Qu. (1) This problem requires a fairly straightforward extension of the procedure that was used in the lecture notes for constructing the Schwarzschild solution, but now incorporating the addition of a cosmological constant so that one is solving $R_{\mu\nu} = \Lambda g_{\mu\nu}$ rather than the vacuum Einstein equation $R_{\mu\nu} = 0$. The metric ansatz one begins with is the same static, spherically-symmetric one that was used for Schwarzschild, with

$$ds^2 = -B(r)\,dt^2 + A(r)\,dr^2 + r^2\,(d\theta^2 + \sin^2 \theta\,d\varphi^2).$$  \hspace{1cm} (1)$$

Thus all the results for the Ricci tensor $R_{\mu\nu}$ that were derived for this ansatz in the discussion of the Schwarzschild solution carry over to the present case too.

Crucially, the inclusion of the $\Lambda$ term does not alter the fact that the Einstein equations still imply $A R_{00} + B R_{11} = 0$, and this gives the very simple equation that implies $A B = \text{constant}$. In the case of the Schwarzschild solution, one could determine the value of this constant by using the fact that at large distances, far away from the massive object, the metric should approach the flat Minkowski metric and so one should have

$$A(r) \rightarrow 1 \text{ and } B(r) \rightarrow 1 \text{ when } r \rightarrow \infty.$$  \hspace{1cm} (2)$$

In the present case, however, the metric is not asymptotic to the Minkowski metric, and so one does not have such a simple guidance to determine the value of the constant $A B$.

In fact, this is really a matter of deciding what one means by “time” in the Schwarzschild-de Sitter case. Notice that if one rescales the time coordinate $t$ in the metric (1) by a factor $k$, where $k$ is a constant, then one could absorb this instead into a rescaling of the function $B(r)$, of the form $B(r) \rightarrow k^{-2} B(r)$. In other words, given that we have derived $A(r) B(r) = \text{constant}$ from the Einstein equations, we can always choose a rescaling of the $t$ coordinate so that this constant can be taken to be equal to 1. If we do this, then the Schwarzschild-de Sitter metric comes out to have the form given in the problem, namely

$$ds^2 = -\left(1 - \frac{2M}{r} - \frac{1}{3} \Lambda r^2\right)dt^2 + \left(1 - \frac{2M}{r} - \frac{1}{3} \Lambda r^2\right)^{-1}dr^2 + r^2\,(d\theta^2 + \sin^2 \theta\,d\varphi^2).$$  \hspace{1cm} (3)$$

The specific asymptotic form that this implies when $r \rightarrow \infty$ is by convention taken as defining what one chooses as a canonical time coordinate in an asymptotically de Sitter or anti-de Sitter metric.

Qu. (2) This question illustrates the fact that if you are unfortunate enough to fall into a black hole then things are pretty bleak, and nothing that you do can prevent you
from hitting the singularity at $r = 0$ in a finite amount of proper time. Note that it is
the proper time $\tau$ that is the time perceived by an observer, or a clock. The calculation here
shows that the maximum possible proper time that can elapse between crossing the event
horizon at $r = 2M$ and hitting the singularity at $r = 0$ is $\pi M$. To put that in terms of more
familiar units, that would be $\pi GM/c$ seconds. A black hole of the mass of the sun would
have a Schwarzschild radius of about 3 kilometres, and so the maximum time you would
have before hitting the singularity would be the time it takes light to travel $\frac{3}{2}\pi$ kilometres,
so about 15 microseconds.

Interestingly, as one finds in part (2c), the lifetime is in fact maximised by following
an infalling radial geodesic path. So even if you are equipped with a rocket when you fall
into a black hole, the best thing to do in order to maximise your limited remaining lifetime
after having crossed the event horizon is to switch off the rocket engine. Notice that not
only would firing the rocket engine to try to head outwards make matters worse, but so too
would firing it to try to orbit around the singularity.

Qu. (3) The steps for performing this calculation are more or less spelt out in the
question. It is sometimes quite helpful, when analysing the properties of the solutions of a
theory, if the theory has the property of being conformally invariant. One familiar example
of a theory that is conformally invariant is Maxwell’s theory of electrodynamics in four
dimensions (provided, at least, that we restrict the discussion to Maxwell theory in the
absence of source currents). In fact the conformal invariance of the source-free Maxwell
theory is especially simple because the vector potential $A_\mu$ does not need to be scaled at
all by the conformal factor. The conformal invariance can be seen as follows:

Consider the Maxwell action

$$I = -\frac{1}{16\pi} \int \sqrt{-g} F^{\mu\nu} F_{\mu\nu} \, d^4x,$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Now consider making a conformal rescaling of the metric, in
which

$$g_{\mu\nu} \rightarrow \bar{g}_{\mu\nu} = \Omega^2 g_{\mu\nu},$$

where $\Omega$ is an arbitrary function of the spacetime coordinates. It follows that

$$\bar{g}^{\mu\nu} = \Omega^{-2} g^{\mu\nu}, \quad \sqrt{-\bar{g}} = \Omega^4 \sqrt{-g}.$$

As a consequence, we have that

$$\sqrt{-\bar{g}} F_{\mu\nu} F_{\rho\sigma} g^{\mu\rho} g^{\nu\sigma} = \sqrt{-g} F_{\mu\nu} F_{\rho\sigma} g^{\mu\rho} g^{\nu\sigma};$$
and so therefore the Maxwell action $I$ given in eqn (4) is completely unchanged if one evaluates it in the background of the conformally rescaled metric $\tilde{g}_{\mu\nu}$ or instead in the original metric $g_{\mu\nu}$. Consequently, if one finds a solution of the Maxwell equations in the original metric, then it also provides the solution in the conformally-related metric.

This property of the Maxwell equations and its solutions has been exploited quite extensively in a number of contexts, including discussions by Roger Penrose and others when studying the asymptotic behaviour of electromagnetic fields near infinity. We shall be encountering some discussions of asymptotic infinity in the context of black hole geometries later in the course.

Notice that in the above discussion the gauge potential $A_\mu$ was taken to be unaffected in the conformal scaling, and so $F_{\mu\nu}$ was also unaffected. Of course when the indices are raised using the metric, one then picks up powers of the conformal factor.

Notice also that this property of conformal invariance of the Maxwell equations only holds in four dimensions. It depends upon the fact that the two factors of $\Omega^{-2}$ that come from raising the two indices on the first field strength tensor in the action (4) are precisely cancelled by the $\Omega^4$ factor that comes from the $\sqrt{-g}$ in four dimensions. If we were instead in $n$ dimensions, we would have

$$\sqrt{-\tilde{g}} F_{\mu\nu} F_{\rho\sigma} \tilde{g}^{\mu\rho} \tilde{g}^{\nu\sigma} = \Omega^{n-4} \sqrt{-g} F_{\mu\nu} F_{\rho\sigma} g^{\mu\rho} g^{\nu\sigma},$$

and so there would be no conformal invariance except when $n = 4$.

In the case of a scalar field, the calculations in Qu. (3) show that if we merely considered a scalar obeying the massless Klein-Gordon equation $\Box \psi = 0$ (this would sometimes be called the “minimally-coupled” massless wave equation), then there would be no conformal invariance in any dimension. However, by adding the Ricci scalar term and considering the non-minimally coupled wave equation $(-\Box + \alpha R) \psi = 0$, one does then find that there is a choice of the constant $\alpha$ in any dimension $n$, namely

$$\alpha = \frac{n-2}{4(n-1)},$$

for which the equation is conformally invariant. In this case, however, unlike that of the four-dimensional Maxwell equation, one must now make a conformal rescaling of the field as well: The conformal invariance requires that one rescales $\psi$ to $\tilde{\psi} = \Omega^\beta \psi$, where

$$\beta = -\frac{1}{2}(n-2).$$

It is worth remarking that the calculations demonstrating the conformal invariance of the scalar wave equation $(-\Box + \alpha R) \psi = 0$ can actually be done a little more easily by
following the same method that we used for the Maxwell theory earlier. Namely, rather than looking at the conformal transformation at the level of the wave equation, we can look at the action instead. It is easy to see that the equation \((\Box + \alpha R) \psi = 0\) can be derived from the action

\[
I_{\text{scal}} = \int \sqrt{-g} \left( \frac{1}{2} g^\mu\nu \partial_\mu \psi \partial_\nu \psi + \frac{1}{2} \alpha R \psi^2 \right) d^n x.
\]  
(11)

(The overall normalisation of the action is unimportant in this discussion.) One can now verify that this action is invariant under the conformal rescaling found in this question. (Some integration by parts is necessary in order to show this.)

Qu. (4) Showing that the solutions of the massless Klein-Gordon equation \(\Box \psi = 0\) in the Schwarzschild metric are separable, by looking first for factorised solutions of the form \(\psi(t, r, \theta, \varphi) = T(t) R(r) S(\theta) \Phi(\varphi)\), is rather straightforward. If unsure of how to do it, just review first how to separate the massless Klein-Gordon equation in the case of a Minkowski spacetime background. The only new feature in the Schwarzschild background is that the metric functions \(g_{00} = -(1 - \frac{2M}{r})\) and \(g_{11} = (1 - \frac{2M}{r})^{-1}\) now make the radial equation (for \(R(r)\)) a bit more complicated. Since one is not being asked to solve the radial equation, but merely to derive what it is, the fact that it is a slightly more complicated equation than in Minkowski is not a big deal, as far as question 4 is concerned.

Of course if one wants to construct the solutions explicitly, then the greater complexity of the radial equation in the Schwarzschild background does lead to some complications. Specifically, the radial equation now has an extra singularity, at \(r = 2M\), that is absent in the case of the Minkowski background. In fact the radial equation now has a regular singular point at \(r = 0\), a regular singular point at \(r = 2M\), and an irregular singular point (of order 2) at \(r = \infty\). This means that the equation lies outside the usual kinds one commonly encounters in physics, where there are at most three regular singular points (possibly with a confluence of two of these so that there would be one regular singular point and an irregular singular point of order 2). This means that the solutions to the Schwarzschild radial equation are not given by any hypergeometric function.

Studying the properties of the solutions of wave equations in the Schwarzschild spacetime geometry is of enormous interest for many reasons (for example, studying Hawking radiation by black holes). The difficulty in obtaining explicit solutions for the Schwarzschild radial functions has meant that a variety of ingenious approximate methods have been developed over the years.

Whether one calls a solution “explicit” or not is, in a certain sense, dependent on exactly
what one means by the term “explicit.” If we have an equation whose solutions are given in terms of trigonometric functions, we would call them explicit, and this is in part because we are very familiar with the trigonometric functions. But also, and very importantly, we essentially know everything we could possibly want to know about them; what they look like at small argument, what they look like at large argument, and so on. The same is true really for any hypergeometric function.

In a certain sense one could always say that one can obtain an “explicit” solution to any equation; all that would be necessary would be to give one’s own name to “the function that is the solution of the equation one wants to solve.” But other than being able to attach one’s own name to a function, this would not do any good unless one had ways of determining the properties of the “Pope function,” or whatever one called it.

In fact, there is a class of already-named functions that include the solutions of the Schwarzschild radial equation. They were invented by Karl Heun (1859 – 1929), and are called Heun functions. They are the solutions of the Heun equation, which is the most general second-order linear ODE with four regular singular points. If the parameters in the equation are adjusted so that two of these point coalesce, then this gives the Confluent Heun equation. The radial equation arising in the separation of variables in the Klein-Gordon equation in the Schwarzschild background is in fact a particular case of the confluent Heun equation. Quite a lot is actually known about the properties of the Heun functions and confluent Heun functions, although until recently they have not been much appreciated by physicists. Until recently, there was a tendency to view an equation that turned out to be a Heun equation as pretty much equivalent to “an equation that can’t be solved.” This may be changing soon, not least because in Mathematica version 12.1, the Heun functions have been implemented for the first time.

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¹He was German, and his name is pronounced to rhyme with “coin.”