614—Maths Methods in Theoretical Physics

Problem Sheet 1

(1) Prove that if one defines the derivative of \( f(z) = u(x, y) + iv(x, y) \) at \( z \) by taking the limit of \( (f(z + \delta z) - f(z))/\delta z \), where \( \delta z \) approaches zero along a line at angle \( \psi \) in the complex plane (i.e. take \( \delta z = \epsilon e^{i\psi} \) as the real constant \( \epsilon \) goes to zero), then the answer is independent of \( \psi \) if the Cauchy-Riemann equations hold.

(2a.) Show that each of the functions

\[
\begin{align*}
    u &= x^4 - 6x^2y^2 + y^4, \\
    u &= e^{x^2 - y^2} \cos(2xy)
\end{align*}
\]

satisfies the Laplace equation in two dimensions. In each case, taking \( u \) to be the real part of an analytic function, use the Cauchy-Riemann equations to find \( v(x, y) \), and hence find \( f = u + iv \). (Express \( f \) eventually entirely in terms of \( z \).)

(2b.) Which of the following can be the real part of an analytic function?

\[
\begin{align*}
    u &= x + y, \\
    u &= x^2 + y^2, \\
    u &= \cos x
\end{align*}
\]

For those which can be, find \( v \) and hence \( f \).

For recreation:

(3a) Write two quaternions \( p \) and \( q \) as ordered pairs of complex numbers, \( p = (a, b) \) and \( q = (c, d) \). Using the multiplication rule \( pq = (ac - db, da + bc) \), show that in general multiplication is non-commutative, i.e. \( pq \neq qp \). Show that multiplication of any three quaternions is associative; \( p(qr) = (pq)r \). With the conjugate defined by \( \bar{p} = (a, -b) \), show that \( \bar{p}p = pp\bar{p} = (a\bar{a} + b\bar{b}, 0) \), which is just the real number \( a\bar{a} + b\bar{b} \).

(3b) Show, using the definitions of multiplication and conjugation of quaternions given in part (3a), that the conjugate of the product \( pq \), i.e. \( \overline{pq} \), is equal to \( \overline{q} \overline{p} \).

(3c) Define octonions \( A \), \( B \) and \( C \) by \( A = (a, b) \), \( B = (c, d) \), \( C = (e, f) \), where \( a, b, c, d, e \) and \( f \) are quaternions. Using the multiplication rule for the octonions, \( AB = (ac - db, da + bc) \), show that multiplication is non-associative.

(3d) The conjugate for octonions is defined by \( \overline{A} = (\bar{a}, -b) \). Show that

\[
\overline{A}(AB) = (\overline{A}A)B,
\]

which is needed to establish that the octonions form a division algebra, as discussed in the lectures.

Due Thursday 12th November
1) \( f = u + iv \)

\[
\frac{df}{dz} = \frac{ux + uy + ivx + ivy \frac{dy}{dx}}{c + is} = \frac{ux (c + is) + ivx (c + is)}{c + is} = \frac{ux + ivx}{c + is} \text{ independent of } \Psi.
\]

2(a) \( u = x^4 - 6x^2y^2 + iy^4 \)

\[
u = \int x^3y - 12x^2y + 4y^3 dx' + \alpha(x) = \int (4x^2 - 12xy)' dy' + \alpha(x) = 4x^3y - 4xy^3 + \alpha(x)
\]

\[
u = -\int x^4 - 4y^3 dx' + \beta(y) = \int (x^4 - 4y^3) dy' + \beta(y) = x^4y - 4xy^3 + \beta(y)
\]

\[
\therefore \quad \alpha(x) = \beta(y) = \gamma.
\]

\[
\therefore \quad f = x^4 - 6x^2y^2 + iy^4 + 4ix^3y - 4iy^3 + i\gamma
\]

\[
\therefore \quad f = (x + iy)^4 + i\gamma = z^4 + i\gamma
\]

\[
\therefore \quad f = e^{z^4 + i\gamma}
\]

Alternative calculation:

\[
u = e^{x^2 - y^2} \cos 2xy
\]

\[
u = \frac{1}{2} e^{x^2 - y^2} (e^{2ixy} + e^{-2ixy}) = \frac{1}{2} (e^{x+iy} + e^{x-iy}) = \frac{1}{2} (e^{x+iy})^2 + \frac{1}{2} (e^{x-iy})^2
\]

\[
\therefore \quad f = e^{z^2} + ie
\]
Similarly, \( u_1 = i(x+iy)e^{(x+iy)^2} - i(x-iy)e^{(x-iy)^2} \)
\[= \frac{i}{2} x e^{(x+iy)^2} - \frac{i}{2} x e^{(x-iy)^2} \]
\[= x \left( -\frac{i}{2} e^{2ixy} + \frac{i}{2} e^{-2ixy} \right) = e^{x^2-y^2} \sin 2xy + \theta \]

2b) \( u = x+iy \)
\( u_x = 1, \ u_{xx} = 0, \ u_y = 1, \ u_{yy} = 0 \)
\( u_{xx} + u_{yy} = 0 \) so yes
\( v = \int y dy' + \int (x+y) \) dx = \( x + \beta y \)
\( v = -\int x dx' + \beta (y) = -x + \beta y \)
\( \therefore \quad d(x) + x = \beta (y) - y = \text{const} = \delta \)
\[\sqrt{v = y + d(x)} = y + x - x \]
\[\therefore \ f = u + iv = x+iy + iy - ix + iv = (x+iy) - i(x+iy) + iv \]
\( \therefore \ (f = (1-i)z + i \delta) \]

3a) \( p = (a, b), \ q = (c, d) \)
\( pq = (ac - \bar{a}b, da + b\bar{c}) \)
\( qp = (ca - \bar{c}d, bc + da) \neq pq \) because complex conjugations are different.

3b) \( p = (a, b), \ q = (c, d) \)
\( r = (e, f) \)
\( qr = \left( ce-fd, fc+de \right) = \left( ce-fd, fc+de \right) = (ace- \bar{a}e, fca + 2de + b \bar{c} - b \bar{f}) \)
\( (pq)r = (ac-db, da+bc) \)
\( p(qr) = (a, b)(ce-fd, fc+de) = (ace- \bar{a}e, fca + 2de + b \bar{c} - b \bar{f}) \)

Thus \( p(qr) = (pq)r \) since \( a, b, c, d, e, f \) are complex numbers, which commute.

3b) \( p = (a, b) \)
\( \overline{p} = (a, -b) \)
\( \overline{p}p = (a, -b)(a, b) = (aa + \bar{b}b, b\bar{a} - ba) = (a^2 + \bar{b}b, a) = \overline{a}a + b \bar{b} \)
\( \overline{p} = (a, b)(a, -b) = (aa + \bar{b}b, b\bar{a} - ba) = (a^2 + b\bar{a} - ba, \overline{a}a + \bar{b}b) = \overline{p}p \)

3b) \( p = (ac- \bar{a}b, da + b\bar{c}) \)
\( \overline{p} = ( \bar{a}, -b) \)
\( \overline{p} \overline{p} = (\bar{a}, -b)(\bar{a}, b) = (\bar{a}^2 + \bar{b}b, -b\bar{a} - \bar{b}b) = (b\bar{a} + \bar{b}b, \overline{a}a + \bar{b}b) = \overline{p}p \)
3c) \( A = (a, b) \)  \( B = (c, d) \)  \( C = (e, f) \)

\[ (a, b)(c, d) = (ac-bd, da+bc) \]

Calculation in (3a), where we were careful about ordering, shows \( p(g) \neq p(f) \) if 
\( a, b, c, d, e, f \) do not commute.

3d) \( \bar{A} = (\bar{a}, -\bar{b}) \).

\( \bar{A} (AB) = (\bar{a}, -\bar{b}) (ac-bd, da+bc) \)
\[ = (\bar{a}c - \bar{a}b \bar{d}, \bar{a}b + \bar{b}c \bar{d}) \]
\[ = (\bar{a}c + \bar{b} \bar{d}) \cdot (c, d) \]

\( (\bar{A}A)B = (\bar{a}, -\bar{b})(a, b)B = (\bar{a}c + \bar{b} \bar{d}, b\bar{a} - b\bar{c})B = (\bar{a}c + \bar{b} \bar{d}, 0)B \)
\[ = (\bar{a}c + \bar{b} \bar{d}) \cdot (c, d) \]

So \( \bar{A} (AB) = (\bar{A}A)B \).

So if \( AB = C \), we can solve for \( B \):

\[ \bar{A} (AB) = \bar{A} C \]
\[ \therefore (\bar{A}A)B = \bar{A} C \]
\[ \therefore B = \frac{\bar{A} C}{(\bar{A}A)} \]

Since \( \bar{A}A \) is just a real number.