615–Maths Methods in Theoretical Physics

Problem Sheet 4

(1a) Find the types of all the singular points of the equation

$$xy'' - 2y' + xy = 0\,.$$

Consider series solutions of the form $y(x) = \sum_{n\geq 0} a_n\, x^{n+\sigma}$. Obtain the recursion relation for the coefficients a_n, and show that although the indicial equation for σ has roots differing by an integer, the two roots generate linearly independent solutions.

(1b) Show, by verifying that the coefficients below obey the recursion relation obtained in part (1a), that the two solutions can be taken to be

$$y_1(x) \;=\; \sum_{n\geq 1} \frac{(-1)^{n+1}\, 2n\, x^{2n+1}}{(2n+1)!}\,, \qquad y_2(x) = \sum_{n\geq 0} \frac{(-1)^{n+1}\,(2n-1)\, x^{2n}}{(2n)!}\,.$$

(1c) Show $y_1(x)$ is equal to $\sin x - x\,\cos x$, by expanding the trigonometric functions. Then, using the Wronskian method, find the closed-form expression for the second solution (i.e. $y(x) = y_1(x)\int^x dt/(f(t)y_1^2(t))$ as in the lecture notes).

[**Hint**: Write t^2 as $(t/\sin t)(t\sin t)$ and integrate by parts in order to evaluate the integral.]

(1d) Explicitly calculate the Wronskian of the two closed-form solutions in part (1c).

(2a) Look for solutions of the Legendre equation $(1-x^2)y'' - 2xy' + \lambda y = 0$ in the form of a series expansion $y(x) = \sum_{n\geq 0} a_n\,(x-1)^{n+\sigma}$ around the regular singular point $x = 1$. Obtain the indicial equation for σ, and the recursion relation for the coefficients a_n.

(2b) Observe that only one solution is obtained by this method. Calculate its radius of convergence, and show that this is consistent with the locations of the singular points of the equation.

(3a) Show the derivatives of the Legendre polynomials satisfy the orthogonality relation,

$$\int_{-1}^{1} dx\,(1-x^2)\,P_m'(x)\,P_n'(x) = 0\,, \qquad m \neq n\,.$$

(3b) Show the "second solution" to the Legendre equation $(1-x^2)y'' - 2xy' + \ell(\ell+1)y = 0$ with $\ell = 0$ is $Q_0(x) = \frac{1}{2}\log[(1+x)/(1-x)]$. For $\ell = 1$, the first solution is $P_1(x) = x$. Calculate the orthogonality integral

$$\int_{-1}^{1} dx\, P_1(x)\, Q_0(x)\,.$$

(3c) Explain why the result in part (3b) is non-zero, even though $P_1(x)$ and $Q_0(x)$ are eigenfunctions of the Legendre equation with *different* eigenvalues.

Due Tuesday 4th October in class