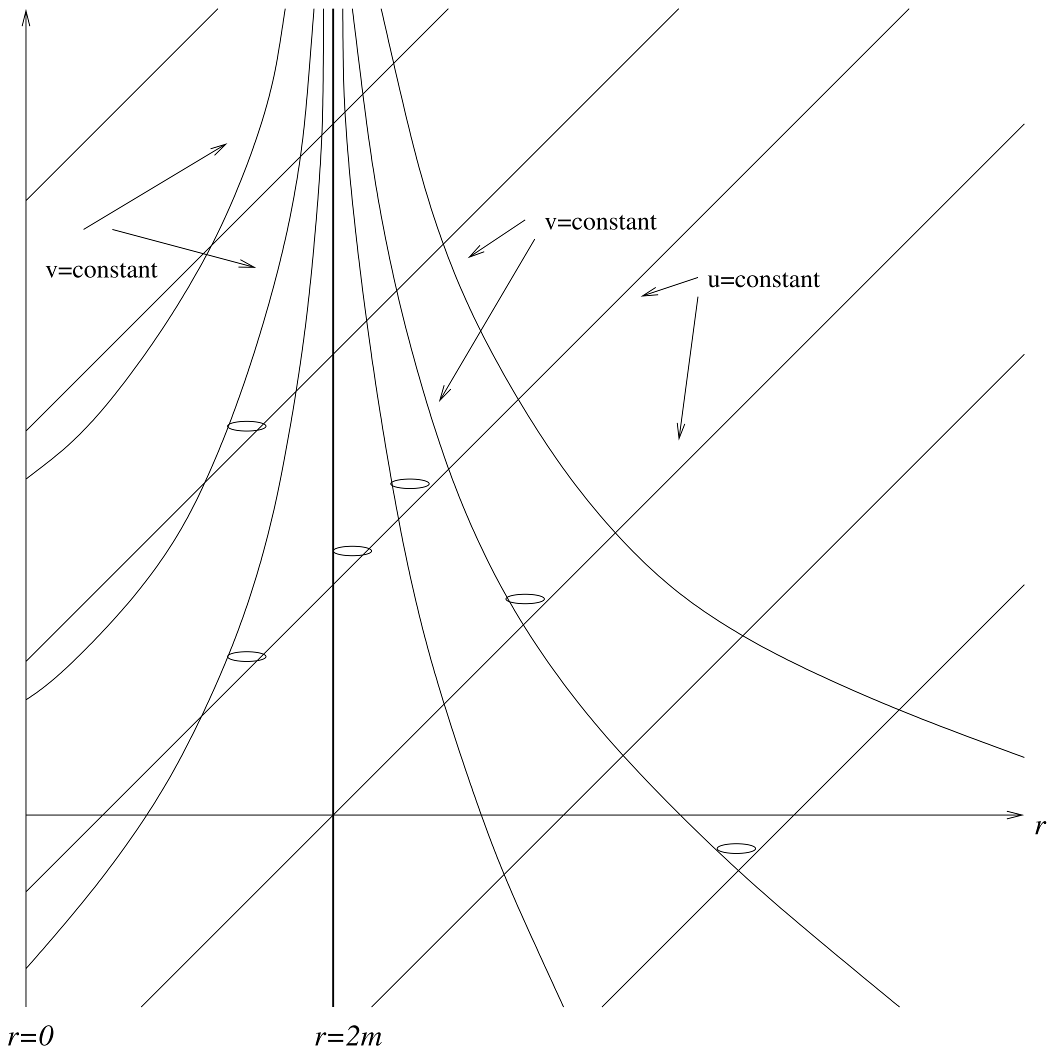

Figure 3: Schwarzschild spacetime $(\mathcal{M}'', \mathbf{g}'')$. The vertical axis is $t'' = u + r$ here.

Note, however, that the U and V coordinates need no longer be restricted by the condition (10.36), and indeed the region $r < 2M$ precisely corresponds to $UV > 0$. The coordinates U and V are now each allowed to range independently over the entire real line:

$$-\infty \le U \le \infty\,, \qquad -\infty \le V \le \infty \tag{10.38}$$

In terms of U and V, and the analytic extension in which r is now taken to be defined implicitly by (10.37), we arrive at the metric $\mathbf{g}^*$, given by

$$ds^2 = -\frac{32M^3\, e^{-\frac{r}{2M}}}{r}\, dV\, dU + r^2\, d\Omega^2\ , \tag{10.39}$$

As one can easily verify, with r now defined implicitly by (10.37) we still find that the metric (10.39) satisfies the vacuum Einstein equations. (This must, of course, be the case since we have merely performed coordinate transformations, and if a tensor, such as $R_{\mu\nu}$, vanishes in one coordinate frame it must vanish in all coordinate frames.) The restrictions (10.36) on the signs of U and V are now removed, which means that we have effectively quadrupled the extent of the region over which the metric is defined.

It is useful also to define

$$\tilde t = \tfrac{1}{2}(V+U)\ , \qquad \tilde x = \tfrac{1}{2}(V-U)\ , \tag{10.40}$$

in terms of which the metric $\mathbf{g}^*$ becomes

$$ds^2 = -\frac{16M^3\, e^{-\frac{r}{2M}}}{r}\, (-d\tilde t^2 + d\tilde x^2) + r^2\, d\Omega^2\ . \tag{10.41}$$

On the manifold $\mathcal{M}^*$, defined by the coordinates $(\tilde t, \tilde x, \theta, \varphi)$ such that the solution r of (10.37) obeys $r > 0$, the metric $\mathbf{g}^*$ given by (10.41) has components that are analytic. We may draw a new spacetime diagram, given in Figure 4, to represent the manifold $\mathcal{M}^*$. The pair $(\mathcal{M}^*, \mathbf{g}^*)$ is the *maximal analytic extension* of the original Schwarzschild solution. The region I, defined by $\tilde x > |\tilde t|$, is isometric to the original Schwarzschild spacetime $(\mathcal{M}, \mathbf{g})$, for which $r > 2M$. The region $\tilde x > -\tilde t$, corresponding to regions I and II in Figure 4, is isometric to the advanced analytic extension $(\mathcal{M}', \mathbf{g}')$. Similarly the region $\tilde x > \tilde t$, corresponding to regions I and II$'$ in Figure 4, is isometric to the retarded analytic extension $(\mathcal{M}'', \mathbf{g}'')$. (I have no idea why there are curious bumps in some of the $r =$ constant curves in this figure. It appears to be some anomaly in exporting a figure constructed in xfig as a pdf file.)

There is also a region I$'$, defined by $\tilde x < -|\tilde t|$, which again is isometric to the exterior spacetime $(\mathcal{M}, \mathbf{g})$. This is another asymptotically-flat universe, separated from "our" universe by a "throat" where the area $4\pi\, r^2$ of the 2-spheres in the (θ, φ) directions has shrunk

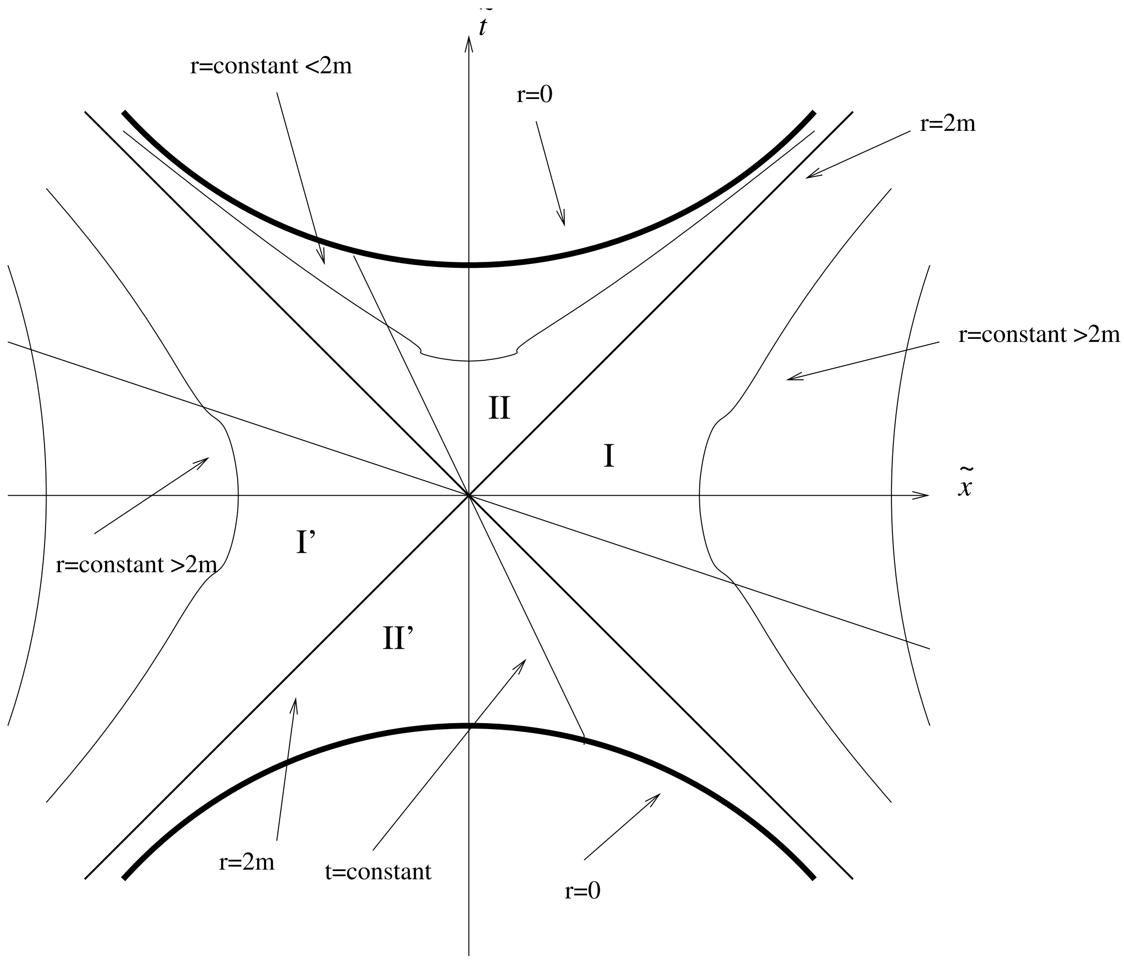

Figure 4: Schwarzschild spacetime $(\mathcal{M}^*, \mathbf{g}^*)$. The U axis runs along the diagonal from bottom right to top left. The V axis runs along the diagonal from bottom left to top right.

down to a minimum value of $16\pi M^2$ (i.e. $r = 2M$), and then expanded out again. In fact one can see from Figure 4 that the regions I′ and II are isometric to the advanced Finkelstein extension of region I′, and that the regions I′ and II′ are isometric to the retarded Finkelstein extension of I′. No timelike or null curves can cross from region I to region I′; in fact any such curve that crosses from I′ into the region where $r < 2M$ will necessarily end up at the (upper) singularity at $r = 0$. So neither material objects, nor information, can cross from I′ to I.

It is instructive to look at the Killing vector

$$K = \frac{\partial}{\partial t} \tag{10.42}$$

in a little more detail. K is timelike outside the horizon, that is, $K^\mu K_\mu = -(1 - 2M/r)$, which is negative when $r > 2M$. It asymptotically satisfies $K^\mu K_\mu \to -1$ as r goes to infinity, which implies that it is the generator of canonically-normalised time translations in the asymptotic region at large r. K becomes null on the horizon, i.e. $K^\mu K_\mu = 0$ at $r = 2M$. In terms of the Eddington-Finkelstein coordinates u and v it is given by

$$K = \frac{\partial}{\partial u} + \frac{\partial}{\partial v}\,, \tag{10.43}$$

and in terms of the Kruskal coordinates U and V, it is given by

$$K = \frac{1}{4M}\Big(V\,\frac{\partial}{\partial V} - U\,\frac{\partial}{\partial U}\Big)\,. \tag{10.44}$$

Now, the horizon is located on the entirity of the two 45-degree cross-lines on the Kruskal diagram depicted in figure 4, that is to say, on the line $U = 0$ for all V, and on the line $V = 0$ for all U. There is a bifurcation point at $U = V = 0$ on the diagram (at the origin), where the two disjoint 45-degree lines describing the horizon intersect. A black hole with this kind of geometry is said to have a *bifurcate horizon.* Note from (10.44) that the Killing vector K actually vanishes at the bifurcation point. (Of course, as always, there is really a suppressed 2-sphere of radius r sitting over each point in the two-dimensional diagram.)

Finally, in our analysis of the maximal analytic extension of the Schwarzschild solution we can make one further transformation of the coordinates, which has the effect of bringing infinity in to a finite distance, so that the entire spacetime can be fitted onto the back of a postage stamp (times a 2-sphere sitting over each point, of course). We do this by making use of the arctangent function, which has the property of mapping the entire real line into the interval between $-\frac{1}{2}\pi$ and $+\frac{1}{2}\pi$. Thus we define new coordinates $\widetilde V$ and $\widetilde U$, in place of V and U, where

$$\widetilde V = \arctan V\ , \qquad \widetilde U = \arctan U\ , \tag{10.45}$$

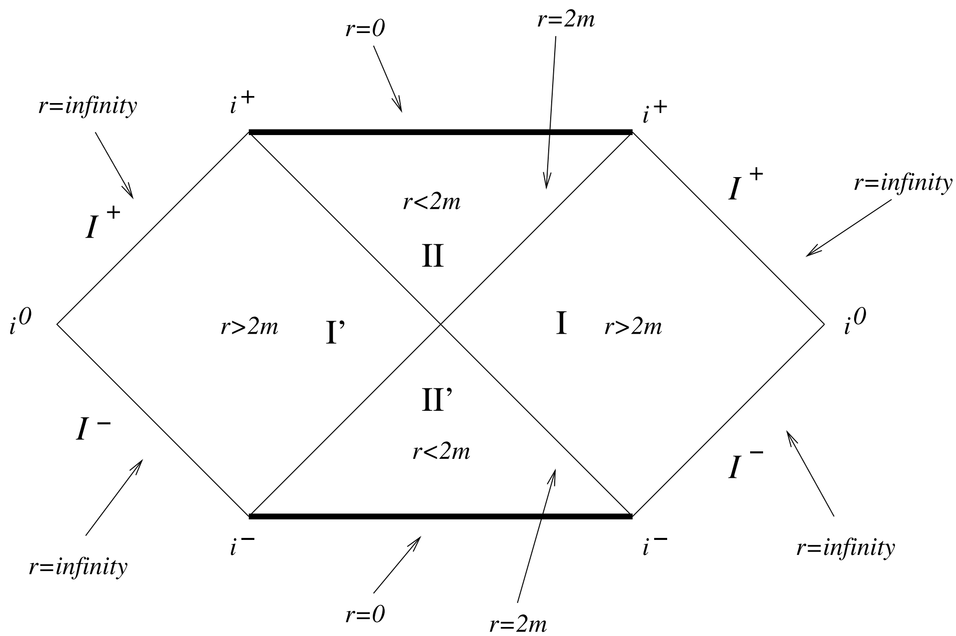

Figure 5: The Penrose diagram for the Schwarzschild spacetime $(\mathcal{M}^*, \mathbf{g}^*)$. The $\widetilde{U}$ axis runs along the diagonal from bottom right to top left, while the $\widetilde{V}$ axis runs along the diagonal from bottom left to top right. (The slanting I^+ and I^- should be $\mathscr{I}^+$ and $\mathscr{I}^-$, but xfig (or the user!) wasn't able to achieve that.)

where

$$-\pi < \widetilde{V} + \widetilde{U} < \pi \,, \qquad \text{and} \qquad -\tfrac{1}{2}\pi < \widetilde{V} < \tfrac{1}{2}\pi \,, \quad -\tfrac{1}{2}\pi < \widetilde{U} < \tfrac{1}{2}\pi \,. \tag{10.46}$$

With this mapping, the Kruskal maximal extension of Figure 4 turns into the so-called Penrose diagram for the Schwarzschild spacetime, depicted in Figure 5. Note that we can express r in terms of $\widetilde{U}$ and $\widetilde{V}$ as

$$\tan \widetilde{V}\, \tan \widetilde{U} = -\frac{(r-2M)}{2M}\, e^{\frac{r}{2M}} \,. \tag{10.47}$$

Essentially all that has been done in this last transformation is to bring infinity in to a finite distance. However, by doing so a new feature has come to light, namely that there are a number of different kinds of asymptotic infinity. These can be characterised as the places where the various different kinds of particles come from, and where they end up. Thus we have the places denoted by i^-, which is where massive particles (which follow timelike geodesics) came from at $r = \infty$ in the distant past, and i^+, which is where they end up at $r = \infty$ in the distant future, if they are fortunate enough to have followed paths that keep them away from the event horizon and the singularity of the black hole. The regions denoted by $\mathscr{I}^-$ (and pronounced, regretfully, as "scri") are likewise the places that massless particles (following null geodesics) came from at $r = \infty$ in the distant past, and $\mathscr{I}^+$ is where the lucky ones end up at in the distant future. (Note that in Figure 5 the symbols for scri, appearing on the outer diagonal borders of the diagram, appear just as italic I, owing to the limited xfig skills of the author.) Finally, hypothetical particles of negative mass-squared (tachyons) would follow spacelike geadesics, and these begin and end at i^0. The regions $i^\pm$ are known as future and past timelike infinity, the regions $\mathscr{I}^\pm$ are known as future and past null infinity, and i^0 is known as spacelike infinity. Of course one should remember that the effect of having squeezed the entire universe onto a postage stamp is that one can gain a false impression of distance. In particular, for example, although i^0 looks like a single point in the Penrose diagram, it is actually an entire infinite region. (This is over and above the now-familiar fact that each point in any of our two-dimensional spacetime diagrams really represents a 2-sphere.) Likewise, the "points" labelled i^- and i^+ are infinite in extent. Furthermore, another aspect of the Penrose diagram is that i^+ and i^-, at $r = \infty$, appear to be coincident with the ends of the horizontal $r = 0$ lines, which represent the spacelike curvature singularities. This is again an unfortunate impression created by the foreshortening resulting from the arctangent mapping, and they are in actuality infinitely separated. In the words of Douglas Adams, in *The Hitchhiker's Guide to the Galaxy*, "The universe is a big place."

It should be remarked that the discussion in this section has been somewhat of an idealisation, and the maximal analytic extension of the Schwarzschild solution is not what would arise in a physical situation where a black hole formed as a result of gravitational collapse. In particular, the "south-west" part of the Penrose diagram would be missing in a realistic example where a star collapsed to form a black hole. This is perhaps just as well, because the south-west part of the diagram really describes a "white hole" from our point of view as dwellers in the eastern part of the diagram; particles and null rays can come out of it, but they cannot go in. A Penrose diagram for a star that collapses to form a Schwarzschild black hole is depicted in Figure 6. The shaded area represents the inside of the star.

10.4 Global structure of the Reissner-Nordström solution

The Reissner-Nordström solution that we obtained previoualy has some features in common with the Schwarzschild solution. There are also some important differences, and, as we shall see, the global structure of the maximal analytic extension of the Reissner-Nordström

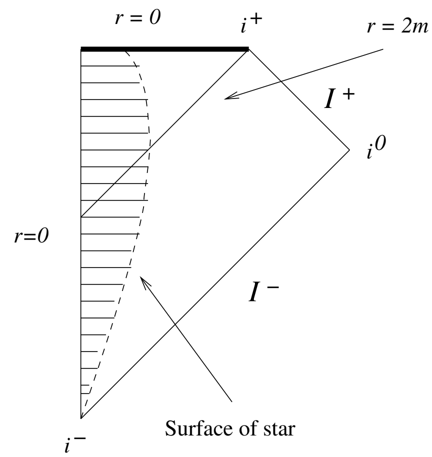

Figure 6: The Penrose diagram for a collapsing spherically-symmetric star. (Again, $I^{\pm}$ should be $\mathscr{I}^{\pm}$.)

spacetime is quite different from that of the Schwarzschild spacetime.

First, we give again the Reissner-Nordström metric:

$$ds^2 = -\Big(1 - \frac{2M}{r} + \frac{q^2}{r^2}\Big)\, dt^2 + \Big(1 - \frac{2M}{r} + \frac{q^2}{r^2}\Big)^{-1}\, dr^2 + r^2\, d\Omega^2\,, \tag{10.48}$$

where, as usual, $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\varphi^2$ is the metric on the unt 2-sphere. Like Schwarzschild, the metric is free of curvature singularities everywhere except at $r = 0$, and in fact a straightforward calculation shows that

$$|\text{Riem}|^2 = \frac{48M^2}{r^6} - \frac{96\, q^2\, M}{r^7} + \frac{56q^4}{r^8}\ . \tag{10.49}$$

The function $(1 - \frac{2M}{r} + \frac{q^2}{r^2})$ appearing in the metric has roots, possibly complex, of the form $r = r_{\pm}$, where

$$r_+ = M + \sqrt{M^2 - q^2} \qquad r_- = M - \sqrt{M^2 - q^2}\ . \tag{10.50}$$

Consequently, we have three different regimes to consider, namely $q^2 < M^2$, $q^2 = M^2$ and $q^2 > M^2$. For $q^2 < M^2$ there are two distinct real, positive, roots; these coalesce to one double root at $r = M$ if $q^2 = M^2$. Finally, if $q^2 > M^2$, the two roots are complex.

Let us first calculate the analogue of the Regge-Wheeler "tortoise" coordinate for the Reissner-Nordström metric. In other words, we solve for radial null geodesics in the Reissner-Nordström geometry, with $0 = ds^2 = -(1 - \frac{2M}{r} + \frac{q^2}{r^2})\, dt^2 + (1 - \frac{2M}{r} + \frac{q^2}{r^2})^{-1}\, dr^2$. It follows by integrating this that we shall have ingoing and outgoing null geodesics with $r^* = -t$ and $r^* = +t$ respectively, where

$$q^2 < M^2: \qquad r^* = r + \frac{r_+^2}{r_+ - r_-}\, \log(r - r_+) - \frac{r_-^2}{r_+ - r_-}\, \log(r - r_-)\ , \tag{10.51}$$

$$q^2 = M^2 : \qquad r^* = M \log\Big((r-M)^2\Big) - \frac{M^2}{r-M} \ , \tag{10.52}$$

$$q^2 > M^2 : \qquad r^* = r + M \log\Big((r-M)^2 + q^2 - M^2\Big) - \frac{2(q^2-2M^2)}{\sqrt{q^2-M^2}} \arctan\Big[\frac{r-M}{\sqrt{q^2-M^2}}\sqrt{q^2-M^2}\Big] \ . \tag{10.53}$$

We can dispose of the case $q^2 > M^2$ rather easily. The roots $r_\pm$ are complex, and hence the function $(1-\frac{2M}{r}+\frac{q^2}{r^2})$ has no zeros for $r > 0$. This means that the curvature singularity at $r = 0$ is not hidden behind an horizon, and it can in fact be seen from infinity. This can be demonstrated by looking at the r^* coordinate given in (10.53). We see that an outgoing null geodesic, which will satisfy $r^* = t$, requires only a finite amount of coordinate time to travel from $r = 0$ to any finite distance r. In other words, one can stand at a safe distance from the singularity and look at it. More technically, we can say that null geodesics can emanate from the singularity and end up at $\mathscr{I}^+$. When this circumstance arises, the singularity is called a *Naked Singularity*. By contrast, in the Schwarzschild solution, we saw that the singularity was hidden behind the event horizon at $r = 2M$, and no timelike or null curves could pass from $r = 0$ to the "outside." In the 1960's a conjecture was formulated, known as the "Cosmic Censorship Hypothesis," which asserted that no physically-realistic collapsing matter system could ever end up having naked singularities; they would always be decently clothed behind event horizons. This has subsequently been proven. In particular, it can be shown that no realistic system can evolve to give a $q^2 > M^2$ Reissner-Nordström black hole. In the dimensionless natural units which we are using it is sometimes easy to forget what the scales of the various quantities are. It is worth remarking, therefore, that if a macroscopic black hole with $q^2 > M^2$ did exist, it would be a fearsome object carrying a gargantuan amount of charge.

Let us postpone the discussion of the intermediate case $q^2 = M^2$ for now, and look next at the situation when $q^2 < M^2$. The function $(1 - \frac{2M}{r} + \frac{q^2}{r^2})$ now has two distinct, real, positive, roots $r_\pm$, given by (10.50). This means that there are in fact two distinct horizons; the *outer horizon* at $r = r_+$, and the *inner horizon* at $r = r_-$. These mark the boundaries where the function $(1 - \frac{2M}{r} + \frac{q^2}{r^2})$ passes through zero and changes sign, implying that the time coordinate t is spacelike for $r_- < r < r+$, while it is genuinely timelike for $r > r_+$ and for $0 < r < r_-$. We may short-circuit some of the intermediate steps paralleling our discussion for the Schwarzschild metric, and first go directly to the double-null coordinates

$$v = t + r^* \ , \qquad u = t - r^* \ , \tag{10.54}$$

in terms of which the Reissner-Nordström metric becomes

$$ds^2 = -\Big(1 - \frac{2M}{r} + \frac{q^2}{r^2}\Big)\, dv\, du + r^2\, d\Omega^2 \ . \tag{10.55}$$

At this stage, things start to get a little tricky. First, to simplify the formulae a bit, let us define two constants $\kappa_\pm$, by

$$\kappa_\pm = \frac{r_\pm - r_\mp}{2r_\pm^2} \ . \tag{10.56}$$

The expression for the r^* coordinate (10.51) now becomes

$$r^* = r + \frac{1}{2\kappa_+}\, \log(r - r_+) + \frac{1}{2\kappa_-}\, \log(r - r_-) \ . \tag{10.57}$$

Now introduce coordinates V_+ and U_+, defined by

$$V_+ = e^{\kappa_+ v} \ , \qquad U_+ = -e^{-\kappa_+ u} \ . \tag{10.58}$$

These are analogous to the Kruskal coordinates (V, U) that we used in the Schwarzschild maximal analytic extension. Note that

$$V_+\, U_+ = -(r - r_+)\, (r - r_-)^{\kappa_+/\kappa_-}\, e^{2\kappa_+ r} \ , \quad dV_+\, dU_+ = -\kappa_+^2\, V_+\, U_+\, dv\, du \ , \tag{10.59}$$

$$\tag{10.60}$$

so $V_+\, U_+$ is negative wher $r > r_+$ and positive when $r_- < r < r_+$.

Substituting into (10.55), we see that the metric becomes

$$ds^2 = -\frac{(r - r_-)^{1-\kappa_+/\kappa_-}}{\kappa_+^2\, r^2}\, e^{-2\kappa_+ r}\, dV_+\, dU_+ + r^2\, d\Omega^2 \ , \tag{10.61}$$

and so it is non-singular for $r > r_-$, with a coordinate singularity at $r = r_-$. In fact these (V_+, U_+) coordinates cover a region looking very like the Kruskal diagram (Figure 4) for Schwarzschild, except that the genuine $r = 0$ curvature singularity in Figure 4 is now relabelled as the $r = r_-$ coordinate singularity, and the $r = 2M$ lines in Figure 4 become $r = r_+$. This is depicted in Figure 7.

Unlike Schwarzschild, where the Kruskal coordinates (U, V) covered the entire region $r > 0$, here in Reissner-Nordström the (U_+, V_+) coordinates only cover the region $r > r_-$. We need another coordinate system to cover the rest of the region with $r > 0$. To do this, we define another pair of Kruskal-type coordinates, which we shall call (V_-, U_-), where

$$\begin{aligned} V_- &= e^{\kappa_- \tilde v} \ , \qquad U_- = -e^{-\kappa_- \tilde u} \ , \qquad \tilde v = t + \tilde r^* \ , \qquad \tilde u = t - \tilde r^* \ , \\ \tilde r^* &= r + \frac{r_+^2}{r_+ - r_-}\, \log(r_+ - r) - \frac{r_-^2}{r_+ - r_-}\, \log(r_- - r) \ , \end{aligned} \tag{10.62}$$

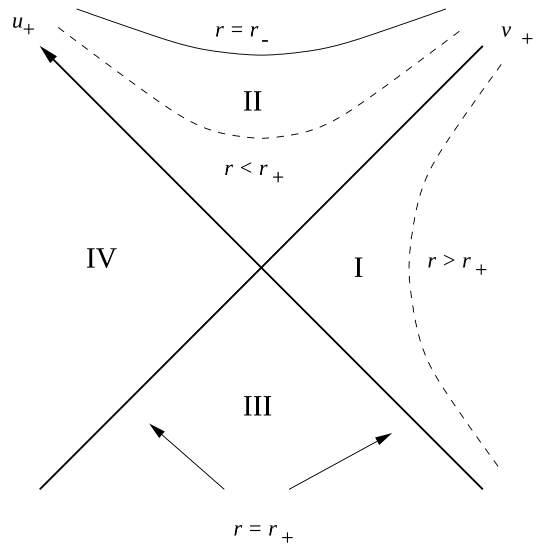

Figure 7: The region $r > r_-$ in Reissner-Nordström.

(note that relative to the definition of r^* in (10.57), a different constant of integration has been chosen here) and so

$$V_-\,U_- = -(r_- - r)\,(r_+ - r)^{\kappa_-/\kappa_+}\,e^{2\kappa_-\,r}\,,\qquad dV_-\,dU_- = -\kappa_-^2\,V_-\,U_-\,dv\,du\,. \tag{10.63}$$

Note that these coordinates are well defined for $r < r_+$, and that $V_-\,U_-$ is positive for $r_- < r < r_+$ and negative for $0 < r < r_-$. In terms of (V_-, U_-), the Reissner-Nordström metric becomes

$$ds^2 = -\frac{(r - r_+)^{1-\kappa_-/\kappa_+}}{\kappa_-^2\,r^2}\,e^{-2\kappa_-\,r}\,dV_-\,dU_- + r^2\,d\Omega^2\,, \tag{10.64}$$

This is non-singular for $r < r_+$, with a coordinate singularity at $r = r_+$. Crucially, since $r_+ > r_-$, this means that the (V_+, U_+) and $V_-, U_-)$ coordinate patches overlap in the region $r_- < r < r_+$. The Kruskal-type diagram for the (V_-, U_-) coordinates is depicted in Figure 8. Now, the two main diagonals represent $r = r_+$, and the singularity at $r = 0$ corresponds to the two vertical arcs on the left and right hand sides of the diagram. The crucial point is that there is the region of overlap between the validity of the (V_+, U_+) and the (V_-, U_-) coordinates, when $r_- < r < r_+$. This means that region II in Figure 7 is actually the same as region II in Figure 8. On the other hand, region III in Figure 7 is distinct from region III′ in Figure 8. However, since region II in Figure 7 connects to an exterior spacetime in the past (namely regions I, III and IV), it follows by time-reversal invariance that region III′ in Figure 8 must connect to an exterior spacetime in its future. This argument then repeats indefinitely, so that we must go on stacking up copies of Figure 7, then Figure 8, then Figure 7 again, and so on, into the infinite past and future.

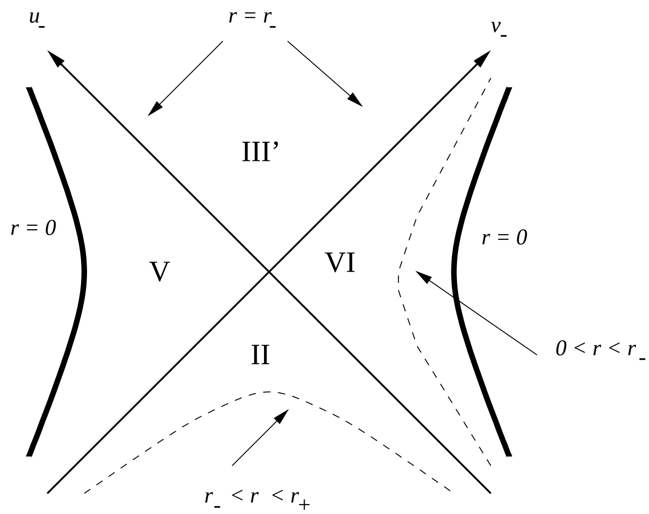

Figure 8: The region $0 < r < r_+$ in Reissner-Nordström.

If we now make arctangent transformations of the kind we used for Schwarzschild, we can make an entire Figure 7 plus Figure 8 pair fit onto a finite-sized piece of paper. However, since we have to stack up an infinite number of such pairs, we will still have a Penrose diagram that streches off to infinity along the vertical axis. We might say that if Schwarzschild spacetime can be fitted onto a postage stamp, then for Reissner-Nordström we need an infinite roll of stamps. This is depicted in Figure 9.

The most striking difference between the Reissner-Nordström and the Schwarzschild maximal analytical extensions is that for Reissner-Nordström, the curvature singularities at $r = 0$ are *timelike*, rather than spacelike. This means that an infalling timelike curve can in fact avoid the singularity, and come out into another asymptotic region. For example, in Figure 9 a particle (or observer) can start in region I, pass through regions II, VI and III$'$, and come out into region I$'$. There is no possibility of returning, however, so if we inhabited region I we could never receive reports of what was happening in region I$'$. By the same token, however, it would be possible in principle for an observer to enter our region I from region II, having started out on the next "postage stamp" down on the roll. Such an observer would emerge from the outer horizon of the black hole. One should really view the $r = r_+$ boundary between regions II and I as the outer horizon of a white hole, in fact, since future-directed particles or null rays can only come out of it; they cannot cross inwards. Again, as in the Schwarzschild spacetime of the previous chapter, one should be cautious about taking the entire maximal analytic extension too seriously as a physical spacetime, since a realistic gravitational collapse will not give rise to the entire diagram.

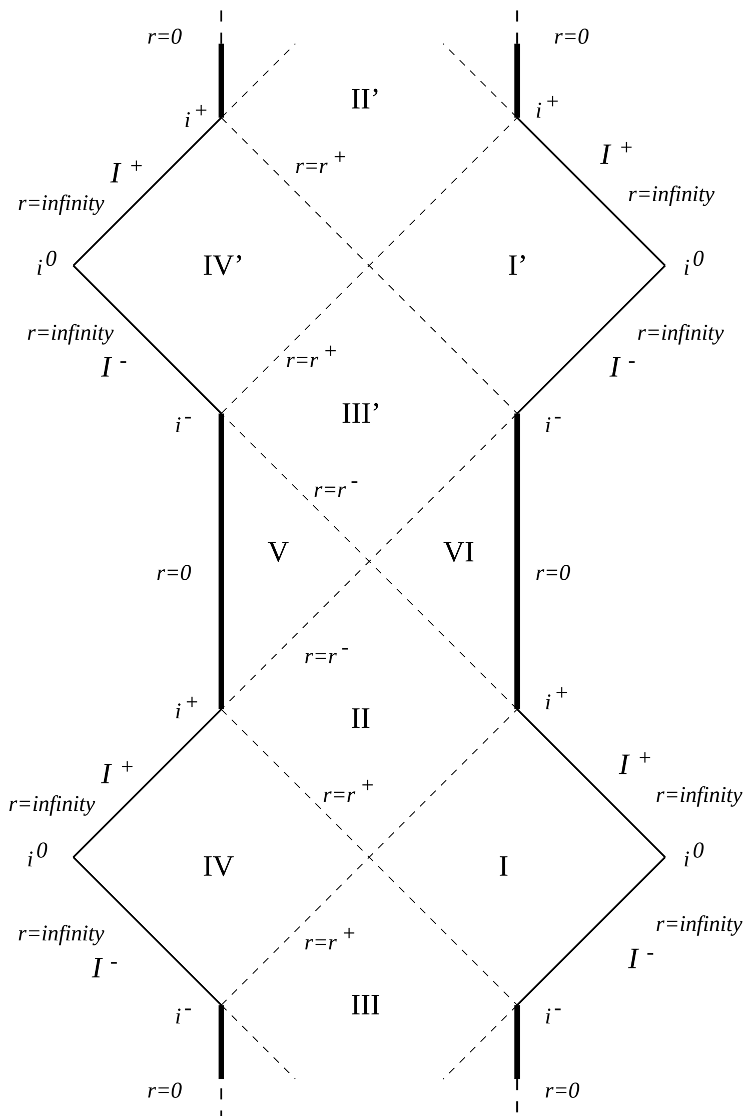

Figure 9: The maximal analytic extension of Reissner-Nordström. ($I^{\pm}$ are again $\mathscr{I}^{\pm}$.)

The remaining case to consider is when $q^2 = M^2$. We see from (10.50) that the inner and outer horizons now coalesce, at $r = M$. The metric in this limit is known as the *Extremal Reissner-Nordström solution*, and in terms of the original coordinates it takes the form

$$ds^2 = -\Big(1 - \frac{M}{r}\Big)^2\, dt^2 + \Big(1 - \frac{M}{r}\Big)^{-2}\, dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)\ . \tag{10.65}$$

This is singular at $r = M$, and so in the now familiar way, we change first to the appropriate ingoing Eddington-Finkelstein type coordinates (v, r), where $v = t + r^*$ and r^* is defined in (10.52). This turns the metric into the form

$$ds^2 = -\Big(1 - \frac{M}{r}\Big)^2\, dv^2 + 2dv\, dr + r^2\, d\Omega^2\ , \tag{10.66}$$

where again we use the abbreviated notation $d\Omega^2$ for the metric on the unit 2-sphere. This is non-singular for all $r > 0$, including, in particular, the horizon at $r = M$. As usual, one can easily show that infalling timelike geodesics can reach and cross the horizon in a finite proper time.

The analysis of the maximal analytic extension proceeds in a similar fashion to the previous discussion for $q^2 < M^2$. Essentially all that changes is that region II and its copies II$'$, *etc.* all disappear, since r_- and r_+ are now both equal to M. Thus we arrive at the maximal analytic extension depicted in Figure 10. This spacetime with $q = M$ is known as the extremal Reissner-Nordström solution. Note that the points marked by a "p" on the left-hand vertical axis in Figure 10 are actually at $r = \infty$, and not at $r = 0$. This is again one of the penalties exacted upon those who would presume to fit the universe onto a scrap of paper.

Note, incidentally, that the horizon at $r = M$, like all those that we have encountered, has the property of being a null surface. A null surface is defined as follows. Suppose we have a surface, or hypersurface, defined by $f(x) = 0$, where x represents the spacetime coordinates x^μ. It follows that the 1-form df, with components $\partial_\mu f$, will be perpendicular to the surface. If one now calculates the norm of this covector, namely $|df|^2 \equiv g^{\mu\nu}\, \partial_\mu f\, \partial_\nu f$, then the surface is defined to be null, timelike or spacelike according to whether this norm is zero, positive or negative. In all our cases the equation defining the event horizon is of the form $f(r) = 0$ (for example, in the present case of the extremal Reissner-Nordström metric, it is $f(r) \equiv r - m = 0$, and so we have $|df|^2 = |dr|^2 = g^{rr}$. It is easily seen, either in the original diagonal forms for the metrics, or in the Eddington-Finkelstein forms where the metric has off-diagonal components, that g^{rr} vanishes at the horizons. For example, in the present case we have $g^{rr} = (1 - M/r)^2$, demonstrating that the event horizon is a null surface.

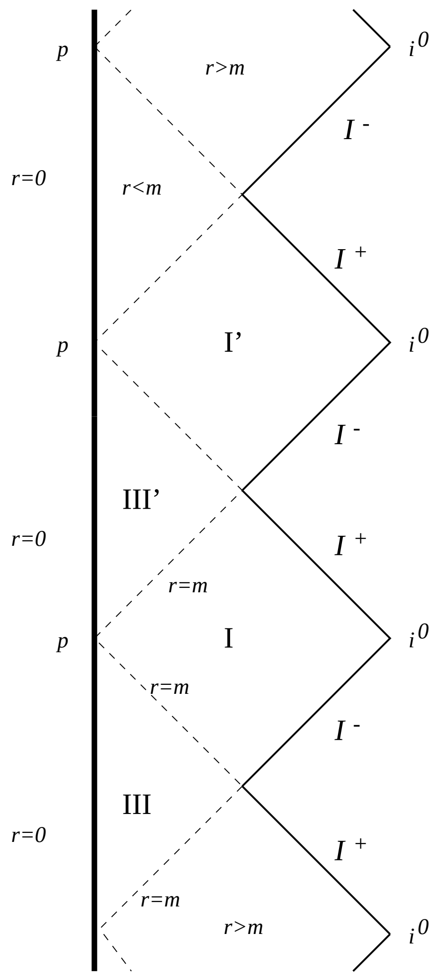

Figure 10: The maximal analytic extension of extremal Reissner-Nordström. ($I^{\pm}$ are again $\mathscr{I}^{\pm}$.)

11 Hamiltonian Formulation of Electrodynamics and General Relativity

For a variety of reasons, it is sometimes advantageous to formulate general relativity as a Hamiltonian dynamical system. This may on the face of it sound like a retrograde step, since one is taking a theory that possesses a beautiful four-dimensionally covariant symmetry, and then brutally breaking it apart into a "3+1" formulation where time is treated on a different footing from the three spatial directions. There can, nevertheless, be good reasons for doing this. For one thing, energy, or mass, is a very important physical concept, as for example in the notion of the mass of the Schwarzschild or Kerr black hole solution. To give a physical meaning to mass, one is, essentially, needing to calculate the Hamiltonian, the generator of time translations, and so the original four-dimensional covariance of the theory is going to have to be broken in the process. (The *solutions*, after all, in any case themselves break the four-dimensional covariance of the theory.) Another reason for introducing a Hamiltonian formulation is for the purposes of trying to quantise the theory. This takes us beyond what will be discuss in this course, but as with any quantum field theory, a proper discussion will more or less inevitably require the introduction of a Hamiltonian formulation at some stage, so that such things as the imposition of canonical commutation relations on constant-time hypersurfaces can be addressed.

By way of an introduction to some of the key ideas, it is instructive first to look at the conceptually simpler example of the Hamiltonian formulation of electrodynamics in Minkowski spacetime. It has some important features in common with the more complicated example of general relativity, arising from the fact that it is described in terms of a vector potential that involves the redundancy associated with the gauge symmetry of the theory. Having described the Hamiltonian treatment of electrodynamics we shall then move on to the case of general relativity. Again, there are redundancies in the description, this time as a consequence of the general-coordinate invariance of the theory.

11.1 Hamiltonian formulation of electrodynamics

Since the overall normalisation of the action will not play an important role here, we shall just make a convenient choice that minimises the occurrence of extraneous factors in the formulae. Accordingly, we shall for now take the action for the source-free Maxwell equations to be

$$S = \int \mathcal{L}\, d^4x\,, \qquad \mathcal{L} = -\tfrac{1}{4} F^{\mu\nu}\, F_{\mu\nu}\,, \tag{11.1}$$

where it is understood that $F_{\mu\nu}$ here is just a short-hand for

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \,. \tag{11.2}$$

(To get back to our canonical normalisation, we should multiply this action by $1/(4\pi)$. At the final stage of this discussion, having obtained the Hamiltonian for the system, we shall re-instate the omitted $1/(4\pi)$ factor.) Note that $\mathcal{L}$ here is the *Lagrangian density*; the Lagrangian L is obtained by integrating $\mathcal{L}$ over all 3-space, so

$$L = \int \mathcal{L}\, d^3x \,. \tag{11.3}$$

In this Lagrangian formulation, the vector field A_μ is viewed as the fundamental field of the theory. As we saw earlier, requiring that S be stationary with respect to infinitesimal variations of A_μ implies the source-free Maxwell equations

$$\partial_\mu F^{\mu\nu} = 0 \,. \tag{11.4}$$

(Recall that we are in Minkowski spacetime here.) We define the electric and magnetic fields through

$$F_{0i} = -E_i \,, \qquad F_{ij} = \epsilon_{ijk}\, B_k \,. \tag{11.5}$$

We now wish to give a Hamiltonian description, and so we begin by calculating the canonical momenta π^μ conjugate to the field variables A_μ, via the standard prescription

$$\pi^\mu = \frac{\delta S}{\delta \dot{A}_\mu} \,, \tag{11.6}$$

where $\dot{A}_\mu$ means $\partial_0 A_\mu = \partial A_\mu/\partial t$. When varying the action (11.1) with respect to A_μ we will get two equal contributions form varying each of the $F_{\mu\nu}$ factors, and so we have

$$\delta S = -\tfrac12 \int F^{\mu\nu}\,(\partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu)\, d^4x = -\tfrac12 \int \Big[F^{ij}\,(\partial_i \delta A_j - \partial_j \delta A_i) + 2F^{0i}\,(\partial_0 \delta A_i - \partial_i \delta A_0)\Big]\, d^4x \,. \tag{11.7}$$

Thus we see that

$$\pi^i = \frac{\delta S}{\delta \dot{A}_i} = -F^{0i} = -E^i \,, \qquad \pi^0 = \frac{\delta S}{\delta \dot{A}_0} = 0 \,. \tag{11.8}$$

Thus there is no canonical momentum π^0 conjugate to A_0; there are only 3 conjugate momenta π^i, conjugate to A_i. The fact that there is one fewer conjugate momentum component than one might have expected is a consequence of the fact that electrodynamics has a gauge invariance under $A_\mu \to A_\mu + \partial_\mu \Lambda$. The one gauge parameter Λ is responsible for knocking out the one canonical momentum π^0.

We can now proceed to construct the Hamiltonian H for the system by following the standard procedure of Legendre transforming the Lagrangian, by writing

$$H = \int \Big[\pi^i\,\dot A_i - \mathcal{L}\Big]\,d^3x\,. \tag{11.9}$$

Using (11.1) this gives

$$\begin{aligned} H &= \int \Big[\pi^i\,\dot A_i + \tfrac14 F^{ij}\,F_{ij} + \tfrac12 F^{0i}\,F_{0i}\Big]\,d^3x\,,\\ &= \int \Big[\pi^i\,\pi^i + \pi^i\,\partial_i A_0 + \tfrac14 F^{ij}\,F_{ij} - \tfrac12 \pi^i\,\pi^i\Big]\,d^3x\,, \end{aligned} \tag{11.10}$$

where in getting to the bottom line we have used (11.8) and also that $\pi^i = -F^{0i} = -E^i = F_{0i} = \dot A_i - \partial_i A_0$, so $\dot A_i = \pi^i + \partial_i A_0$. Thus we can write

$$H = \int \Big[\tfrac12 \pi^i\,\pi^i + \tfrac14 F^{ij}\,F_{ij} - A_0\,\partial_i\pi^i + \partial_i(A_0\,\pi^i)\Big]\,d^3x\,. \tag{11.11}$$

The last term can be turned into a surface integral by using the divergence theorem, and this will give zero for appropriate boundary conditions on the fields. Thus, finally, we have the Hamiltonian

$$H = \int \Big[\tfrac12 \pi^i\,\pi^i + \tfrac14 F^{ij}\,F_{ij} - A_0\,\partial_i\pi^i\Big]\,d^3x\,. \tag{11.12}$$

The Hamilton equations for the dynamical variables A_i and π^i give

$$\dot A_i = \frac{\delta H}{\delta \pi^i} = \pi_i + \partial_i A_0\,, \tag{11.13}$$

and

$$\dot\pi^i = -\frac{\delta H}{\delta A_i} = \partial_j F^{ij}\,. \tag{11.14}$$

Equation (11.13) implies $\pi_i = \partial_0 A_i - \partial_i A_0$, and hence it reproduces $\pi_i = F_{0i} = -E_i$ which we knew already. Equation (11.14) then gives

$$-\dot E_i = -\epsilon_{ijk}\,\partial_j B_k\,, \tag{11.15}$$

which is the source-free Maxwell equation $\vec\nabla\times\vec B - \partial\vec E/\partial t = 0$.

The field A_0 is not a dynamical field at all. As can be seen from (11.12) the Hamilton equations for A_0, which has no conjugate momentum, is just

$$0 = \frac{\delta H}{\delta A_0} = -\partial_i \pi^i\,, \tag{11.16}$$

which is simply $\partial_i E_i = 0$. Thus A_0 is just playing the role of a Lagrange multiplier, enforcing the *Gauss law constraint*

$$\vec\nabla\cdot\vec E = 0\,. \tag{11.17}$$

(Recall that we are considering the source-free Maxwell equations here, so the charge density ρ vanishes.)

Viewing electrodynamics as a dynamical Hamiltonian system, one would specify initial data $(A_i(t_0), \pi^i(t_0))$ on some timelike hypersurface at an initial time $t = t_0$, and then evolve it forwards in time using the Hamilton equations

$$\dot{A}_i = \frac{\delta H}{\delta \pi^i}\,, \qquad \dot{\pi}^i = -\frac{\delta H}{\delta A_i}\,. \tag{11.18}$$

However, one cannot specify the initial data completely arbitrarily, because of the Gauss law constraint (11.17); rather, one must choose initial data that satisfies (11.17) at $t = t_0$. The Hamilton equations will then ensure that this constraint is obeyed at all later times. This can be seen by taking the divergence of the (11.15) dynamical equation $\partial\vec{E}/\partial t = \vec{\nabla} \times \vec{B}$, giving

$$\frac{\partial(\vec{\nabla}\cdot\vec{E})}{\partial t} = \vec{\nabla}\cdot(\vec{\nabla}\times\vec{B}) = 0\,, \tag{11.19}$$

thus showing that if $\vec{\nabla}\cdot\vec{E} = 0$ at the initial time $t = t_0$, then it remains zero for all subsequent times.

Finally, we note that the Hamiltonian (11.12) can be used in order to calculate the energy in the electromagnetic field. The term $-A_0\,\partial_i\pi^i$ in (11.12) vanishes on shell, by virtue of the Gauss law constraint (11.17). From (11.5) and (11.8) we therefore find, after re-instating the $1/(4\pi)$ factor that we suppressed in all of the discussion so far, that the energy in the electromagnetic field is given by

$$\mathcal{E}_{EM} = \frac{1}{8\pi}\int (E^2 + B^2)\, d^3x\,. \tag{11.20}$$

This is the standard, expected, result.

The feature that we have seen here, with the gauge symmetry of the theory leading to the non-dynamical nature of the zero component of the vector potential A_μ and the associated Gauss law constraint, will arise also in a similar way when we look at the Hamiltonian formulation of general relativity. In the GR case it will be considerable more complicated, however. Furthermore, there will now be four non-dynamical components of the gravitational field $g_{\mu\nu}$, since there are four "gauge parameters" corresponding to the four infinitesimal diffeomorphisms $\delta x^\mu = -\xi^\mu$.

11.2 Hamiltonian formulation of general relativity

The key groundwork needed for constructing a Hamiltonian formulation of general relativity was laid down by Arnowitt, Deser and Misner (known universally as ADM) in the late 1950s and early 1960s. The starting point is to make a 3+1 dimensional decomposition of the spacetime, so that one views it as a foliation of $t = $ constant hypersurfaces, with a metric given by

$$ds^2 = -N^2\, dt^2 + h_{ij}\, (dx^i + N^i\, dt)(dx^j + N^j\, dt)\,, \tag{11.21}$$

where *Lapse Function* N, the *Shift Vector* N^i and the 3-metric h_{ij} all depend on the time coordinate t and the three spatial coordinates x^i. Note that the spacetime metric is still completely general; the 10 independent components of the four-dimensional metric $g_{\mu\nu}$ are parameterised now in terms of the 6 independent components of the 3-metric h_{ij}, the 3-component shift vector N^i and the lapse function N. Thus one has

$$g_{00} = -N^2 + N^i N_i\,, \qquad g_{0i} = g_{i0} = N_i\,, \qquad g_{ij} = h_{ij}\,, \tag{11.22}$$

where we define $N_i \equiv h_{ij}\, N^j$. It is easy to verify that the components of the inverse $g^{\mu\nu}$ of the four-dimensional metric are given by

$$g^{00} = -\frac{1}{N^2}\,, \qquad g^{0i} = g^{i0} = \frac{1}{N^2}\, N^i\,, \qquad g^{ij} = h^{ij} - \frac{1}{N^2}\, N^i N^j\,. \tag{11.23}$$

(We leave it as an exercise to check that indeed these components satisfy $g_{\mu\nu}\, g^{\nu\rho} = \delta^\rho_\mu$.) Note that by definition, h^{ij} means the inverse of the 3-dimensional metric h_{ij}, i.e. $h_{ij}\, h^{jk} = \delta^k_i$.

One can then calculate the four-dimensional Christoffel connection $\Gamma^\mu{}_{\nu\rho}$, and then the four-dimensional curvature, in terms of the quantities in the metric decomposition (11.21). Calculating the components of the Christoffel connection is not too challenging; one finds

$$\begin{aligned}
\Gamma^0{}_{00} &= \frac{1}{N}\,(\dot N + N^i\partial_i N) + \frac{1}{N}\, N^i N^j\, K_{ij}\,,\\
\Gamma^i{}_{jk} &= \bar\Gamma^i{}_{jk} - \frac{1}{N}\, N^i\, K_{jk}\,,\\
\Gamma^0{}_{ij} &= \frac{1}{N}\, K_{ij}\,,\\
\Gamma^i{}_{0j} &= -\frac{1}{N}\, N^i\partial_j N - \frac{1}{N}\, N^i N^k\, K_{jk} + \tfrac12 h^{ik}\,(\dot h_{jk} + D_j N_k - D_k N_j)\,,\\
\Gamma^i{}_{00} &= \dot N^i - \frac{1}{N}\, N^i\, \dot N - \frac{1}{N}\, N^i N^j N^k\, K_{jk} + N h^{ij}\,\partial_j N - \frac{1}{N}\, N^i\, N^j\,\partial_j N + h^{ij}\, N^k\,(\dot h_{jk} - D_j N_k)\,,\\
\Gamma^0{}_{0i} &= \frac{1}{N}\,\partial_i N + \frac{1}{N}\, N^j K_{ij}\,.
\end{aligned} \tag{11.24}$$

(Of course, components related to those given above by the symmetry on the lower two indices follow from these in the obvious way.) Note that here we have defined the *second fundamental form*, or *extrinsic curvature*, of the $t = $ constant surfaces by

$$K_{ij} = \frac{1}{2N}\,\Big(\dot h_{ij} - D_i N_j - D_j N_i\Big)\,, \tag{11.25}$$

and a dot denotes a derivative with resepct to time. D_i denotes the 3-dimensional covariant derivative with respect to the 3-metric h_{ij}, so that

$$D_i N_j = \partial_i N_j - \bar{\Gamma}^k{}_{ij}\, N_k\,, \tag{11.26}$$

etc. Note that $\bar{\Gamma}^i{}_{jk}$ denotes the components of the Christoffel connection for the 3-metric h_{ij}, and so

$$\bar{\Gamma}^i{}_{jk} = \tfrac{1}{2} h^{i\ell}\,(\partial_j h_{\ell k} + \partial_k h_{j\ell} - \partial_\ell h_{jk})\,. \tag{11.27}$$

Calculating the curvature is quite a bit more challenging, and we shall merely present a final result here. One finds that the Einstein-Hilbert action, after dropping various total derivative terms that will not affect the equations of motion,[26] can be written in terms of the 3-dimensional quantities as

$$S = \int \sqrt{-g} R\, d^4x = \int \sqrt{h}\, N\,\Big(\bar{R} + K_{ij}\, K^{ij} - K^2\Big)\, d^4x\,, \tag{11.28}$$

where $K \equiv h^{ij}\, K_{ij}$. We have omitted the usual $1/(16\pi)$ prefactor for now, since it plays no essential role in the discussion; we shall restore it at the end of the calculation. Note that here $\bar{R}$ is the Ricci scalar of the 3-metric h_{ij}, and that $\sqrt{-g} = N\,\sqrt{h}$ in terms of the ADM variables. (As usual, $g = \det(g_{\mu\nu})$, and also we define $h = \det(h_{ij})$.) The action S is thus expressed in terms of the 3-dimensional quantities N, N^i and h_{ij}.

We can now follow the standard steps for reformulating the theory as a Hamiltonian system. First, we calculate the canonical momenta, by evaluating the variational derivatives with respect to $\dot{N}$, $\dot{N}^i$ and $\dot{h}_{ij}$. It is easy to see that S in (11.28) does not involve $\dot{N}$ or $\dot{N}^i$ anywhere, and so there are no canonical momenta conjugate to N or N^i:

$$\frac{\delta S}{\delta \dot{N}} = 0\,, \qquad \frac{\delta S}{\delta \dot{N}^i} = 0\,. \tag{11.29}$$

This means that N and N^i are non-dynamical, and are simply like Lagrange multipliers which will impose initial-value constraints. This is the same phenomenon as we saw with the component A_0 of the electromagnetic vector potential in the previous discussion for electrodynamics.

The canonical momentum conjugate to h_{ij}, given by calculating $\pi^{ij} = \partial S/\delta \dot{h}_{ij}$, is

$$\pi^{ij} = \sqrt{h}\,(K^{ij} - K\, h^{ij})\,. \tag{11.30}$$

[26] But see later. The total derivatives that we are ignoring for now integrate to give boundary terms, and these can potentially cause trouble when we are careful about the argument that they should give zero in the variation.

(Note that π^{ij} is a 3-tensor density of weight 1.)

To derive the constraints mentioned above, write K_{ij}, defined in (11.25), as $K_{ij} = N^{-1}\,\widetilde{K}_{ij}$, so that $\widetilde{K}_{ij}$ is independent of N. It follows from (11.28) that

$$S = \int \sqrt{h}\,\Big(N\,\bar{R} + N^{-1}\,\widetilde{K}_{ij}\widetilde{K}^{ij} - N^{-1}\,\widetilde{K}^2\Big)\,d^4x\,, \tag{11.31}$$

and so the variation with respect to N, with $\widetilde{K}_{ij}$ then replaced by $N\,K_{ij}$, gives the initial-value constraint

$$\mathcal{H} \equiv -\bar{R} + K_{ij}\,K^{ij} - K^2 = 0\,. \tag{11.32}$$

The constraints following from the variation of S with respect to N^i can be found easily:

$$\begin{aligned}
\delta S &= \int \sqrt{h}\,N\,[2K^{ij}\,\delta K_{ij} - 2K\delta K]\,d^4x\,,\\
&= \int \sqrt{h}\,[-K^{ij}(D_i\delta N_j + D_j\delta N_i + 2K\,D_j\delta N^j]\,d^4x\,,\\
&= 2\int \sqrt{h}\,[-K^i{}_j\,D_i + K\,D_j]\delta N^j\,d^4x\,,\\
&= 2\int \sqrt{h}\,[D_iK^i{}_j - \partial_j K]\,\delta N^j\,d^4x\,,
\end{aligned} \tag{11.33}$$

whence we obtain

$$\mathcal{H}_i \equiv -2(D_jK^j{}_i - \partial_i K) = 0\,. \tag{11.34}$$

Expressed in terms of the conjugate momenta π^{ij}, the constraints (11.32) and (11.34) become

$$\mathcal{H} = -\bar{R} + h^{-1}\,\pi^{ij}\,\pi_{ij} - \tfrac{1}{2}h^{-1}\,\pi^2 \;= 0\,, \tag{11.35}$$

$$\mathcal{H}_i = -2h_{ik}\,D_j(h^{-1/2}\,\pi^{jk}) = 0\,, \tag{11.36}$$

where $\pi \equiv h_{ij}\,\pi^{ij}$. The Hamiltonian H, calculated in the usual way from the Lagrangian via the Legendre transform

$$H = \int d^3x\,\Big(\pi^{ij}\,\dot{h}_{ij} - \mathcal{L}\Big)\,, \tag{11.37}$$

takes the form

$$H = \int \sqrt{h}\,(N\,\mathcal{H} + N^i\,\mathcal{H}_i)\,d^3x\,, \tag{11.38}$$

It is instructive to compare the Hamiltonian (11.38) for general relativity with the Hamiltonian (11.12) that we obtained previously in electrodynamics. In that case, we had a contribution $(-A_0\,\partial_i\pi^i)$ that was analogous to one of the terms in (11.38); i.e. a term of the form of a Lagrange multiplier times a constraint. In the electrodynamic case, however, we had other terms too in (11.12); these were the E^2 and B^2 terms in the standard Hamiltonian for the Maxwell system. In the case of general relativity, on the other hand, (11.38) contains *only* contribitions of the form (Lagrange multiplier) times (constraint). This means that on-shell, (11.38) actually vanishes. We shall have more to say about this below.[27]

The dynamics of the gravitational system is contained in the fields h_{ij} and their conjugate momenta π^{ij}. Hamilton's equations for these fields give

$$\dot{h}_{ij} = \frac{\delta H}{\delta \pi^{ij}}\,, \qquad \dot{\pi}^{ij} = -\frac{\delta H}{\delta h_{ij}}\,. \tag{11.39}$$

The first equation here just produces, again, the definition of π^{ij} as in (11.30). The second equation here gives the equations of motion for the dynamical fields h_{ij}:

$$\begin{aligned} \dot{\pi}^{ij} &= -N h^{1/2}\,(\bar{R}^{ij} - \tfrac{1}{2}\bar{R}\,h^{ij}) + \tfrac{1}{2}N\,h^{-1/2}\,(\pi^{k\ell}\pi_{k\ell} - \tfrac{1}{2}\pi^2)\,h^{ij} \\ &\quad -2N h^{-1/2}\,(\pi^{ik}\,\pi_k{}^j - \tfrac{1}{2}\pi\,\pi^{ij}) + h^{1/2}\,(D^i D^j N - h^{ij}\,D^k D_k N) \\ &\quad + D_k(\pi^{ij}\,N^k) - \pi^{ki}\,D_k N^j - \pi^{kj}\,D_k N^i\,. \end{aligned} \tag{11.40}$$

The Hamilton equations for the fields N and N^i, which have no conjugate momenta, are

$$\frac{\delta H}{\delta N} = 0\,, \qquad \frac{\delta H}{\delta N^i} = 0\,, \tag{11.41}$$

and these simply reproduce the constraints (11.35) and (11.36) respectively. These constraints are the analogue of the $\partial_i \pi^i = 0$ constraint (11.16) in electrodynamics.

In principle, the idea now is that the energy, or mass, of a solution is given as the on-shell value of the Hamiltonian, just as the energy of the electromagnetic field was given by the on-shell value of the Hamiltonian in the example of electromagnetism we discussed previously. However, we are not quite there yet because naively, as we observed above, if we take the Hamiltonian to be given by (11.38), then we shall always get zero since by definition the constraints (11.35) and (11.36) are satisfied by the solution. The clue to what has gone wrong lies in the cautionary remarks made earlier about our having ignored the issue of boundary terms in the action, and hence in the Hamiltonian. Surface terms do not affect the equations of motion, in the sense that they don't contribute to Hamilton's equations. But in order to have a well-defined variational derivation of the Hamilton equations, one does need to be careful about the surface terms. And furthermore, they certainly can affect the actual on-shell value of the Hamiltonian.

[27] Something rather similar happens at the level of the action. In electrodynamics, the action $S = -\frac{1}{4}\int F^2 d^4x$ implies the field equations $\partial_\mu F^{\mu\nu} = 0$, and the action itself is non-vanishing on-shell. By contrast, the Einstein-Hilbert action $S = \int \sqrt{-g} R d^4x$ in general relativity implies the equations of motion $R_{\mu\nu} = 0$, and so S in fact vanishes on-shell.

The surface terms in question here are the ones associated with the integrations by parts that we have to perform in order to remove derivatives from $\delta\pi^{ij}$ and δh_{ij} when we make the variational derivatives in (11.39). Suppose we are considering a situation where the 3-dimensional hypersurfaces of constant t are asymptotically-flat spatial regions, and so the surface terms of concern to us will be the ones associated with the "sphere at infinity," when we use the 3-dimensional divergence theorem to throw spatial derivatives off the variations $\delta\pi^{ij}$ or δh_{ij} and onto their corresponding co-factors in the integral. We can assume that asymptotic flatness of the metric means that in a suitable coordinate system we shall have

$$h_{ij} \sim \delta_{ij} + \mathcal{O}\Big(\frac{1}{r}\Big) \tag{11.42}$$

at large r, and correspondingly $\pi^{ij} = \mathcal{O}(1/r^2)$. Thus appropriate boundary conditions for the variations are

$$\delta h_{ij} = \mathcal{O}\Big(\frac{1}{r}\Big)\,, \qquad \delta\pi^{ij} = \mathcal{O}\Big(\frac{1}{r^2}\Big)\,. \tag{11.43}$$

With this choice of asymptotically-Minkowskian coordinates we should also have

$$N = 1 + \mathcal{O}\Big(\frac{1}{r}\Big)\,, \qquad N^i = \mathcal{O}\Big(\frac{1}{r}\Big) \tag{11.44}$$

at large r. One can straightforwardly verify that these stated asymptotic forms for the metric functions h_{ij}, N and N^i do indeed occur for the Schwarzschild, Reissner-Norström, Kerr and Kerr-Newman black hole metrics.

When we vary (11.38) with respect to π^{ij}, we can see from (11.36) that the integration by parts for the $N^i\mathcal{H}_i$ term will give rise to a boundary term

$$\int_\Sigma d\Sigma_i(-2N_j h^{-1/2}\,\delta\pi^{ij})\,, \tag{11.45}$$

integrated over the 2-sphere Σ at (large) radius r. Eventually, we push the radius out to infinity. The area element $d\Sigma_i$ on the 2-sphere grows like r^2, but the integrand in (11.45) falls off faster than $1/r^2$, and so there is no contribution from this surface term.

When we vary (11.38) with respect to h_{ij}, an integration by parts will again be needed for the $N^i\,\mathcal{H}_i$ term, and just like the calculation above, this will again give no boundary contribution when we push the radius of the boundary 2-sphere to infinity. Now, however, there will be a need for further integrations by parts, because of the derivatives of δh_{ij} arising from the variation of the 3-dimensional Ricci scalar $\bar R$ in the $N\,\mathcal{H}$ term. The calculation of this variation is just like the one for the variation of the 4-dimensional Ricci scalar, which was obtained in (7.9). Thus here, we shall have

$$\delta\bar R = (-\bar R^{ij} + D^i D^j - h^{ij}\,D^k D_k)\delta h_{ij}\,. \tag{11.46}$$

(The overall sign change here, relative to (7.9), is because here we are using δh_{ij} rather than δh^{ij}.) We have to integrate by parts twice here, on each of the second and the third terms in (11.46), to throw the second derivatives off the δh_{ij} terms. Focusing just on the variations of these terms we shall have, from (11.35) and (11.38), that

$$\begin{aligned}\delta H &= -\int \sqrt{h}\, d^3x\, N\, (D^i D^j\, \delta h_{ij} - h^{ij}\, D^k\, D_k\, \delta h_{ij}) + \cdots \\ &= -\int \sqrt{h}\, d^3x\, \Big[D^i(N\, D^j\, \delta h_{ij}) - D^i N\, D^j \delta h_{ij} - D^k(N h^{ij}\, D_k \delta h_{ij}) + D^k(N\, h^{ij})\, D_k \delta h_{ij}\Big] + \cdots \\ &= -\int_\Sigma d\Sigma^i\, N\, \Big(D^j \delta h_{ij} - D_i(h^{jk}\, \delta h_{jk})\Big) \\ &\quad + \int \sqrt{h}\, d^3x\, \Big[D^i N\, D^j \delta h_{ij} - D^k(N\, h^{ij})\, D_k \delta h_{ij}\Big] + \cdots\,, \end{aligned} \tag{11.47}$$

where the $\cdots$ represents all the other terms that we do not need to look at here, since our goal is just to collect the surface terms arising from the integrations by parts.

The 3-volume terms in the bottom line of (11.47) require a further integration by parts, to throw the remaining derivatives off the δh_{ij}. After doing this and converting the further total derivative terms into surface terms, we arrive from (11.47) at

$$\begin{aligned}\delta H &= -\int_\Sigma d\Sigma^i\, N\, \Big[D^j \delta h_{ij} - D_i(h^{jk}\, \delta h_{jk}) - D^j N\, \delta h_{ij} + D_i N\, h^{jk}\, \delta h_{jk}\Big] \\ &\quad -\int \sqrt{h}\, d^3x\, \Big[D^i D^j\, N - (D^k D_k N)\, h^{ij}\Big]\, \delta h_{ij} + \cdots\,. \end{aligned} \tag{11.48}$$

The first line in (11.48) contains all the surface terms that result from varying the Hamiltonian given in (11.38). The third and fourth terms in the first line of (11.48) give no problem, because they do indeed go to zero as we push the spatial 2-surface Σ out to infinity. This can be seen from the assumptions in (11.42), (11.43) and (11.44) about the asymptotic behaviour of the metric functions. The point is that $D^i N$ must fall like $1/r^2$ and with δh_{ij} falling like $1/r$, the overall $1/r^3$ falloff of these terms in the integrand outweighs the r^2 growth of the 2-surface area element $d\Sigma^i$.

The first two terms in the first line of (11.48) do contribute, however. Here, we have $D\delta h$ terms that fall off like $1/r^2$, exactly balancing the r^2 growth of the area element. Thus as r goes to infinity we find that these contribute

$$\delta H \longrightarrow -\int_\Sigma d\Sigma_i\, (\partial_j \delta h_{ij} - \partial_i \delta h_{jj})\,. \tag{11.49}$$

(We don't need to distinguish between up and down indices here, since at this order the metric is just δ_{ij}.)

Since this boundary term doesn't vanish for the class of variations we wish to consider, it means that in order to make the variational problem well posed, we should have added a boundary term to the Hamiltonian H defined in (11.38), whose job is to cancel (11.49). Clearly, the extra term that will do the job is

$$H_{\rm extra} = \int_\Sigma d\Sigma_i\, (\partial_j h_{ij} - \partial_i h_{jj})\,. \tag{11.50}$$

Thus the proper Hamiltonian we should use is

$$H_{\rm tot} = H + H_{\rm extra}\,, \tag{11.51}$$

where H is the original Hamiltonian defined in (11.38). Since we have only added a surface term, it leaves the Hamilton equations unaltered.

The additional term does, however, make a contribution to the energy when we evaluate the Hamiltonian for a solution of the Einstein equations. As we observed above, the original Hamiltonian vanishes when we impose the equations of motion. Thus the entire contribution to the energy will come from the additional term $H_{\rm extra}$ given in (11.50). This gives an expression which is known as the "ADM mass" of the solution. Restoring the $1/(16\pi)$ prefactor on the original action that we had suppressed earlier, we therefore have

$$M_{ADM} = \frac{1}{16\pi} \int_\Sigma d\Sigma_i\, (\partial_j h_{ij} - \partial_i h_{jj})\,. \tag{11.52}$$

As a check, let us see what this formula gives for the mass of the Schwarzshild black hole, for which the metric is

$$ds^2 = -B dt^2 + B^{-1}\, dr^2 + r^2\, d\Omega^2\,, \qquad B = 1 - \frac{2M}{r}\,. \tag{11.53}$$

This can be written as

$$\begin{aligned} ds^2 &= -B dt^2 + (B^{-1} - 1)\, dr^2 + dr^2 + r^2\, d\Omega^2\,, \\ &= -B dt^2 + (B^{-1} - 1)\, dr^2 + dx^i dx^i\,, \\ &= -B dt^2 + (B^{-1} - 1)\, \frac{x_i x_j}{r^2}\, dx^i dx^j + \delta_{ij}\, dx^i dx^j\,, \end{aligned} \tag{11.54}$$

where x_i are related to r, θ and φ in the standard way for Cartesian and spherical polar coordinates. Thus we have

$$N = \Big(1 - \frac{2M}{r}\Big)^{1/2}\,, \qquad N^i = 0\,, \qquad h_{ij} = \delta_{ij} + \frac{2M}{Br}\, \frac{x_i x_j}{r^2}\,. \tag{11.55}$$

The fall-off conditions we assumed are fulfilled, and after a simple bit of 3-dimensional Cartesian tensor calculus we find that

$$d\Sigma_i\, (\partial_j h_{ij} - \partial_i h_{jj}) = r^2 d\Omega\, \frac{x_i}{r}\, (\partial_j h_{ij} - \partial_i h_{jj}) = \frac{4M}{B}\, d\Omega\,, \tag{11.56}$$

where $d\Omega$ is the area element on the unit 2-sphere. Plugging into (11.52), integrating over the 2-sphere, and sending r to infinity, we then find

$$M_{ADM} = M\,. \tag{11.57}$$

In other words, we have confirmed that the ADM formula for the mass has indeed reproduced the expect result M for the Schwarzschild solution.

12 Black Hole Dynamics and Thermodynamics

We now turn to a discussion that will lead on to the celebrated finding by Stephen Hawking that a black hole is not really black after all, but instead it radiates as if it were a black body with a temperature known, appropriately enough, as the *Hawking Temperature*.

The first stage in this development will be to introduce the notion of the *surface gravity* of a black hole. This will involve a certain amount of intrincate tensor analysis, but the efforts will be rewarded later.

12.1 Killing horizons

We have seen already that the horizon of the Schwarzschild black hole (6.26) can be characterised as the surface on which the Killing vector

$$\xi \equiv \frac{\partial}{\partial t} \tag{12.1}$$

becomes null:

$$\xi^\mu \xi_\mu = g_{\mu\nu}\, \xi^\mu \xi^\nu = g_{00} = -1 + \frac{2M}{r}\,, \tag{12.2}$$

which vanishes at $r = 2M$. More generally, we can define the notion of a *Killing Horizon* as a null hypersurface $\mathcal{N}$ on which a Killing vector ξ satisfies $\xi^\mu \xi_\mu = 0$ and for which ξ^μ is normal to $\mathcal{N}$.

A hypersurface can always be defined as the surface on which a certain function f vanishes. (For example, the $r = 2M$ hypersurface in Schwarzschild can be defined in this way, by taking $f = 1 - 2M/r$.) Vector fields ℓ^μ normal to the hypersurface $f = 0$ all then have the form

$$\ell^\mu = h\, g^{\mu\nu}\, \partial_\nu f\,, \tag{12.3}$$

where h is some non-vanishing function. Consequently, the hypersurface is a Killing horizon of a Killing vector ξ if, firstly, $\ell^\mu\, \ell_\mu = 0$ (i.e. it is null), and secondly $\xi^\mu = \psi\, \ell^\mu$ for some non-vanishing function $\psi(x)$.

Notice that this might look a little puzzling at first sight. If we take the example of Schwarzschild then

$$\ell = \ell^\mu\,\partial_\mu = h\,g^{\mu\nu}\,(\partial_\nu f)|_{\mathcal{N}}\,\partial_\mu = h\,(2M)^{-1}\,g^{\mu r}\,\partial_\mu\,. \tag{12.4}$$

Naively, if one were using the original (t, r, θ, φ) Schwarzschild coordinates then one would think ℓ must be proportional to $\partial/\partial r$, and thus it could certainly not be proportional to $\xi = \partial/\partial t$. However, it should be recalled that t is not a good coordinate on the horizon, and so we should instead use the advanced Eddington-Finkelstein coordinates (v, r, θ, φ), for which the metric is given by (10.27). In these coordinates we have $g^{rv} = g^{vr} = 1$, $g^{rr} = (1 - 2M/r)$ and $g^{vv} = 0$. Furthermore, the Killing vector ξ is now given by

$$\xi = \frac{\partial}{\partial v}\,. \tag{12.5}$$

Thus we find from (12.4) that on $\mathcal{N}$, the normal vector ℓ is given by

$$\ell = \frac{h}{2M}\,\frac{\partial}{\partial v}\,, \tag{12.6}$$

which is indeed proportional to the Killing vector ξ.

A further observation is that the ℓ^μ is not only normal to the null surface $\mathcal{N}$, but it is also *tangent* to $\mathcal{N}$. This follows from the fact that, by definition, any vector t^μ tangent to a surface is orthogonal to the normal vector ℓ^μ, i.e. $t^\mu\,\ell_\mu = 0$. But since ℓ^μ is null here, it follows that it itself satisfies the condition for being a tangent vector. This means that there must exist some curve $x^\mu = x^\mu(\lambda)$ in $\mathcal{N}$ such that

$$\ell^\mu = \frac{dx^\mu}{d\lambda}\,, \tag{12.7}$$

where λ parameterises the curve.

The curves $x^\mu(\lambda)$ are in fact geodesics. To see this, recall that $\ell^\mu = dx^\mu/d\lambda$ is given by (12.3), and now calculate $\ell^\rho\,\nabla_\rho\,\ell^\mu$:

$$\begin{aligned}
\ell^\rho\,\nabla_\rho\,\ell^\mu &= (\ell^\rho\,\partial_\rho h)\,g^{\mu\nu}\,\partial_\mu f + h\,g^{\mu\nu}\,\ell^\rho\,\nabla_\rho\,\partial_\nu\,f\,,\\
&= (\ell^\rho\,\partial_\rho\,\log h)\,\ell^\mu + h\,g^{\mu\nu}\,\ell^\rho\,(\nabla_\nu\,\partial_\rho\,f)\,,\\
&= \ell^\mu\,\frac{d\log h}{d\lambda} + h\,\ell^\rho\,\nabla^\mu(h^{-1}\,\ell_\rho)\,,\\
&= \ell^\mu\,\frac{d\log h}{d\lambda} + \ell^\rho\,\nabla^\mu\,\ell_\rho - \ell^2\,(\partial^\mu\,\log h)\,,\\
&= \ell^\mu\,\frac{d\log h}{d\lambda} + \tfrac{1}{2}\partial^\mu\,(\ell^2) - \ell^2\,(\partial^\mu\,\log h)\,.
\end{aligned} \tag{12.8}$$

(The indices ρ and ν in the second term of the second line could be interchanged on account of the fact that second covariant derivatives commute on scalar fields.) Now, we know that

ℓ^μ is null on $\mathcal{N}$, so $\ell^2 = 0$ there. This does not mean that $\partial^\mu(\ell^2)$ vanishes on $\mathcal{N}$, but the fact that $\ell^2 = 0$, which is constant, on $\mathcal{N}$ does mean that $t^\mu\,\partial_\mu(\ell^2) = 0$ for any vector t^μ tangent to $\mathcal{N}$. In view of the previous discussion, this means that $\partial_\mu(\ell^2)$ must be proportional to ℓ_μ on $\mathcal{N}$, so $\partial_\mu(\ell^2) = \alpha\,\ell_\mu$ for some function α, and hence we have that

$$\ell^\rho\,\nabla_\rho\,\ell^\mu\Big|_{\mathcal{N}} = \tfrac{1}{2}\alpha\,\ell^\mu + \ell^\mu\,\frac{d\log h}{d\lambda}\,. \tag{12.9}$$

Recalling that the function h in (12.3) is still at our disposal, we see that by choosing it appropriately, we can make the right-hand side of (12.9) vanish. This would imply that $x^\mu(\lambda)$ on $\mathcal{N}$ satisfies the geodesic equation

$$\ell^\rho\,\nabla_\rho\,\ell^\mu = \frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu{}_{\nu\,\rho}\,\frac{dx^\nu}{d\lambda}\,\frac{dx^\rho}{d\lambda} = 0 \tag{12.10}$$

on $\mathcal{N}$, with λ being an affine parameter. (The more general equation (12.9) is still the geodesic equation, but with the parameter λ not an *affine* parameter.) One can define the null geodesics $x^\mu(\lambda)$ with affine parameter λ, for which the tangent vectors $\ell^\mu = dx^\mu/d\lambda$ are normal to the null surface $\mathcal{N}$, to be the *generators* of $\mathcal{N}$.

12.2 Surface gravity

We saw in the previous discussion that if $\mathcal{N}$ is a Killing horizon of the vector field ξ, then if ℓ^μ is a normal vector to $\mathcal{N}$ in the affine parametrisation, implying $\ell^\nu\,\nabla_\nu\,\ell^\mu = 0$, then there exists a function ψ such that $\xi^\mu = \psi\,\ell^\mu$. It then follows that on $\mathcal{N}$ we shall have

$$\xi^\nu\,\nabla_\nu\,\xi^\mu = \kappa\,\xi^\mu\ , \tag{12.11}$$

where

$$\kappa = \xi^\nu\,\partial_\nu\,\log|\psi|\ . \tag{12.12}$$

The surface gravity κ may be expressed in a variety of different ways, which can be derived from (12.11). First, observe that if we view ξ as the covector $\xi = \xi_\mu\,dx^\mu$, then the fact that ξ is normal to $\mathcal{N}$ means that

$$\xi_{[\mu}\,\partial_\nu\,\xi_{\rho]}\Big|_{\mathcal{N}} = 0\ . \tag{12.13}$$

That is to say, it is obvious that if $\xi_\mu = u\,\partial_\mu f$, for any functions u and f, then (12.13) is satisfied. (In our case, we have $u = h\psi$.) Conversely, it can be shown, with a little more work, that if (12.13) is satisfied then there exist functions u and f such that $\xi_\mu = u\,\partial_\mu f$. This is known as Frobenius' theorem. Now since ξ is a Killing vector, it follows from the Killing vector equation

$$\nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu = 0 \tag{12.14}$$

that

$$\nabla_\mu \xi_\nu = \nabla_{[\mu} \xi_{\nu]} = \partial_{[\mu} \xi_{\nu]} \ , \tag{12.15}$$

and hence (12.13) can be rewritten as

$$\xi_\rho \nabla_\mu \xi_\nu = \xi_\nu \nabla_\mu \xi_\rho - \xi_\mu \nabla_\nu \xi_\rho \ . \tag{12.16}$$

Multiplying by $\nabla^\mu \xi^\nu$, we obtain

$$\begin{aligned} \xi_\rho (\nabla^\mu \xi^\nu) (\nabla_\mu \xi_\nu)\Big|_{\mathcal{N}} &= -2(\xi_\mu \nabla^\mu \xi^\nu) (\nabla_\nu \xi_\rho)\Big|_{\mathcal{N}} \ , \\ &= -2\,\kappa\, (\xi^\nu \nabla_\nu \xi_\rho)\Big|_{\mathcal{N}} \ , \\ &= -2\kappa^2\, \xi_\rho\Big|_{\mathcal{N}} \ , \end{aligned} \tag{12.17}$$

where we have twice made use of the equation (12.11). Thus aside from singular points on $\mathcal{N}$ where ξ_ρ vanishes, we have

$$\kappa^2 = -\tfrac{1}{2} (\nabla^\mu \xi^\nu) (\nabla_\mu \xi_\nu)\Big|_{\mathcal{N}} \ . \tag{12.18}$$

In fact points where ξ_ρ vanishes are arbitrarily close to points where it is non-zero, so by continuity the expression (12.18) for κ is valid everywhere on $\mathcal{N}$.

We can in fact obtain a simpler expression for κ, namely

$$\kappa^2 = (\partial^\mu \sigma) (\partial_\mu \sigma)\Big|_{\mathcal{N}} \ , \tag{12.19}$$

where $\sigma^2 \equiv -|\xi|^2 = -\xi^\mu \xi_\mu$. Note that this can be written also as

$$\kappa^2 = -\frac{g^{\mu\nu} (\partial_\mu \xi^2) (\partial_\nu \xi^2)}{4\xi^2} \ , \tag{12.20}$$

and this is often the easiest way to calulate the surface gravity.

The proof of (12.19) is surprisingly tricky. The reason for this is that although (12.19) is evaluated on the Killing horizon $\mathcal{N}$, the fact that the expression involves derivatives of σ means that one must first carry out manipulations that are valid *away* from $\mathcal{N}$, and only move onto the horizon *after* the derivatives are taken.

First, we rewrite the Frobenius condition (12.13) as $\xi_{[\mu} \nabla_\nu \xi_{\rho]} = 0$. On the other hand, since ξ_μ satisfies the Killing-vector condition $\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0$ *everywhere*, we can write

$$3\xi_{[\mu} \nabla_\nu \xi_{\rho]} = \xi_\mu \nabla_\nu \xi_\rho + \xi_\nu \nabla_\rho \xi_\mu + \xi_\rho \nabla_\mu \xi_\nu \ , \tag{12.21}$$

and this is valid both on $\mathcal{N}$ and away from $\mathcal{N}$. Multiplying this equation by $\xi^\mu \nabla^\nu \xi^\rho$, we see that after making use of the antisymmetry of $\nabla_\rho \xi_\mu$ in the second term on the right-hand side, and also the antisymmetry of the multiplier $\nabla^\nu \xi^\rho$ when writing out the third term on the right-hand side, we shall have

$$3(\xi^{[\mu}\,\nabla^\nu\,\xi^{\rho]})(\xi_{[\mu}\,\nabla_\nu\,\xi_{\rho]}) = \xi^\mu\,\xi_\mu\;(\nabla^\nu\,\xi^\rho)(\nabla_\nu\,\xi_\rho) - 2(\xi^\mu\,\nabla^\nu\,\xi^\rho)(\xi_\nu\,\nabla_\mu\,\xi_\rho)\,. \tag{12.22}$$

Again, we emphasise that this is valid everywhere, and not just on $\mathcal{N}$. Now since $\xi_{[\mu}\,\nabla_\nu\,\xi_{\rho]}$ vanishes on the horizon, it follows that the gradient of the left-hand side of (12.22) vanishes on the horizon.[28] On the other hand, we know from (12.11) that the gradient of $|\xi|^2$ does not vanish on the horizon, provided that κ is non-zero. This means that by l'Hospital's rule, it must be that we can divide (12.22) by $|\xi|^2$ and then take the limit as we approach the horizon, and the left-hand side will still vanish. Thus we are able to deduce that in the limit of approaching the horizon, we have

$$(\nabla^\nu\,\xi^\rho)(\nabla_\nu\,\xi_\rho) = \frac{2(\xi^\mu\,\nabla^\nu\,\xi^\rho)(\xi_\nu\,\nabla_\mu\,\xi_\rho)}{|\xi|^2}\;. \tag{12.23}$$

Having successfully negotiated this tricky step, the rest is plain sailing. The right-hand side in (12.23) can be immediately rewritten as

$$\frac{\partial^\rho(\xi^\nu\,\xi_\nu)\,\partial_\rho(\xi^\mu\,\xi_\mu)}{2|\xi|^2}\;, \tag{12.24}$$

which is nothing but $-\frac{1}{2}\partial^\rho\sigma\,\partial_\rho\sigma$. From (12.18), the result (12.19) now immediately follows.

Note that from its definition so far, the normalisation for κ is undetermined, since it scales under constant scalings of the Killing vector ξ. Once cannot normalise ξ at the horizon, since $\xi^2 = 0$ there, but its normalisation can be specified in terms of the behaviour of ξ at infinity. There is a unique Killing vector (up to scale) that is timelike at arbitrarily large distances in the asymptotically flat regions. (In Schwarzschild, it is simply $K = \partial/\partial t$.) This vector, which we shall denote generically by K, may be normalised canonically by requiring that it have magnitude-squared equal to -1 at infinity, and that it be future-directed (this fixes the sign choice). Then the Killing vector ξ of the Killing horizon is defined to be $\xi = K + \cdots$, where the ellipses denote whatever additional spacelike Killing vectors appear in the calculated expression for ξ.

Let us now examine why the quantity κ is called the *surface gravity*. It has the interpretation of being the acceleration of a static particle near the horizon, as measured at spatial infinity. One can see this as follows. Let us consider a particle near the horizon, moving on an orbit of ξ^μ; this means that its 4-velocity $u^\mu = dx^\mu/d\tau$ is proportional to ξ^μ. Since the

[28] The left-hand side is of the form $3W^{\mu\nu\rho}\,W_{\mu\nu\rho}$, where $W_{\mu\nu\rho} = \xi_{[\mu}\,\nabla_\nu\,\xi_{\rho]}$, and so $\nabla_\sigma(3W^{\mu\nu\rho}\,W_{\mu\nu\rho}) = 6W^{\mu\nu\rho}\,\nabla_\sigma\,W_{\mu\nu\rho}$, which therefore vanishes on $\mathcal{N}$ because the undifferentiated factor $W^{\mu\nu\rho}$ vanishes on $\mathcal{N}$.

4-velocity must satisfy $u^\mu\, u_\mu = -1$, this means that we must have

$$u^\mu = \sigma^{-1}\, \xi^\mu \ , \tag{12.25}$$

where, as above, we have defined the function σ by $\sigma^2 = -\xi^\mu\, \xi_\mu$. Now, the 4-acceleration of the particle is given by

$$a^\mu = \frac{Du^\mu}{D\tau} \equiv \frac{dx^\nu}{d\tau}\, \nabla_\nu\, u^\mu = u^\nu\, \nabla_\nu\, u^\mu \ . \tag{12.26}$$

Using (12.25), we see that this gives

$$\begin{aligned} a^\mu &= \sigma^{-2}\, \xi^\nu\, \nabla_\nu\, \xi^\mu - \sigma^{-3}\, \xi^\mu\, \xi^\nu\, \nabla_\nu\, \sigma \\ &= -\sigma^{-2}\, \xi^\nu\, \nabla^\mu\, \xi_\nu - \tfrac{1}{2}\sigma^{-4}\, \xi^\mu\, \xi^\nu\, \nabla_\nu(\xi^\rho\, \xi_\rho) \\ &= -\tfrac{1}{2}\sigma^{-2}\, \partial^\mu\, (\xi^\nu\, \xi_\nu) - \sigma^{-4}\, \xi^\mu\, \xi^\nu\, \xi^\rho\, \nabla_\nu\, \xi_\rho \\ &= \sigma^{-1}\, \partial^\mu\, \sigma \ . \end{aligned} \tag{12.27}$$

In the steps above, we have used the fact that $\nabla_\mu\, \xi_\nu$ is antisymmetric in μ and ν, since ξ is a Killing vector. The upshot from this is that the magnitude of the 4-acceleration is given by

$$|a| = \sqrt{g^{\mu\nu}\, a_\mu\, a_\nu} = \sigma^{-1}\, \sqrt{g^{\mu\nu}\, \partial_\mu\sigma\, \partial_\nu\sigma} \ . \tag{12.28}$$

As the particle approaches the horizon, the factor $\sqrt{g^{\mu\nu}\, \partial_\mu\sigma\, \partial_\nu\sigma}$ becomes equal to the surface gravity (see (12.19)), but the prefactor σ^{-1} diverges, owing to the fact that ξ becomes null on the horizon. Thus the *proper acceleration* of a particle on an orbit of ξ diverges on the horizon (which is why the particle is inevitably drawn through the horizon). However, suppose we measure the acceleration as seen by a static observer at infinity. For such an observer, there will be a scaling factor relating the proper time τ of the particle to the time t measured by the observer at infinity. If the black hole were non-rotating, such as the Schwarzschild solution, ξ would simply be equal to $\partial/\partial t$, and would have $d\tau^2 = -g_{00}\, dt^2$, which could be written nicely as $d\tau^2 = -\xi^\mu\, \xi^\nu\, g_{\mu\nu}\, dt^2$. Since this expression is generally covariant, it provides a natural way of writing the rescaling of the time interval in all cases, and so we shall always have $d\tau = \sigma\, dt$. Consequently, the acceleration of a particle near to the horizon that is on an orbit of ξ, as measured by a static observer at infinity, will be equal to κ. This explains why κ is called the surface gravity.

12.3 First law of black-hole dynamics

To begin, we shall collect some results on the calculation of conserved quantities in general relativity. Specifically, the quantities of interest to us here are the mass, the angular momentum, and the electric charge of a solution such as a black hole.

We already saw, in chapter 11, how the mass of an asymptotically flat spacetime could be calculated by means of the ADM formalism, leading to the formula (11.52). One can show that there is another way in which the mass can be evaluated, by means of a so-called *Komar integral*. Let K be the (unique) asymptotically-timelike Killing vector that generates (canonically-normalised) time translations at infinity. The mass can then be obtained by evaluating the integral

$$M_{Komar} = -\frac{1}{16\pi} \int_{S^2} \epsilon_{\mu\nu}{}^{\rho\sigma}\, \partial_\rho K_\sigma\, d\Sigma^{\mu\nu} \tag{12.29}$$

over the 2-sphere at infinity that forms the boundary of the 3-dimensional spatial volume of the spacetime, where $\epsilon_{\mu\nu\rho\sigma}$ is the Levi-Civita tensor, defined in eqn (7.37). In the examples of the Schwarzschild metric (6.26), the Reissner-Nordström metric (8.11), the Kerr metric (8.14) or the Kerr-Newman metric (8.17), the relevant components of the area element $d\Sigma^{\mu\nu}$ (which is antisymmetric in μ and ν) are $d\Sigma^{23} = -d\Sigma^{32} = d\theta d\varphi$, and the Killing vector K will be $\partial/\partial t$ in each case.[29] We shall not present a derivation of the Komar formula (12.29) for the mass here; a proof can be found in Wald's book.

A Komar formula can also be given for the angular momentum of an isolated asymptotically-flat spacetime (such as the Kerr metric for a rotating black hole). If we denote the azimuthal Killing vector that generates (canonically-normalised) angular translations around the rotation axis by L, the the Komar result is that the angular momentum is given by

$$J_{Komar} = \frac{1}{32\pi} \int_{S^2} \epsilon_{\mu\nu}{}^{\rho\sigma}\, \partial_\rho L_\sigma\, d\Sigma^{\mu\nu}\,, \tag{12.30}$$

again integrated over the boundary 2-sphere at infinity. In the Kerr metric (8.14) and Kerr-Newman metric (8.17), the Killing vector L is given by $L = \partial/\partial\varphi$.

Finally, the conserved electric charge of an asymptotically-flat solution of the Einstein-Maxwell equations will be given by a Gaussian integral, just as in flat space, leading to

$$Q = \frac{1}{16\pi} \int_{S^2} \epsilon_{\mu\nu}{}^{\rho\sigma}\, F_{\rho\sigma}\, d\Sigma^{\mu\nu}\,, \tag{12.31}$$

again integrated over the boundary 2-sphere at infinity.

It can be shown that the conserved mass, angular momentum and electric charge are the three quantities that uniquely characterise a stationary black hole. This result, which is essentially proved by methods analogous to how one proves the uniqueness theorem for the electrostatic potential in electrodynamics, is known as the *No Hair* theorem.

By the early 1970's, it had been established that black holes obey certain relations that are closely analogous to the laws of thermodynamics. We shall only give a brief overview

[29]For those familiar with differential forms, $d\Sigma^{\mu\nu} = dx^\mu \wedge dx^\nu$.

of these properties here, and largely without giving proofs. Details can be found in many textbooks, including those by Wald, and by Hawking and Ellis. At that time these laws of black hole dynamics were just viewed as being analogues of the laws of thermodynamics. In 1974 that all changed, when Hawking published his paper showing that black holes emit thermal radiation.

The law that we shall focus on here is the one known as the first law of black hole dynamics. Let us consider first the Kerr solution for a rotating black hole, in order to illustrate this law. For convenience, we reproduce the Kerr metric (8.14) here:

$$ds^2 = -\frac{\Delta}{\rho^2}\,(dt - a\sin^2\theta\,d\varphi)^2 + \rho^2\Big(\frac{dr^2}{\Delta} + d\theta^2\Big) + \frac{\sin^2\theta}{\rho^2}\,[(r^2+a^2)\,d\varphi - a\,dt]^2\,, \qquad (12.32)$$

where

$$\rho^2 \equiv r^2 + a^2\,\cos^2\theta\,, \qquad \Delta \equiv r^2 - 2m\,r + a^2\,. \qquad (12.33)$$

As mentioned above, this metric has two Killing vectors, namely $\partial/\partial t$ and $\partial/\partial\varphi$, associated respectively with the time-translation symmetry and the azimuthal symmetry around the axis of rotation of the black hole. Using the ADM formula (11.52) or the Komar formula (12.29) to calculate the mass, we can easily see that this is just given by

$$M = m\,, \qquad (12.34)$$

where m in the parameter in the Kerr metric. Using the Komar formula (12.30) for the angular momentum, one finds that this is given by

$$J = am\,. \qquad (12.35)$$

We now define the Killing vector

$$\xi = \frac{\partial}{\partial t} + \Omega\,\frac{\partial}{\partial\varphi}\,, \qquad (12.36)$$

where Ω is a constant. It is straightforward to see that ξ becomes null on the outer horizon, located at $r = r_+$,

$$r_+ = m + \sqrt{m^2 - a^2}\,, \qquad (12.37)$$

the larger of the two roots of $\Delta = 0$, if Ω is given by

$$\Omega = \frac{a}{r_+^2 + a^2}\,. \qquad (12.38)$$

The quantity Ω has the interpretation of being the angular velocity of the horizon of the black hole, as measured from an asymptotically static coordinate frame. The Killing vector

ξ is then the null generator of the outer horizon, which is a Killing horizon as defined in the previous discussion of the surface gravity.

We may also calculate the area of the event horizon. We can do this by looking at the metric on the surface $r = r_+$ at constant time. In other words, we first set $dr = 0$ and $dt = 0$ in (12.32), giving the two-dimensional metric

$$ds^2 = \rho^2\, d\theta^2 + \frac{\Big((r^2 + a^2)^2 - \Delta\, a^2\, \sin^2\theta\Big)\, \sin^2\theta}{\rho^2}\, d\varphi^2\,. \tag{12.39}$$

We now set $r = r_+$, obtaining the metric

$$ds^2 = \rho_+^2\, d\theta^2 + \Big(\frac{2m\, r_+}{\rho_+}\Big)^2\, \sin^2\theta\, d\varphi^2 \tag{12.40}$$

on the outer horizon, where $\rho_+^2 = r_+^2 + a^2\, \cos^2\theta$. The horizon area is therefore given by

$$\mathcal{A} = 2m\, r_+ \int \sin\theta\, d\theta\, d\varphi = 8\pi\, m\, r_+\,. \tag{12.41}$$

Finally, we may calculate the surface gravity κ, which can be done using the formula (12.20). The result, which is fairly straightforward to evaluate and which we leave as an exercise for the reader, is that

$$\kappa = \frac{\sqrt{m^2 - a^2}}{2m\, r_+}\,. \tag{12.42}$$

Note that the surface gravity is constant on the horizon. That this would be the case is obvious in the case of a spherically-symmetric black hole such as Schwarzschild, but it is not a priori obvious in a case such as Kerr, where the horizon, which is topologically a 2-sphere, is not metrically a round sphere. One might have thought κ could have depended on the co-latitude coordinate θ in this case, but it doesn't. In fact there is a general theorem that the surface gravity is necessarily constant over a Killing horizon.

The Kerr black hole metric (12.32) depends on two independent parameters, namely the mass m and the rotation parameter a. The radius r_+ of the outer horizon is then given in terms of these by (12.37). It is often more convenient to use instead the radius outer horizon r_+ and the rotation parameter a as the two indpendent parameters, with m now expressed in terms of these by

$$m = \frac{r_+^2 + a^2}{2r_+}\,. \tag{12.43}$$

This has the advantage of avoiding the need for square roots. Either way, it is now a straightforward matter to verify that if one makes infinitesimal changes to the two independent parameters, then the following equation holds:

$$dM = \frac{\kappa}{8\pi}\, d\mathcal{A} + \Omega\, dJ\,. \tag{12.44}$$

This is known as the first law of black hole dynamics, for the case of (uncharged) rotating black holes. A straightforward extension of the calculations above to the case of the Kerr-Newman black hole solution (8.17), which depends on three independent parameters (mass, rotation parameter and electric charge) leads to the result that in this case we shall have

$$dM = \frac{\kappa}{8\pi}\, d\mathcal{A} + \Omega\, dJ + \Phi\, dQ\,, \tag{12.45}$$

where Φ is the value of the electrostatic potential on the horizon. (To be more precise, Φ is the potential difference between the horizon and infinity.)

We have "derived" the first law of black hole dynamics here by considering the explicit example of the Kerr or Kerr-Newman black hole. One can in fact give a very general derivation of (12.45) that makes no reference to any actual explicit solution, but instead obtains the result from an abstract consideration of the variations of the conserved quantities (mass, angular momentum and charge) that we defined earlier. The derivation is described in detail in Wald's book.

The similarity between (12.45) and the first law of thermodynamics is very striking. If we consider a closed thermodynamic system with energy E, temperature T, entropy S, chemical potentials X_i and their conjugate thermodynamic variables Y_i, then the first law of thermodynamics is

$$dE = T\, dS + \sum_i X_i\, dY_i\,, \tag{12.46}$$

Specific examples of chemical potentials and their conjugate variables are the pair $X = \Omega$, $Y = J$ for a system with angular velocity and angular momentum, and the pair $X = \Phi$ and $Y = Q$ for a system with electric potential and electric charge. What is, thus far, lacking in the comparison between (12.45) and (12.46) is any parallelism between the conjugate pair $(\kappa, \mathcal{A})$ for black holes and the conjugate pair (T, S) in thermodynamics. This missing link was suppplied by Stephen Hawking.

12.4 Hawking radiation in the Euclidean approach

Hawking first derived the black hole radiation by means of a semi-classical analysis, in all fields except gravity are treated as quantum fields, while gravity is still treated classically. This was done because there was no known way, at that time, of successfully treating gravity beyond the classical level.[30] Thus, in the semi-classical approach one essentially studies

[30] More recently, string theory has emerged as a possible way of unifying gravity and the other forces in nature at the full quantum level. And indeed, this has provided some valuable new insights into some of the previously mysterious aspects of Hawking's semi-classical results.

quantum field theories in the curved spacetime background that describes the gravitational field.

Hawking's derivation of black hole radiation required a very careful analysis of what is meant by the vacuum in a quantum field theory in the curved spacetime background of a black hole, and in particular, how the vacuum for an observer at $\mathscr{I}^+$ is related to the vacuum for an observer at $\mathscr{I}^-$. The outcome from this analysis is that in the black-hole background, a zero-particle initial state becomes a state populated by a thermal distribution of particles with respect to the observer at $\mathscr{I}^+$. Rather than going into the details of this derivation, which is quite involved, let us instead follow a route that was developed a little later, once the thermodynamic implications had been digested. The groundwork for this was laid in a paper by Hartle and Hawking, soon after Hawking's original work on black hole radiation, in which they showed that the Green functions for particle wave equations in the black hole background were periodic in imaginary time, with a period $\beta = 1/T$, where T is the Hawking temperature of the black hole.

Such a periodicity is well known in the context of statistical thermodynamics, and is characteristic of a system in thermal equilibrium at temperature $T = 1/\beta$. Roughly speaking, the one considers the two-point amplitude formed between a state $|n, t\rangle$ of energy E_n at time t and the same state at time $t - \mathrm{i}\,\beta$:

$$Z_n = \langle n, t | n, t - \mathrm{i}\,\beta\rangle \,. \tag{12.47}$$

In the Heisenberg picture $e^{-\mathrm{i} H\, t}$ is the time evolution operator, where H is the Hamiltonian and we have chosen units where $\hbar = 1$. Thus

$$|n, t - \mathrm{i}\,\beta\rangle = e^{-\beta H}\, |n, t\rangle \,, \tag{12.48}$$

and so summing over a complete set of energy eigenstates gives

$$\begin{aligned} Z(\beta) &= \sum_n \langle n, t| e^{-\beta H}\, |n, t\rangle \,, \\ &= \sum_n e^{-\beta E_n} \,. \end{aligned} \tag{12.49}$$

This can be recognised as the partition function for a thermal state in equilibrium at temperature $T = 1/\beta$. (We have also chosen units where Boltzmann's constant k_B is set equal to 1.)

The idea of working from the outset in a "Euclidean regime," in which time is replaced by imaginary time, was developed soon after Hawking's original derivation of the black hole radiation, principally by Stephen Hawking and Gary Gibbons.

Let us begin by considering the Schwarzschild solution. We then perform a Wick rotation of the time coordinate, by writing $t = -\mathrm{i}\,\tau$. The original metric (6.26) then becomes

$$ds^2 = \Big(1 - \frac{2m}{r}\Big)\, d\tau^2 + \Big(1 - \frac{2m}{r}\Big)^{-1}\, dr^2 + r^2\, d\Omega^2\ . \tag{12.50}$$

Now, consider the following transformation of the radial coordinate:

$$\rho = 4m\,\Big(1 - \frac{2m}{r}\Big)^{1/2}\ , \tag{12.51}$$

in terms of which the metric (12.50) becomes

$$ds^2 = \Big(\frac{r}{2m}\Big)^4\, d\rho^2 + \rho^2\,\Big(\frac{d\tau}{4m}\Big)^2 + r^2\, d\Omega^2\ . \tag{12.52}$$

Now the coordinate ρ vanishes as r approaches the "horizon" at $r = 2m$. If we look at the form of the metric (12.52) near $r = 2m$, we see that it approaches

$$ds^2 = d\rho^2 + \rho^2\,\Big(\frac{d\tau}{4m}\Big)^2 + 4m^2\, d\Omega^2\ . \tag{12.53}$$

This has a singularity at $\rho = 0$, but under appropriate conditions, namely if $\tau/(4m)$ has period 2π, this is nothing but the familar coordinate singularity at the origin of two-dimensional polar coordinates. (Compare with $ds^2 = dr^2 + r^2\, d\theta^2$.) Of course, if τ is assigned any other period there will be a genuine curvature singularity at $\rho = 0$, since then the metric is like the metric on a cone, which has a delta-function singularity in its curvatuure at the apex. However, if we proceed by making the assumption that this calculation is trying to tell us something, then we would naturally choose to take τ to have the special periodicity for which the nice singularity-free interpretation can be given. The upshot is that we arrive at the interpretation of the Euclideanised Schwarzschild metric as the metric on a smooth manifold defined by

$$0 \le \tau \le 8\pi\, m\ , \qquad 2m \le r \le \infty\ , \tag{12.54}$$

with the angular coordinates θ and φ on the 2-sphere precisely as usual.

This Euclideanised Schwarzschild manifold is completely free of curvature singularities; it makes no more sense to ask what happens for r less than $2m$ here than it does to ask what happens for r less than zero in plane-polar coordinates. The manifold with $r \ge 2m$ is complete. The interesting point is that in terms of the original Schwarzschild spacetime, we have been led to perform a periodic identification in imaginary time, with period $8\pi\, m$. Now, those as we indicated above, a periodicity β in imaginary time is associated with a statistical ensemble in thermal equalibrium at temperature $T = 1/\beta$. Thus we arrive at the conclusion that the Euclideanised Schwarzschild manifold is describing a system in thermal equilibrium at temperature

$$T = \frac{1}{8\pi\, m}\ . \qquad (12.55)$$

This is precisely the temperature already found by Hawking for the black-body radiation emitted by the Schwarzschild black hole. Recall that in the Schwarzschild spacetime, we saw previously that the surface gravity on the future horizon is given by $\kappa = 1/(4m)$, and so indeed the temperature is $T = \kappa/(2\pi)$.

A similar calculation can easily be performed for the Reissner-Nordström solution. In fact, it is quite instructive to do the calculation for a more general class of static metrics, in order to bring out the relation between the surface gravity and the periodicity of τ more transparently. Consider, therefore, a metric of the form

$$\text{Minkowskian}: \qquad ds^2 = -f\, dt^2 + f^{-1}\, dr^2 + r^2\, d\Omega^2\ , \qquad (12.56)$$

$$\text{Euclidean}: \qquad ds^2 = f\, d\tau^2 + f^{-1}\, dr^2 + r^2\, d\Omega^2\ , \qquad (12.57)$$

where we give both its original Minkowskian-signature form, and its form after Euclideanisation. Let us suppose that f, which is taken to be a function only of r, approaches 1 asymptotically as r goes to infinity, and has a simple zero at some point $r = r_0$. (In the case that $f(r)$ has more than one zero, we assume that r_0 is the largest zero.) Thus r_0 corresponds to an event horizon. Let us then define a new radial coordinate $R = f^{1/2}$. Thus we have $dR = \frac{1}{2} f^{-1/2}\, f'\, dr$, and hence, in the vicinity of $r = r_0$, the metric (12.57) approaches

$$ds^2 = \frac{4}{f'(r_0)^2}\Big(dR^2 + \tfrac{1}{4} f'(r_0)^2\, R^2\, d\tau^2\Big) + r_0^2\, d\Omega^2\ . \qquad (12.58)$$

Thus we see that $R = 0$ is like the origin of polar coordinates provided that we identify τ with period $\Delta\tau$ given by

$$\Delta\tau = \frac{4\pi}{f'(r_0)}\ . \qquad (12.59)$$

(The assumption that r_0 is the largest zero of $f(r)$ means that $f'(r_0)$ is positive.)

On the other hand, we can perform a calculation of the surface gravity on the horizon at $r = r_0$ in the metric (12.56). This is a Killing horizon with respect to the timelike Killing vector $K = \partial/\partial t$. Using the expression (12.19) we have $\lambda^2 = -g_{\mu\nu}\, K^\mu\, K^\nu = -g_{tt} = f$, and hence from (12.20)

$$\kappa^2 = \tfrac{1}{4} g^{\mu\nu} f^{-1}\, \partial_\mu f\, \partial_\nu f\Big|_{r=r_0}\ . \qquad (12.60)$$

Thus we see that $\kappa = \frac{1}{2} f'(r_0)$, and so comparing with (12.59) we have the relation

$$\Delta\tau = \frac{2\pi}{\kappa}\ . \qquad (12.61)$$

For a metric such as Kerr, which is stationary but not static, the calculation is a little more tricky. The "Euclidean philosophy" now would be that we should consider operators that are sandwiched between in and out states that have coordinate values related by $(t, r, \theta, \varphi) \sim (t + \mathrm{i}\,\beta, r, \theta, \varphi + \mathrm{i}\,\Omega_H\,\beta)$. Thus in the Euclideanised metric we should make everything real by taking $t = -\mathrm{i}\,\tau$ and $\Omega_H = \mathrm{i}\,\widetilde{\Omega}_H$, where $\widetilde{\Omega}_H$ is real. This means that we should take the rotation parameter a to be imaginary, $a = \mathrm{i}\,\alpha$. Thus the Kerr metric (8.14) Euclideanises to become

$$
\begin{aligned}
ds^2 &= \frac{(\Delta + \alpha^2\,\sin^2\theta)}{\rho^2}\,d\tau^2 - \frac{4m\,\alpha\,r\,\sin^2\theta}{\rho^2}\,d\tau\,d\varphi \\
&\quad + \frac{\Big((r^2-\alpha^2)^2 + \Delta\,\alpha^2\,\sin^2\theta\Big)\,\sin^2\theta}{\rho^2}\,d\varphi^2 + \frac{\rho^2}{\Delta}\,dr^2 + \rho^2\,d\theta^2\ , \qquad (12.62)
\end{aligned}
$$

where

$$
\rho^2 = r^2 - \alpha^2\,\cos^2\theta\ , \qquad \Delta = r^2 - 2m\,r - \alpha^2\ . \qquad (12.63)
$$

We shall want to examine the behaviour of this metric in the vicinity of $r_+ = m + \sqrt{m^2+\alpha^2}$, where Δ first vanishes as one approaches from large r. We shall introduce a new radial coordinate R, defined by $R = \Delta^{1/2}$, and then take the limit when R is very small. We can in fact judiciously set $r = r_+$ at the outset in certain places in the metric (12.62), namely in those places where no singularity will result from doing so. Thus near to $r = r_+$, the metric approaches

$$
ds^2 = \frac{(\Delta + \alpha^2\,\sin^2\theta)}{\rho_+^2}\,d\tau^2 - \frac{4m\,\alpha\,r_+\,\sin^2\theta}{\rho_+^2}\,d\tau\,d\varphi + \frac{4m^2\,r_+^2\,\sin^2\theta}{\rho_+^2}\,d\varphi^2 + \frac{\rho_+^2}{\Delta}\,dr^2 + \rho_+^2\,d\theta^2\ , \qquad (12.64)
$$

where $\rho_+^2 = r_+^2 - \alpha^2\,\cos^2\theta$, and we have used the fact that $r_+^2 - \alpha^2 = 2m\,r_+$. Note that ρ_+^2 is non-vanishing for all θ. The metric (12.64) can be reorganised, by completing the square, so that it becomes

$$
ds^2 = \frac{\rho_+^2}{\Delta}\,dr^2 + \frac{\Delta}{\rho_+^2}\,d\tau^2 + \frac{4m^2\,r_+^2\,\sin^2\theta}{\rho_+^2}\,(d\varphi - \widetilde{\Omega}_H\,d\tau)^2 + \rho_+^2\,d\theta^2\ , \qquad (12.65)
$$

where $\widetilde{\Omega}_H = \alpha/(2m\,r_+)$ is the "angular momentum" on the horizon in his Euclideanised metric (see (12.38)). Now, making our substitution $R = \Delta^{1/2}$, and noting that near to $r = r_+$ we can consequently write $2R\,dR = d[(r-r_+)(r-r_-)] \sim dr\,(r_+ - r_-) = 2\sqrt{m^2+\alpha^2}\,dr$, we see that near $r = r_+$ the Euclideanised Kerr metric approaches

$$
ds^2 = \frac{\rho_+^2}{m^2+\alpha^2}\,dR^2 + \frac{R^2}{\rho_+^2}\,d\tau^2 + \frac{4m^2\,r_+^2\,\sin^2\theta}{\rho_+^2}\,(d\varphi - \widetilde{\Omega}_H\,d\tau)^2 + \rho_+^2\,d\theta^2\ . \qquad (12.66)
$$

We now have to examine in detail what happens as R approaches zero. If θ is equal to 0 or π, the prefactor of $(d\varphi - \widetilde{\Omega}_H\,d\tau)^2$ vanishes, and consequently we shall have a conical singularity at $R=0$ in the (R,τ) plane unless τ has the appropriate periodicity. Noting that at $\theta=0$ or $\theta=\pi$ we have $\rho_+^2 = r_+^2 - \alpha^2 = 2m\, r_+$, we see that the relevant two-dimensional part of the metric is

$$ds^2 = \frac{2m\, r_+}{m^2+\alpha^2}\, \Big[dR^2 + R^2\, \Big(\frac{m^2+\alpha^2}{4m^2\, r_+^2}\Big)\, d\tau^2\Big]\ , \tag{12.67}$$

and thus the conical singularity is avoided if τ is identified periodically with period

$$\Delta\tau = \frac{4\pi\, m\, r_+}{\sqrt{m^2+\alpha^2}}\ . \tag{12.68}$$

If θ takes any other generic value $0<\theta<\pi$, the prefactor of $(d\varphi - \widetilde\Omega_H\, d\tau)^2$ in (12.66) is non-zero, and no further conditions arise.

Comparing (12.68) with the expression for the surface gravity for the Kerr metric that we obtained in (12.42), we see that the periodicity of τ is again given by

$$\Delta\tau = \frac{2\pi}{\kappa}\ , \tag{12.69}$$

where κ is given by (12.42) with $a = \mathrm{i}\,\alpha$.

The upshot from the discussions above is that for all the black hole examples, the Euclideanised metrics extend smoothly onto singularity-free manifolds provided that the imaginary time coordinate is assigned the period $\Delta\tau = 2\pi/\kappa$, where κ is the surface gravity. By the general arguments presented earlier, this periodicity in imaginary time corresponds to a system in thermal equilibrium at temperature

$$T = \frac{\kappa}{2\pi}\,. \tag{12.70}$$

This is the same as the result Hawking first derived by purely Lorentian-signature quantum field theory, for the temperature at which black holes radiate.

The first law of black hole dynamics (12.45), with κ replaced by $2\pi\, T$, now becomes the first law of thermodynamics,

$$dM = T\, dS + \Omega\, dJ + \Phi\, dQ\,, \tag{12.71}$$

provided that we identify the entropy S as

$$S = \frac{1}{4}\, A\,, \tag{12.72}$$

where A is the area of the event horizon.

13 Differential Forms

Here, we shall give an introduction to the theory of *differential forms*, and some of their applications in general relativity. One application in particular is that they can provide a convenient way of calculating the curvature tensor of a given metric, which is often easier and less tedious than the methods we have seen so far.

13.1 Definitions

A particularly important class of tensors comprises cotensors whose components are totally antisymmetric;

$$U_{\mu_1\cdots\mu_p} = U_{[\mu_1\cdots\mu_p]}\ . \tag{13.1}$$

Here, we are using the notation intoduced previously, that square brackets enclosing a set of indices indicate that they should be totally antisymmetrised, with strength one. Thus we have have

$$\begin{aligned} U_{[\mu\nu]} &= \frac{1}{2!}\,(U_{\mu\nu} - U_{\nu\mu})\ , \\ U_{[\mu\nu\sigma]} &= \frac{1}{3!}\,(U_{\mu\nu\sigma} + U_{\nu\sigma\mu} + U_{\sigma\mu\nu} - U_{\mu\sigma\nu} - U_{\sigma\nu\mu} - U_{\nu\mu\sigma})\ , \end{aligned} \tag{13.2}$$

etc. Generally, for p indices, there will be $p!$ terms, comprising the $\frac{1}{2}p!$ even permutations of the indices, which enter with plus signs, and the $\frac{1}{2}p!$ odd permutations, which enter with minus signs. The $1/p!$ prefactor ensures that the antisymmetrisation is of strength one. In particular, this means that antisymmetrising twice leaves the tensor the same: $U_{[[\mu_1\cdots\mu_p]]} = U_{[\mu_1\cdots\mu_p]}$.

Recall that geometrically, we may think of any p-index cotensor W (not necessarily antisymmetric) as an object

$$W = W_{\mu_1\mu_2\cdots\mu_p}\, dx^{\mu_1}\otimes dx^{\mu_2}\otimes\cdots\otimes dx^{\mu_p}\ , \tag{13.3}$$

where $W_{\mu_1\cdots\mu_p}$ are its components with respect to the basis $dx^{\mu_1}\otimes dx^{\mu_2}\otimes\cdots\otimes dx^{\mu_p}$. Clearly, if the cotensor is antisymmetric in its indices it will make an antisymmetric projection on the tensor product of basis 1-forms dx^μ. Since antisymmetric cotensors are so important in differential geometry, a special symbol is introduced to denote an antisymmetrised product of basis 1-forms. This symbol is the wedge product, $\wedge$. Thus we define

$$\begin{aligned} dx^\mu\wedge dx^\nu &= dx^\mu\otimes dx^\nu - dx^\nu\otimes dx^\mu\ , \\ dx^\mu\wedge dx^\nu\wedge dx^\sigma &= dx^\mu\otimes dx^\nu\otimes dx^\sigma + dx^\nu\otimes dx^\sigma\otimes dx^\mu + dx^\sigma\otimes dx^\mu\otimes dx^\nu \\ &\quad - dx^\mu\otimes dx^\sigma\otimes dx^\nu - dx^\sigma\otimes dx^\nu\otimes dx^\mu - dx^\nu\otimes dx^\sigma\otimes dx^\mu \end{aligned} \tag{13.4}$$

and so on. (Note that there is no $1/p!$ combinatoric factor in these definitions.)

Cotensors antisymmetric in p indices are called p-forms. Suppose we have such an object A, with components $A_{\mu_1\cdots\mu_p}$. It therefore has the expansion

$$A = \frac{1}{p!}\, A_{\mu_1\cdots\mu_p}\, dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}\ . \tag{13.5}$$

Note that a function is a special case of a p-form with $p = 0$. It is quite easy to see from the definitions above that if A is a p-form, and B is a q-form, then they satisfy

$$A \wedge B = (-1)^{pq}\, B \wedge A\ . \tag{13.6}$$

13.2 Exterior derivative

The exterior derivative d is defined to act on a p-form field and produce a $(p+1)$-form field. It is defined as follows. On functions (i.e. 0-forms), it is just the operation of taking the differential; we met this earlier:

$$df = \partial_\mu f\, dx^\mu\ . \tag{13.7}$$

More generally, on a p-form $\omega = (1/p!)\, \omega_{\mu_1\cdots\mu_p}\, dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}$, it is defined by

$$d\omega = \frac{1}{p!}\, (\partial_\nu \omega_{\mu_1\cdots\mu_p})\, dx^\nu \wedge dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}\ . \tag{13.8}$$

Note that from our definition of p-forms, it follows that the components of the $(p+1)$-form $d\omega$ are given by

$$(d\omega)_{\nu\mu_1\cdots\mu_p} = (p+1)\, \partial_{[\nu}\, \omega_{\mu_1\cdots\mu_p]}\ . \tag{13.9}$$

By this we mean that the expansion of the $(p+1)$-form $d\omega$ in the coordinate basis we are using takes the form

$$d\omega = \frac{1}{(p+1)!}\, (d\omega)_{\mu_1\cdots\mu_{p+1}}\, dx_1^\mu \wedge \cdots \wedge dx^{\mu_{p+1}}\ . \tag{13.10}$$

It is easily seen from the definitions that if A is a p-form and B is a q-form, then the following Leibnitz rule holds:

$$d(A \wedge B) = dA \wedge B + (-1)^p\, A \wedge dB\ . \tag{13.11}$$

It is also easy to see from the definition of d that if it acts twice, it automatically gives zero, i.e. $dd\omega = 0$ where ω is any differential form of any degree p. This just follows from (13.8), which shows that d is an *antisymmetric* derivative, while on the other hand partial derivatives *commute*.

A simple, and important, example of differential forms and the use of the exterior derivative can be seen in Maxwell theory. The vector potential is a 1-form, $A = A_\mu\, dx^\mu$. The Maxwell field strength is a 2-form, $F = \frac{1}{2} F_{\mu\nu}\, dx^\mu \wedge dx^\nu$, and we can construct it from A by taking the exterior derivative:

$$F = dA = \partial_\mu\, A_\nu\, dx^\mu \wedge dx^\nu = \tfrac{1}{2} F_{\mu\nu}\, dx^\mu \wedge dx^\nu\ , \tag{13.12}$$

from which we read off that $F_{\mu\nu} = 2\, \partial_{[\mu}\, A_{\nu]} = \partial_\mu\, A_\nu - \partial_\nu\, A_\mu$. The fact that $d^2 \equiv 0$ means that $dF = 0$, since $dF = d^2 A$. The equation $dF = 0$ is nothing but the Bianchi identity in Maxwell theory, since from the definition (13.8) we have

$$dF = \tfrac{1}{2} \partial_\mu\, F_{\nu\rho}\, dx^\mu \wedge dx^\nu \wedge dx^\rho\ , \tag{13.13}$$

hence implying that $\partial_{[\mu}\, F_{\nu\rho]} = 0$.

The Bianchi identity Maxwell equation $dF = 0$ can always be (locally) solved by introducing the vector potential (1-form) A and writing $F = dA$. It is guaranteed that this satisfies $dF = 0$, since, as we saw in general, d^2 is identically zero when acting on any differential form. The qualification that we can in general only solve $dF = 0$ by writing $F = dA$ *locally* is a little more subtle. We shall discuss this in greater detail a bit later.

13.3 Hodge dual

We can also express the Maxwell field equation elegantly in terms of differential forms. This requires the introduction of the Hodge dual operator $*$, which is defined in terms of the totally-antisymmetric Levi-Civita tensor that we introduced earlier. This requires the introduction of a metric tensor $g_{\mu\nu}$, which we have not needed until now in this discussion of differential forms. Recall that we defined the totally-antisymmetric tensor density $\varepsilon_{\mu_1 \cdots \mu_n}$ in n dimensions, whose components are completely specified, given its antisymmetry, by saying that

$$\varepsilon_{012\cdots n-1} = +1\,. \tag{13.14}$$

The totally-antisymmetric Levi-Civita *tensor* is then defined to be

$$\epsilon_{\mu_1 \cdots \mu_n} = \sqrt{-g}\, \varepsilon_{\mu_1 \cdots \mu_n}\,. \tag{13.15}$$

(We actually defined these previously just in the four-dimensional case, but the generalisation to n dimensions that we are presenting here is immediate.) It is a straightforward exercise to show that if we write $n = p + q$, and take the product of two epsilon tensors contracted on p indices as shown here:

$$\epsilon_{\mu_1\cdots\mu_p\nu_1\cdots\nu_q}\,\epsilon^{\mu_1\cdots\mu_p\rho_1\cdots\rho_q} = -p!\,q!\,\delta^{\rho_1\cdots\rho_q}_{\nu_1\cdots\nu_q}\,, \tag{13.16}$$

where the multi-index Kronecker delta tensors are defined by

$$\delta^{\nu_1\cdots\nu_q}_{\mu_1\cdots\mu_q} \equiv \delta^{\nu_1}_{[\mu_1}\,\delta^{\nu_2}_{\mu_2}\cdots\delta^{\nu_q}_{\mu_q]}\,. \tag{13.17}$$

(Note that having antisymmetrised the Kronecker deltas in the product over their lower indices, antisymmetrisation over their upper indices is automatic.) Note also that in eqn (13.16), the indices on the second epsilon tensor have been raised using the inverse metric $g^{\mu\nu}$. The minus sign in (13.16) arises because of the negative eigenvalue of the metric tensor in a spacetime of signature $(-++\cdots+)$.

The Hodge dual operator $*$ is now defined as follows:

$$*(dx^{\mu_1}\cdots dx^{\mu_p}) \equiv \frac{1}{(n-p)!}\,\epsilon_{\nu_1\cdots\nu_{n-p}}{}^{\mu_1\cdots\mu^p}\,dx^{\nu_1}\wedge\cdots\wedge dx^{\nu_{n-p}}\,. \tag{13.18}$$

Thus $*$ is a map from p-forms to $(n-p)$-forms: Acting on a p-form ω, expanded as in (13.9), we have

$$*\omega = \frac{1}{p!\,(n-p)!}\,\epsilon_{\nu_1\cdots\nu_{n-p}}{}^{\mu_1\cdots\mu^p}\,\omega_{\mu_1\cdots\mu_p}\,dx^{\nu_1}\wedge\cdots\wedge dx^{\nu_{n-p}}\,, \tag{13.19}$$

and so the $(n-p)$-form $*\omega$ has the components

$$(*\omega)_{\mu_1\cdots\mu_q} = \frac{1}{p!}\,\epsilon_{\mu_1\cdots\mu_q}{}^{\nu_1\cdots\nu_p}\,\omega_{\nu_1\cdots\nu_p}\,, \tag{13.20}$$

where, as before, we are writing $n = p+q$, and so $q = n-p$.

It is straightforward to see from the previous definitions, and from (13.16), that if applied twice to a p-form one again gets a p-form, and in fact if we start with the p-form ω then

$$**\,\omega = (-1)^{pq+1}\,\omega\,, \tag{13.21}$$

where again $n = p+q$.

It is also evident that if we start from a p-form ω, then $*d*\omega$ is a $(p-1)$ form. In fact, it is related to the divergence of ω, and

$$(*d*\omega)_{\mu_1\cdots\mu_{p-1}} = (-1)^{pq+p}\,\nabla_\nu\,\omega^\nu{}_{\mu_1\cdots\mu_{p-1}}\,, \tag{13.22}$$

where ∇_ν is the usual covariant derivative built using the Christoffel connection.[31] (We leave it as an exercise to derive this result.)

[31] In the case of an n-dimensional space with t time directions, eqn (13.21) reads $**\,\omega = (-1)^{pq+t}\,\omega$, and eqn (13.22) reads $(*d*\omega)_{\mu_1\cdots\mu_{p-1}} = (-1)^{pq+p+t+1}\,\nabla_\nu\,\omega^\nu{}_{\mu_1\cdots\mu_{p-1}}$. The usual spacetime of general relativity corresponds to $t=1$, whilst the case of a space with positive definite metric signature corresponds to $t=0$.

With these preliminaries, it can be seen that the source-free Maxwell field equation $\nabla_\mu F^{\mu\nu} = 0$ can be written in the language of differential forms as

$$d * F = 0\,. \tag{13.23}$$

13.4 Vielbein, spin connection and curvature 2-form

We begin by observing that we may "take the square root" of a metric $g_{\mu\nu}$, by introducing a vielbein,[32] which is a basis of 1-forms $e^a = e^a_\mu\, dx^\mu$, with components e^a_μ, having the property

$$g_{\mu\nu} = \eta_{ab}\, e^a_\mu\, e^b_\nu\ . \tag{13.24}$$

Here the indices a are a new type, different from the coordinate indices μ we have encountered up until now. They are called local-Lorentz indices, or tangent-space indices, and η_{ab} is a "flat" metric, with constant components. The language of "local-Lorentz" indices stems from the situation when the metric $g_{\mu\nu}$ has Minkowskian signature (which is $(-,+,+,\ldots,+)$ in sensible conventions). The signature of η_{ab} must be the same as that of $g_{\mu\nu}$, so if we are working in general relativity with Minkowskian signature we will have

$$\eta_{ab} = \mathrm{diag}\,(-1,1,1,\ldots,1)\ . \tag{13.25}$$

If, on the other hand, we are working in a space with Euclidean signature $(+,+,\ldots,+)$, then η_{ab} will just equal the Kronecker delta, $\eta_{ab} = \delta_{ab}$, or in other words

$$\eta_{ab} = \mathrm{diag}\,(1,1,1,\ldots,1)\ . \tag{13.26}$$

Of course the choice of vielbeins[33] e^a as the square root of the metric in (13.24) is to some extent arbitrary. Specifically, we could, given a particular choice of vielbein e^a, perform an (pseudo)orthogonal transformation to get another equally-valid vielbein e'^a, given by

$$e'^a = \Lambda^a{}_b\, e^b\ , \tag{13.27}$$

where $\Lambda^a{}_b$ is a matrix satisfying the (pseudo)orthogonality condition

$$\eta_{ab}\, \Lambda^a{}_c\, \Lambda^b{}_d = \eta_{cd}\ . \tag{13.28}$$

Note that $\Lambda^a{}_b$ can be coordinate dependent. If the n-dimensional manifold has a Euclidean-signature metric then $\eta = \mathbb{1}$ and (13.28) is literally the orthogonality condition $\Lambda^T\,\Lambda = \mathbb{1}$.

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[32] German for "many legs."

[33] Strictly speaking, if we recall its German origin, the plural of vielbein would be vielbeine, and in fact, as with any noun in German, we should have used an upper case first letter for Vielbein or Vielbeine, but this would perhaps be carrying pedantry a little far.

Thus in this case the arbitrariness in the choice of vielbein is precisely the freedom to make local $O(n)$ rotations in the tangent space. If the metric signature is Minkowskian, then instead (13.28) is the condition for Λ to be an $O(1, n-1)$ matrix; in other words, one then has the freedom to perform Lorentz transformations in the tangent space. The Lorentz transformation matrix may depend upon the spacetime coordinates, and so (13.28) is called a *local Lorentz transformation.* We shall typically use the words "local Lorentz transformation" regardless of whether we are working with metrics of Minkowskian or Euclidean signature.

Briefly reviewing the next steps, we introduce the spin connection, or connection 1-forms, ${\omega^a}_b = {\omega^a}_{b\mu}\, dx^\mu$, and the torsion 2-forms $T^a = \frac{1}{2} T^a_{\mu\nu}\, dx^\mu \wedge dx^\nu$, defining

$$T^a = de^a + {\omega^a}_b \wedge e^b\ . \tag{13.29}$$

Next, we define the curvature 2-forms ${\Theta^a}_b$, *via* the equation

$$\Theta^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b\ . \tag{13.30}$$

Note that if we adopt the obvious matrix notation where the local Lorentz transformation (13.27) is written as $e' = \Lambda\, e$, then we have the property that ${\omega^a}_b$, T^a and ${\Theta^a}_b$ transform as follows:

$$\begin{aligned} \omega' &= \Lambda\, \omega\, \Lambda^{-1} + \Lambda\, d\Lambda^{-1}\ , \\ T' &= \Lambda\, T\ , \qquad \Theta' = \Lambda\, \Theta\, \Lambda^{-1}\ . \end{aligned} \tag{13.31}$$

Thus the torsion 2-forms T^a and the curvature 2-forms ${\Theta^a}_b$ both transform nicely, in a covariant way, under local Lorentz transformations, while the spin connection does not; it has an extra inhomogeneous term in its transformation rule. This is the characteristic way in which connections transform. Because of this, we can define a Lorentz-covariant exterior derivative D as follows:

$$D V^a{}_b \equiv dV^a{}_b + \omega^a{}_c \wedge V^c{}_b - \omega^c{}_b \wedge V^a{}_c\ , \tag{13.32}$$

where ${V^a}_b$ is some set of p-forms carrying tangent-space indices a and b. One can easily check that if ${V^a}_b$ itself transforms covariantly under local Lorentz transformations, then so does $D{V^a}_b$. In other words, the potentially-troublesome terms where the exterior derivative lands on the transformation matrix Λ are cancelled out by the contributions from the inhomgeneous second term in the transformation rule for ${\omega^a}_b$ in (13.31). We have taken the example of ${V^a}_b$ with one upstairs and one downstairs tangent space index for simplicity, but the generalisation to arbitrary numbers of indices is immediate. There is one term like the second term on the right-hand side of (13.32) for each upstairs index, and a term like the third term on the right-hand side of (13.32) for each downstairs index.

The covariant exterior derivative D will commute nicely with the process of contracting tangent-space indices with η_{ab}, provided that we require

$$D\,\eta_{ab} \equiv d\eta_{ab} - \omega^c{}_a\,\eta_{cb} - \omega^c{}_b\,\eta_{ac} = 0\ . \tag{13.33}$$

Since we are taking the components of η_{ab} to be literally constants, meaning that $d\eta_{ab} = 0$, it follows from this equation, which is known as the equation of *metric compatibility*, that

$$\omega_{ab} = -\omega_{ba}\ , \tag{13.34}$$

where ω_{ab} is, by definition, $\omega^a{}_b$ with the upper index lowered using η_{ab}: $\omega_{ab} \equiv \eta_{ac}\,\omega^c{}_b$. With this imposed, it is now the case that we can take covariant exterior derivatives of products, and freely move the local-Lorentz metric tensor η_{ab} through the derivative. This means that we get the same answer if we differentiate the product and then contract some indices, or if instead we contract the indices and then differentiate.

In addition to the requirement of metric compatibiilty we usually also choose a *torsion-free* spin-connection, meaning that we demand that the torsion 2-forms T^a defined by (13.29) vanish. If we assume $T^a = 0$ for now, then equation (13.29), together with the metric-compatibility condition (13.34), determine $\omega^a{}_b$ uniquely. In other words, the two conditions

$$de^a = -\omega^a{}_b \wedge e^b\ , \qquad \omega_{ab} = -\omega_{ba} \tag{13.35}$$

have a unique solution. It can be given as follows. Let us say that, by definition, the exterior derivatives of the vielbeins e^a are given by

$$de^a = -\tfrac{1}{2} c_{bc}{}^a\, e^b \wedge e^c\ , \tag{13.36}$$

where the structure functions $c_{bc}{}^a$ are, by definition, antisymmetric in bc. Then the solution for ω_{ab} is given by

$$\omega_{ab} = \tfrac{1}{2}(c_{abc} + c_{acb} - c_{bca})\, e^c\ , \tag{13.37}$$

where $c_{abc} \equiv \eta_{cd}\, c_{ab}{}^d$. It is easy to check by direct substitution that this indeed solves the two conditions (13.35).

The procedure, then, for calculating the curvature 2-forms for a metric $g_{\mu\nu}$ with vielebeins e^a is the following. We write down a choice of vielbein, and by taking the exterior derivative we read off the coefficients ${c_{bc}}^a$ in (13.36). Using these, we calculate the spin connection using (13.37). Then, we substitute into (13.30), to calculate the curvature 2-forms.

Each curvature 2-form ${\Theta^a}_b$ has, as its components, a tensor that is antisymmetric in two coordinate indices. This is the Riemann tensor, defined by

$$ {\Theta^a}_b = \tfrac{1}{2} {R^a}_{b\mu\nu}\, dx^\mu \wedge dx^\nu \ . \tag{13.38} $$

We may always use the vielbein e^a_μ, which is a non-degenerate $n \times n$ matrix in n dimensions, to convert between coordinate indices μ and tangent-space indices a. For this purpose we also need the inverse of the vielbein, sometimes denoted by E^μ_a, and satisfying the defining properties[34]

$$ E^\mu_a\, e^a_\nu = \delta^\mu_\nu \ , \qquad E^\mu_a\, e^b_\mu = \delta^a_b \ . \tag{13.39} $$

Then we may define Riemann tensor components entirely within the tangent-frame basis, as follows:

$$ {R^a}_{bcd} \equiv E^\mu_c\, E^\nu_d\, {R^a}_{b\mu\nu} \ . \tag{13.40} $$

Note that we use the same symbol for the tensors, and distinguish them simply by the kinds of indices that they carry. (This requires that one pay careful attention to establishing unambiguous notations, which keep track of which are coordinate indices, and which are tangent-space indices!) In terms of ${R^a}_{bcd}$, it is easily seen from the various definitions that we have

$$ {\Theta^a}_b = \tfrac{1}{2} {R^a}_{bcd}\, e^c \wedge e^d \ . \tag{13.41} $$

From the Riemann tensor two further quantities can be defined; the Ricci tensor R_{ab} and the Ricci scalar R:

$$ R_{ab} = {R^c}_{acb} \ , \qquad R = \eta^{ab}\, R_{ab} \ . \tag{13.42} $$

Note that the Riemann tensor and Ricci tensor have the following symmetries, which can be proved straightforwardly from the definitions:

$$ \begin{aligned} & R_{abcd} = -R_{bacd} = -R_{abdc} = R_{cdab} \ , \\ & R_{abcd} + R_{acdb} + R_{adbc} = 0 \ , \\ & R_{ab} = R_{ba} \ . \end{aligned} \tag{13.43} $$

[34]Note that introducing the new symbol E for the inverse vielbein is not really necessary, since it is just what one gets by raising or lowering coordinate or local-Lorentz indices with the coordinate or local-Lorentz metrics. Thus $E^\mu_a = g^{\mu\nu}\, \eta_{ab}\, e^b_\nu$, and so there is no ambiguity in simply writing E^μ_a as e^μ_a. Often, it is more convenient to do this.

13.5 Relation to coordinate-frame calculation

The description of torsion and curvature in terms of the vielbein and differential forms can be related to the previous coordinate-frame metric description of connections and curvature. Recall that in that earlier discussion, we declared more or less from the outset that we would take the connection $\Gamma^\mu{}_{\nu\rho}$ to be *symmetric* in ν and ρ, and this, together with the assumption of *metric compatibility* $\nabla_\mu\, g_{\nu\rho} = 0$, led to the unique solution for $\Gamma^\mu{}_{\nu\rho}$ as the Christoffel connection, as in eqn (4.48). Similarly, in our discussion in terms of differential forms above, we made the assumption that the torsion 2-form T^a vanished, and that, together with the assumption of local-Lorentz metric compatibility $d\eta_{ab} = 0$, led to the unique solution (13.37) for the spin connection $\omega^a{}_b$. In demonstrating the relation between the vielbein description and the metric description, we shall not make any assumptions about the vanishing of torsion.[35] In what follows we shall denote the Christoffel connection by $\bar\Gamma^\mu{}_{\nu\rho}$, and the torsion-free spin connection by $\bar\omega^a{}_b$.

We begin by writing a general spin connection $\omega^a{}_b$ in terms of the torsion-free spin connection $\bar\omega^a{}_b$ plus an additional term:

$$\omega_\mu{}^a{}_b = \bar\omega_\mu{}^a{}_b + K_\mu{}^a{}_b\,, \tag{13.44}$$

where we are now writing the connection 1-forms in terms of their coordinate-frame components:

$$\omega^a{}_b = \omega_\mu{}^a{}_b\, dx^\mu\,, \qquad \bar\omega^a{}_b = \bar\omega_\mu{}^a{}_b\, dx^\mu\,. \tag{13.45}$$

Thus $\bar\omega^a{}_b$ is what we were previously calling simply $\omega^a{}_b$ when we were assuming that there was no torsion; it is defined (uniquely) by

$$de^a + \bar\omega^a{}_b \wedge e^b = 0\,, \qquad \bar\omega_{ab} = -\bar\omega_{ba}\,, \tag{13.46}$$

where, as always, local-Lorentz indices are lowered or raised using the local-Lorentz metric η_{ab} or its inverse η^{ab}. The quantity $K_\mu{}^a{}_b$ in (13.44) is called the *Contorsion*.[36] We shall

[35]Torsion usually plays no role in discssions of general relativity, but it *is* important in the context of supergravity. Specifically, it turns out that many of the equations in supergravity can be written more compactly and elegenatly by using a covariant derivative defined using a connection with torsion. The torsion is "generated" by terms bilinear in the fermion fields. A good introduction to supergravity may be found in the book "Supergravity" by D.Z. Freedman and A. Van Proeyen.

[36]There is some disagreement in the literature as to whether it is called contorsion or contortion. Since it is closely related to the torsion the former seems to be more appropriate. Although we are following Freeman and Van Proeyen in their book on Supergravity for mathematical conventions on this topic, we are not going to follow their lingusitic convention of calling it contortion.

require that not only $\bar{\omega}^a{}_b$ but also $\omega^a{}_b$ should be compatible with the local-Lorentz metric η_{ab}, so

$$D\eta_{ab} = d\eta_{ab} - \omega^c{}_a\,\eta_{cb} - \omega^c{}_b\,\eta_{ac} = 0\,, \tag{13.47}$$

and hence $\omega_{ab} = -\omega_{ba}$. Thus it follows from (13.44) that

$$K_{\mu ab} = -K_{\mu ba}\,, \tag{13.48}$$

where again, the upper local-Lorentz index is lowered using the local-Lorentz metric. It is very important to keep track of the ordering of indices on $K_{\mu ab}$; the first index is the coordinate index while the second and third indices are the local-Lorentz indices.

From the definition (13.29) of the torsion, and the definition (13.44) of the contorsion, it follows, using (13.46), that

$$\begin{aligned} \tfrac{1}{2}T^a_{\mu\nu}\,dx^\mu\wedge dx^\nu &= de^a + \bar{\omega}^a{}_b\wedge e^b + K_\mu{}^a{}_b\,e^b_\nu\,dx^\mu\wedge dx^\nu\,,\\ &= K_\mu{}^a{}_b\,e^b_\nu\,dx^\mu\wedge dx^\nu\,, \end{aligned} \tag{13.49}$$

and so

$$T^a_{\mu\nu} = K_\mu{}^a{}_b\,e^b_\nu - K_\nu{}^a{}_b\,e^b_\mu\,. \tag{13.50}$$

(Here, as always, one must be careful when reading off the components of tensors that are contracted onto wedge products of coordinate differentials to remember that the wedge product is antisymmetric, and so it enforces a projection onto the antisymmetric part of the contracted tensor.) We can use the vielbein and its inverse to map back and forth between coordinate indices and local-Lorentz indices, and so if we define[37]

$$T_{\mu\nu\rho} \equiv T^a_{\mu\nu}\,e_{a\rho}\,, \tag{13.51}$$

then we see that (13.50) implies

$$T_{\mu\nu\rho} = K_{\mu\rho\nu} - K_{\nu\rho\mu}\,. \tag{13.52}$$

(Here $K_{\mu\nu\rho} \equiv K_{\mu ab}\,e^a_\nu\,e^b_\rho$.)

[37]As with the definition of the index ordering in $K_{\mu ab}$, here one must also be very careful about the index ordering. Note that when it is lowered as a coordinate index using $e_{a\rho}$, the local-Lorentz index a on $T^a_{\mu\nu}$ becomes the *third* index on $T_{\mu\nu\rho}$. Thus the torsion tensor $T_{\mu\nu\rho}$ is automatically antisymmetric in its *first two* indices; $T_{\mu\nu\rho} = -T_{\nu\mu\rho}$, while the contorsion tensor $K_{\mu\nu\rho}$ is automatically antisymmetric in its *last two* indices, $K_{\mu\nu\rho} = -K_{\mu\rho\nu}$.

A simple calculation, making use of (13.52) and the antisymmetry properties of the torsion and contorsion tensors as stated in footnote 37, shows that $T_{\mu\nu\rho} - T_{\nu\rho\mu} + T_{\rho\mu\nu}$ is equal to $-2K_{\mu\nu\rho}$, and so we can express the contorsion in terms of the torsion as

$$K_{\mu\nu\rho} = -\tfrac{1}{2}(T_{\mu\nu\rho} - T_{\nu\rho\mu} + T_{\rho\mu\nu})\,. \tag{13.53}$$

We are now ready to establish the relation between the vielbein formulation and the metric formulation of connections and curvatures. To do this we begin by extending the previous notions of the covariant derivative to include the case where the covariant derivative, which we shall call D_μ, acts on an object carrying both coordinate indices and local-Lorentz indices. Thus for each coordinate index we have a connection term as in (4.42), and for each local-Lorentz index we have a term as in (13.32). In particular, acting on the vielbein e^a_ν we shall have

$$D_\mu\, e^a_\nu = \partial_\mu\, e^a_\nu + \omega_\mu{}^a{}_b\, e^b_\nu - \Gamma^\rho{}_{\mu\nu}\, e^a_\rho\,. \tag{13.54}$$

Note that we are not yet making any assumption about $\Gamma^\rho{}_{\mu\nu}$; in particular, we are *not* assuming it is the Christoffel connection. However, for the same reasons that motivated our previous imposition of metric compatibility (so that raising or lowering indices would commute with covariant differentiation), here we shall impose the requirement of *vielbein compatibility*, namely $D_\mu\, e^a_\nu = 0$. This ensures not only that raising or lowering coordinate indices or local-Lorent indices commutes with covariant differentiation, but also that converting between local-Lorentz indices and coordinate indices by using the vielbein commutes with covariant differentiation.

Consider first the contraction of (13.54) with $dx^\mu \wedge dx^\nu$, which, from the previous definitions, means

$$\begin{aligned} De^a &= de^a + \omega^a{}_b \wedge e^b - \Gamma^\rho{}_{\mu\nu}\, e^a_\rho\, dx^\mu \wedge dx^\nu\,, \\ &= de^a + \bar\omega^a{}_b \wedge e^b + K_\mu{}^a{}_\nu\, dx^\mu \wedge dx^\nu - \Gamma^\rho{}_{\mu\nu}\, e^a_\rho\, dx^\mu \wedge dx^\nu\,, \\ &= K_\mu{}^a{}_\nu\, dx^\mu \wedge dx^\nu - \Gamma^\rho{}_{\mu\nu}\, e^a_\rho\, dx^\mu \wedge dx^\nu\,. \end{aligned} \tag{13.55}$$

(We have used (13.46) in getting to the third line here.) Thus from $D_\mu\, e^a_\nu = 0$ it follows that $De^a = 0$ and so

$$K_{[\mu}{}^\rho{}_{\nu]} = \Gamma^\rho{}_{[\mu\nu]}\,. \tag{13.56}$$

Now, we can write

$$\Gamma^\rho{}_{\mu\nu} = \bar\Gamma^\rho{}_{\mu\nu} + L^\rho{}_{\mu\nu}\,, \tag{13.57}$$

where $\bar{\Gamma}^\rho{}_{\mu\nu}$ is the Christoffel connection, given as usual by

$$\bar{\Gamma}^\rho{}_{\mu\nu} = \tfrac{1}{2} g^{\rho\lambda}\left(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}\right), \tag{13.58}$$

and $L^\rho{}_{\mu\nu}$ is just a name for the tensor[38] $\Gamma^\rho{}_{\mu\nu} - \bar{\Gamma}^\rho{}_{\mu\nu}$. Going back now to eqn (13.54) and imposing the vielbein compatibility condition $D_\mu\, e^a_\nu = 0$, we see that it implies

$$\begin{aligned} \Gamma^\rho{}_{\mu\nu} &= e^\rho_a\, \partial_\mu\, e^a_\nu + \omega_\mu{}^a{}_b\, e^\rho_a\, a^b_\nu\,, \\ &= e^\rho_a\, \partial_\mu\, e^a_\nu + \bar{\omega}_\mu{}^a{}_b\, e^\rho_a\, a^b_\nu + K_\mu{}^a{}_\nu\, e^\rho_a\,. \end{aligned} \tag{13.59}$$

Now, in the absence of torsion (and hence contorsion), eqn (13.56) implies $\Gamma^\rho{}_{\mu\nu}$ is symmetric in μ and ν, and therefore it is just the usual Christoffel connection. Thus eqn (13.59) then tells us that

$$\bar{\Gamma}^\rho{}_{\mu\nu} = e^\rho_a\, \partial_\mu\, e^a_\nu + \bar{\omega}_\mu{}^a{}_b\, e^\rho_a\, a^b_\nu\,. \tag{13.60}$$

In general, therefore, when the torsion and contorsion are non-zero, eqn (13.59) implies

$$\Gamma^\rho{}_{\mu\nu} = \bar{\Gamma}^\rho{}_{\mu\nu} + K_\mu{}^\rho{}_\nu\,. \tag{13.61}$$

Going back to (13.56), and using (13.52), we see that

$$\Gamma^\rho{}_{[\mu\nu]} = \tfrac{1}{2} T_{\mu\nu}{}^\rho\,. \tag{13.62}$$

(Recall that $T_{\mu\nu}{}^\rho$ is antisymmetric in μ and ν.) Thus the antisymmetric part of the conection $\Gamma^\rho{}_{\mu\nu}$ is directly proportional to the torsion tensor. One can also see that

$$\Gamma^\rho{}_{\mu\nu} = \bar{\Gamma}^\rho{}_{\mu\nu} + T^\rho{}_{(\mu\nu)} + \tfrac{1}{2} T_{\mu\nu}{}^\rho\,. \tag{13.63}$$

(Recall that round brackets denote symmetrisation.) Note that if there is torsion, the symmetric part $\Gamma^\rho{}_{(\mu\nu)}$ of $\Gamma^\rho{}_{\mu\nu}$ is not simply equal to the Christoffel connection $\bar{\Gamma}^\rho{}_{\mu\nu}$, since it receives the additional contribution $T^\rho{}_{(\mu\nu)}$.

13.6 Stokes' Theorem

In three-dimensional Cartesian vector analysis there are two familiar integral identities, known respectively as the divergence theorem and Stokes' theorem, which relate an integral over a certain domain to an integral over the boundary of that domain. In the case of the divergence theorem and integral over a 3-volume V is related to an integral over the 2-surface S that bounds V. Thus for any vector A one has

$$\int_V \vec{\nabla}\cdot\vec{A}\, dV = \int_S \vec{A}\cdot d\vec{S}\,. \tag{13.64}$$

[38] Recall that the *difference* between two connections is always a tensor.

For Stokes' theorem, an integral over a 2-dimensional area Σ is related to an integral over the 1-dimensional boundary C of Σ. For any vector $\vec{A}$ one has

$$\int_\Sigma (\vec{\nabla} \times \vec{A}) \cdot d\vec{S} = \int_C \vec{A} \cdot d\vec{\ell}. \tag{13.65}$$

These two identities are in fact just special cases of a much more general theorem in differential geometry, which can be stated as follows. Suppose that we have a p-form ω in an n-dimensional manifold M, and that there is some $(p+1)$-dimensional submanifold Σ in M, with a p-dimensional boundary that will be denoted by $\partial\Sigma$. The general theorem, which is known as Stokes' theorem, states that

$$\int_\Sigma d\omega = \int_{\partial\Sigma} \omega\,. \tag{13.66}$$

Note that in general we can integrate a p-form over a p-dimensional surface, to get a number. An example would be to integrate the 2-form $\omega = \sin\theta\, d\theta \wedge d\varphi$ over the 2-sphere, to get

$$\int_{S^2} \omega = \int_{S^2} \sin\theta\, d\theta \wedge d\varphi = \int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\varphi = 4\pi\,. \tag{13.67}$$

We should actually qualify the statement of Stokes' theorem in eqn (13.66) by saying that the p-form ω must be globally defined in order for the theorem to be valid. Let us assume for now that this is the case. Consider now what happens if our p-form ω is actually itself the exterior derivative of a globally-defined $(p-1)$-form σ:

$$\omega = d\sigma\,. \tag{13.68}$$

Now we know that d^2 always gives zero, and so that means $d\omega = d^2\,\sigma = 0$. Plugging into (13.66) we therefore get

$$0 = \int_{\partial\Sigma} d\sigma\,. \tag{13.69}$$

We can now use Stokes' theorem for a second time, to turn this integral into an integral over the boundary of $\partial\Sigma$, thus giving

$$0 = \int_{\partial\Sigma} d\sigma = \int_{\partial^2\Sigma} \sigma = 0\,. \tag{13.70}$$

This result holds for any globally-defined $(p-1)$-form σ, and any $(p+1)$-dimensional surface Σ. It must therefore be the case that the surface $\partial^2\Sigma$ is in fact non-existent. And indeed this makes perfect sense. If you think about it, you can see that the boundary of a boundary of a surface is always empty. For example, think of a unit-radius ball in Euclidean 3-space. The boundary of the ball is the 2-dimensional surface (the "unit 2-sphere"). And the boundary of the 2-sphere is empty; it has no boundary.

By means of integration of forms over surfaces, we see that we can establish a mapping between statements about exterior derivatives of forms, and statements about the boundaries of surfaces. For example, the statement $d^2 = 0$ for forms is dual, in this sense, to the statement that $\partial^2 = \emptyset$ for surfaces. The one-to-one mapping between statements about integrals of differential forms over surfaces, and exterior derivatives of differential forms, is known as *Poincaré Duality.*

We should consider, at this point, the significance of the qualification we inserted in the statement about Stokes' theorem (13.66) that the p-form ω should be globally defined. What does this mean, and what might go wrong if it isn't?

The example of the 2-form $\omega = \sin\theta\, d\theta \wedge d\varphi$ that we looked at earlier actually illustrates this nicely. We *can* in fact write $\sin\theta\, d\theta \wedge d\varphi$ as the exterior derivative of a 1-form:

$$\omega = \sin\theta\, d\theta \wedge d\varphi = d\sigma\,, \qquad \sigma = -\cos\theta\, d\varphi\,. \tag{13.71}$$

So, if we didn't pay heed to the requirement that σ should be globally defined, we would conclude that since we can write $\omega = d\sigma$ in this case, we must have $\int_{S^2} \omega = \int_{\partial S^2} \sigma = 0$, since S^2 has no boundary. This contradicts the fact that, as seen in (13.67), $\int_{S^2} \omega = 4\pi$ for this 2-form $\omega = \sin\theta\, d\theta \wedge d\varphi$.

The flaw in the argument is precisely that $\sigma = -\cos\theta\, d\varphi$ is not a globally-defined 1-form. The reason for this is that it is singular at the north and south poles of the sphere, at $\theta = 0$ and $\theta = \pi$ respectively. The problem is not that it itself is becoming infinite, but that it is ill-defined at the poles of the sphere. The 1-form $d\varphi$ describes a displacement along the direction of increasing φ, that is to say, a displacement along a line of constant latitude. In other words, it is like saying "move east at fixed latitude." That is fine at a generic latitude, but it is meaningless at the north or the south pole. "East" is not defined at either of the poles.

There is a way to "patch things up" (literally, in fact!) in this example. To do this, it is useful to note that we can make two other choices for a 1-form σ whose exterior derivative in each case gives our ω. Calling them σ_+ and σ_-, they are

$$\sigma_+ = (1 - \cos\theta)\, d\varphi\,, \qquad \sigma_- = -(1 + \cos\theta)\, d\varphi\,. \tag{13.72}$$

Clearly we have $d\sigma_+ = d\sigma_- = \omega = \sin\theta\, d\theta \wedge d\varphi$. The 1-form σ_+ is perfectly regular at the north pole of the sphere, because the prefactor $(1 - \cos\theta)$ vanishes there, thus resolving the "which way is east?" dilemma. It is still singular at the south pole, however. On the other hand, σ_- is non-singular at the south pole while being singular at the north pole.

We can therefore split the sphere up into two patches; S_+ which denotes the entire sphere except for the point at the south pole, and S_- which denotes the entire sphere except the point at the north pole. Crucially, the two patches overlap, and the two together provide a covering of the entire sphere. The point is that σ_+ is globally defined in S_+, and σ_- is globally defined in S_-.

The overlap region where both σ_+ and σ_- are non-singular is in fact "almost everywhere" on the sphere; just the two poles are excluded. We don't in fact need such a lot of overlap, and it is sufficient to know that there is certainly an overlap of validity in a thin little strip around the equator of the sphere. Let us define $\widetilde{S}_+$ to be the surface of the northern hemisphere, and $\widetilde{S}_-$ to be the surface of the southern hemisphere. Thus we can write

$$\int_{S^2} \omega = \int_{\widetilde{S}_+} d\sigma_+ + \int_{\widetilde{S}_-} d\sigma_- \,. \tag{13.73}$$

The 1-formsσ_+ and σ_- are both perfectly well-defined and nonsingular in their respective integrals on the right-hand side, and so we can apply Stokes' theorem to each of them with complete confidence. Thus we have

$$\int_{S^2} \omega = \int_{\partial\widetilde{S}_+} \sigma_+ + \int_{\partial\widetilde{S}_-} \sigma_- \,. \tag{13.74}$$

Now the boundary of the northern hemisphere $\widetilde{S}_+$ is the equatorial great circle, and the boundary of the southern hemisphere $\widetilde{S}_-$ is also the equatorial great circle, but with the opposite orientation. Thus we have

$$\begin{aligned} \int_{S^2} \omega &= \int_0^{2\pi} \Big(\sigma_+\Big)_{\theta=\frac{\pi}{2}} + \int_{2\pi}^0 \Big(\sigma_-\Big)_{\theta=\frac{\pi}{2}} \,, \\ &= \int_0^{2\pi} d\varphi - \int_{2\pi}^0 d\varphi \,, \\ &= 2\pi + 2\pi = 4\pi \,, \end{aligned} \tag{13.75}$$

and we have correctly recovered the result (13.67) for the integral of $\omega = \sin\theta\, d\theta \wedge d\varphi$ over the 2-sphere.

Note that there is nothing special about the choice of the equator in the calculation above. We could equally well choose to split the sphere in any other way, into an upper part where σ_+ is well-defined, and a lower part where σ_- is well defined. For example, one can easily check that the same answer $\int_{S^2} = 4\pi$ is obtained if one divides the sphere into an upper region with $0 \le \theta \le \theta_0$ and a lower region with $\theta_0 \le \theta \le \pi$, and then uses Stokes' theorem to turn the two surface integrals into closed line integrals around the line of co-latitude $\theta = \theta_0$. One would also get the same answer $\int_{S^2} = 4\pi$ if one chose an arbitrary wiggly boundary separating the upper and the lower regions.

The important lesson to be learned from the discussion above is that there can exist circumstances where a p-form ω obeys $d\omega = 0$ (as in the 2-form example with $\omega = \sin\theta\, d\theta \wedge d\varphi$), and yet we cannot write ω *globally* as $\omega = d\sigma$. In our example above it was necessary to use two different expressions, σ_+ and σ_-, in order to write ω as the exterior derivative of something. Neither σ_+ nor σ_- alone is well-defined over the entire sphere.

The underlying reason for this is that the 2-sphere is topologically nontrivial. Specifically, this is reflected in the fact that there exists a non-contractible closed 2-surface (known technically as a 2-cycle), namely the sphere itself. If one draws a closed loop (a 1-cycle) on the surface of the sphere it can always be *contracted*; that is to say, it can be continuously shrunk down to a point. (Imagine an infinitely stretchy rubber band lying on the surface of the sphere.) But a closed 2-cycle on the sphere cannot be continuously contracted. (Imagine putting a balloon around the sphere, with the air-inlet sealed off; it cannot be stretched or deformed to shrink it to a point, without breaking it.)

By Poincaré duality, the statement about the topological nontriviality of a p-cycle in a manifold translates into a statement about differential forms on the manifold. First, a bit of terminology: A p-form ω is called *closed* if it satisfies $d\omega = 0$. It is called *exact* if it can be written as $\omega = d\sigma$, for some *globally-defined* $(p-1)$-form σ. If there exists a topologically nontrivial p-cycle in the manifold then by Poincaré duality this means that there exists a *closed p-form that is not exact.* Such a form is called a *harmonic form.* We saw an example of such a harmonic form in the earlier discussion; the 2-form $\omega = \sin\theta\, d\theta \wedge d\varphi$ is closed ($d\omega = 0$), but it is not exact since there does not exist a globally-defined 1-form σ such that we can write $\omega = d\sigma$.

There is a general result that can be proven, stating that an arbitrary p-form ω can always be written as

$$\omega = d\sigma + {*d*}\,\rho + \omega_H\,, \tag{13.76}$$

where the $(p-1)$-form σ and the $(p+1)$-form ρ are both *globally well-defined*, and ω_H is harmonic. This is known as the *Hodge decomposition.* If the manifold has no topologically nontrivial p-cycles, then there is no ω_H.