Complex Manifolds, Kähler Geometry and Special Holonomy

ABSTRACT

Lecture notes for short course given at Beijing Normal University, June 2017.

Contents

1 Introduction

This course follows on from Geometry and Topology in Physics I, in which the basic notions and formalism of differential geometry and topology were introduced. The aim of the second part of this course is to go on to apply the formalism in a number of contexts of physics interest, also developing the basic ideas further as and when the need arises during the course. To begin, we present a brief overview of the essential aspects of differential forms, which provide the basic tools we shall be using in the course. This is essentially material covered in depth in Part I, and reference can be made to the course notes for that course.

1.1 Vectors and tensors

In physics we encounter vectors and tensors in a variety of contexts; for example the notion of the *position vector* in three-dimensional vector analysis and its four-dimensional spacetime analogue; the 4-vector potential in Maxwell theory; the metric tensor in general relativity, and so on. The language in which all of these can be described is the language of differential geometry. The first examples listed were rather special ones, in that the *position vector* is a concept that is applicable only in the restricted case of a flat Euclidean space or Minkowskian spactime. In general, the line joining one point to another in the space or spacetime is not a vector. Rather, one must pass to the limit where one considers two points that are *infinitesimally separated.* Now, in the limit where the separation tends to zero, the line joining the two points can be viewed as a vector. The reason for this need to use a limiting procedure is easily understood if one thinks of a familar non-Euclidean space, the surface of the Earth. For example, the line joing New York to London is not a vector, from the point of view of transformations on the surface of the Earth (i.e. on the 2-sphere). But in the limit where one considers a line joining two nearby points on a street in New York, one approaches more and more closely to a genuine vector on the 2-sphere. We shall make this precise below.

With the observation that a vector is defined in terms of an arrow joining two points that are infinitesimally separated, it is not surprising that the natural mathematical quantity that describes the vector is the *derivative.* Thus we define a vector V as the tangent vector to some curve in the manifold. Suppose that the manifold M has coordinates x^μ in some patch, and that we have a curve described by $x^\mu = x^\mu(t)$, where t is some parameter along the path. Then we may define the tangent vector

$$V = \frac{\partial}{\partial t}\ . \tag{1.1}$$

Note that V is defined here in a coordinate-independent fashion. However, using the chain rule we may express V as a linear combination of the basis vectors $\partial/\partial x^i$:

$$V = V^i \frac{\partial}{\partial x^i} = V^i \partial_i \ , \tag{1.2}$$

where $V^i = dx^i/dt$. Note that in the last expression in (1.2), we are using the shorthand notation of ∂_i to mean $\partial/\partial x^i$. Einstein summation convention is always understood, so the index i in (1.2) is understood to be summed over the n index values labelling the coordinates on M. The components V^i, unlike the vector V itself, *are* coordinate dependent, and we can calculate their transformation rule under general coordinate transformations $x^i \longrightarrow x'^i = x'^i(x^j)$ by using the chain rule again:

$$V = V^j \partial_j = V^j \frac{\partial x'^i}{\partial x^j} \partial'_i \ , \tag{1.3}$$

where ∂'_i means $\partial/\partial x'^i$. By definition, the coefficients of the ∂'_i in (1.3) are the components of V with respect to the primed coordinate system, and so we read off

$$V'^i = \frac{\partial x'^i}{\partial x^j} V^j \ . \tag{1.4}$$

This is the standard way that the components of a vector transform. Straightforward generalisation to multiple indices gives the transformation rule for tensors. A p-index tensor T will have components $T^{i_1 \cdots i_p}$, defined by

$$T = T^{i_1 \cdots i_p} \partial_{i_1} \otimes \cdots \otimes \partial_{i_p} \ . \tag{1.5}$$

From this, it follows by analogous calculations to those described above that the components will transform as

$$T'^{i_1 \cdots i_p} = \frac{\partial x'^{i_1}}{\partial x^{j_1}} \cdots \frac{\partial x'^{i_p}}{\partial x^{j_p}} T^{j_1 \cdots j_p} \ , \tag{1.6}$$

under a change of coordinate frame.

1.2 Covectors and cotensors

We may also define quantities whose components carry downstairs indices. The idea here is best introduced by considering a function f on the manifold. Using the chain rule, we see that its differential df can be written as

$$df = \partial_i f \, dx^i \ . \tag{1.7}$$

We may think of df as a geometrical, coordinate-independent quantity, whose components in a given coordinate basis are the derivatives $\partial_i f$. In fact df is a special case of a *covector*. More generally, we can consider a covector U, with components U_i, and define

$$U = U_i \, dx^i \; . \tag{1.8}$$

With U itself being a coordinate-independent construct, we may deduce how its components U_i transform under general coordinate transformations by following steps analogous to those that we used above for vectors:

$$U = U_j \, \frac{\partial x^j}{\partial x'^i} \, dx'^i \; . \tag{1.9}$$

By definition, the coefficients of dx'^i are the components U'_i in the primed coordinate frame, and so we read off the transformation rule for 1-form components:

$$U'_i = \frac{\partial x^j}{\partial x'^i} \, U_j \; . \tag{1.10}$$

One may again generalise to multiple-index objects, or cotensors. Thus, for example, we can consider an object U with p-index components,

$$U = U_{i_1 \cdots i_p} \, dx^{i_1} \otimes \cdots \otimes dx^{i_p} \; . \tag{1.11}$$

The transformation rule for the components $U_{i_1 \cdots i_p}$ under general coordinate transformations is again easily read off:

$$U'_{i_1 \cdots i_p} = \frac{\partial x^{j_1}}{\partial x'^{i_1}} \cdots \frac{\partial x^{j_p}}{\partial x'^{i_p}} \, U_{j_1 \cdots j_p} \; . \tag{1.12}$$

It is easy to see that because ω_i transforms "inversely" to the way V^i transforms (compare (1.4) and (1.10)), the quantity $\omega_i \, V^i$ will be *invariant* under general coordinate transformations:

$$\begin{aligned} \omega'_i \, V'^i &= \frac{\partial x^j}{\partial x'^i} \, \frac{\partial x'^i}{\partial x^k} \, \omega_j \, V^k \\ &= \frac{\partial x^j}{\partial x^k} \, \omega_j \, V^k \\ &= \delta^j_k \, \omega_j \, V^k = \omega_j \, V^j \; . \end{aligned} \tag{1.13}$$

This is the scalar product, or *inner product*, of V with ω. It can be expressed more "geometrically," without reference to specific coordinates, as $\langle \omega, V \rangle$. The coordinate bases ∂_i and dx^i for objects with upstairs and downstairs indices are defined to be orthonormal, so that

$$\langle dx^i, \partial_j \rangle = \delta^i_j \; . \tag{1.14}$$

It follows from this that

$$\langle \omega, V \rangle = \omega_i \, V^j \, \langle dx^i, \partial_j \rangle = \omega_i \, V^j \, \delta^i_j = \omega_i \, V^i \ , \tag{1.15}$$

and so indeed this gives the hoped-for inner product. Note that if we apply this inner product to the differential df, we get

$$\langle df, V \rangle = \partial_i f \, V^j \, \langle dx^i, \partial_j \rangle = V^i \, \partial_i f = V(f) \ . \tag{1.16}$$

In other words, recalling the original definition of V as a differential operator (1.1), we see that in this case the inner product of df and V is nothing but the directional derivative of the function f along the curve parameterised by t; *i.e.* $\langle df, V \rangle = V(f) = \partial f / \partial t$.

1.3 Differential forms

A particularly important class of cotensors are those whose components are totally antisymmetric;

$$U_{i_1 \cdots i_p} = U_{[i_1 \cdots i_p]} \ . \tag{1.17}$$

Here, we are using the notation that square brackets enclosing a set of indices mean that they should be totally antisymmetrised. Thus we have

$$\begin{aligned} U_{[ij]} &= \frac{1}{2!} \, (U_{ij} - U_{ji}) \ , \\ U_{[ijk]} &= \frac{1}{3!} \, (U_{ijk} + U_{jki} + U_{kij} - U_{ikj} - U_{kji} - U_{jik}) \ , \end{aligned} \tag{1.18}$$

etc. Generally, for p indices, there will be $p!$ terms, comprising the $\frac{1}{2}p!$ even permutations of the indices, which enter with plus signs, and the $\frac{1}{2}p!$ odd permutations, which enter with minus signs. The $1/p!$ prefactor ensures that the antisymmetrisation is of strength one. In particular, this means that antisymmetrising twice leaves the tensor the same: $U_{[[i_1 \cdots i_p]]} = U_{[i_1 \cdots i_p]}$.

Clearly, if the cotensor is antisymmetric in its indices it will make an antisymmetric projection on the tensor product of basis 1-forms dx^i. Since antisymmetric cotensors are so important in differential geometry, a special symbol is introduced to denote an antisymmetrised product of basis 1-forms. This symbol is the wedge product, $\wedge$. Thus we define

$$\begin{aligned} dx^i \wedge dx^j &= dx^i \otimes dx^j - dx^j \otimes dx^i \ , \qquad (1.19) \\ dx^i \wedge dx^j \wedge dx^k &= dx^i \otimes dx^j \otimes dx^k + dx^j \otimes dx^k \otimes dx^i + dx^k \otimes dx^i \otimes dx^j \\ &\quad - dx^i \otimes dx^k \otimes dx^j - dx^k \otimes dx^j \otimes dx^i - dx^j \otimes dx^i \otimes dx^k \ , \end{aligned}$$

and so on.

Cotensors antisymmetric in p indices are called p-forms. Suppose we have such an object A, with components $A_{i_1\cdots i_p}$. Then we expand it as

$$A = \frac{1}{p!}\, A_{i_1\cdots i_p}\, dx^{i_1} \wedge \cdots \wedge dx^{i_p} \ . \tag{1.20}$$

It is quite easy to see from the definitions above that if A is a p-form, and B is a q-form, then they satisfy

$$A \wedge B = (-1)^{pq}\, B \wedge A \ . \tag{1.21}$$

1.4 Exterior derivative

The exterior derivative d acts on a p-form field, and produces a $(p+1)$-form. It is defined as follows. On functions (i.e. 0-forms), it is just the operation of taking the differential; we met this earlier:

$$df = \partial_i f\, dx^i \ . \tag{1.22}$$

More generally, on a p-form $\omega = 1/p!\, \omega_{i_1\cdots i_p}\, dx^{i_1} \wedge \cdots \wedge dx^{i_p}$, it is defined by

$$d\omega = \frac{1}{p!}\, (\partial_j \omega_{i_1\cdots i_p})\, dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p} \ . \tag{1.23}$$

Note that from our definition of p-forms, it follows that the components of the $(p+1)$-form $d\omega$ are given by

$$(d\omega)_{j i_1\cdots i_p} = (p+1)\, \partial_{[j}\, \omega_{i_1\cdots i_p]} \ . \tag{1.24}$$

It is easily seen from the definitions that if A is a p-form and B is a q-form, then the following Leibnitz rule holds:

$$d(A \wedge B) = dA \wedge B + (-1)^p\, A \wedge dB \ . \tag{1.25}$$

It is also easy to see from the definition of d that if it acts twice, it automatically gives zero, i.e. $d^2 \equiv 0$. This just follows from (1.23), which shows that d is an *antisymmetric* derivative, while on the other hand partial derivatives *commute*.

A simple, and important, example of differential forms and the use of the exterior derivative can be seen in Maxwell theory. The vector potential is a 1-form, $A = A_i\, dx^i$. The Maxwell field strength is a 2-form, $F = \frac{1}{2} F_{ij}\, dx^i \wedge dx^j$, and we can construct it from A by taking the exterior derivative:

$$F = dA = \partial_i\, A_j\, dx^i \wedge dx^j = \tfrac{1}{2} F_{ij}\, dx^i \wedge dx^j \ , \tag{1.26}$$

from which we read off that $F_{ij} = 2\,\partial_{[i}\, A_{j]} = \partial_i\, A_j - \partial_j\, A_i$. The fact that $d^2 \equiv 0$ means that $dF = 0$, since $dF = d^2 A$. The equation $dF = 0$ is nothing but the Bianchi identity in Maxwell theory, since from the definition (1.23) we have

$$dF = \tfrac{1}{2}\partial_i\, F_{jk}\, dx^i \wedge dx^j \wedge dx^k \ , \tag{1.27}$$

hence implying that $\partial_{[i}\, F_{jk]} = 0$. We can also express the Maxwell field equation elegantly in terms of differential forms. This requires the introduction of the Hodge dual operator $*$. This was discussed at length in Part I of the course, and we will not revisit all the details again here. See the course notes for Part I for details.

For now, we shall move on to a very brief review of the basic notions of metrics, vielbeins, spin connections and curvatures, which we shall then use extensively in the subsequent chapters.

2 Metrics, Connections and Curvature

A metric tensor provides a rule for measuring distances between neighouring points on a manifold. It is an additional piece of structure that was not needed up until this point in the discussion. The metric is a symmetric 2-index cotensor $g_{\mu\nu}$, and in general it is a field on the manifold M, which depends upon the coordinates x^μ. The distance squared between two infinitesimally-separated points is denoted by ds^2, and thus we have, generalising Pythagoras' theorem,

$$ds^2 = g_{ij}\, dx^i\, dx^j \ . \tag{2.1}$$

2.1 Spin connection and curvature 2-forms

Here, we gather together some basic results from part I of the course. We begin by "taking the square root" of the metric g_{ij} in (2.1), by introducing a vielbein, which is a basis of 1-forms $e^a = e^a_i\, dx^i$, with components e^a_i, having the property

$$g_{ij} = \eta_{ab}\, e^a_i\, e^b_j \ . \tag{2.2}$$

Here the indices a are a new type, different from the coordinate indices i we have encountered up until now. They are called local-Lorentz indices, or tangent-space indices, and η_{ab} is a "flat" metric, with constant components. The language of "local-Lorentz" indices stems from the situation when the metric g_{ij} has Minkowskian signature (which is $(-,+,+,\ldots,+)$ in sensible conventions). The signature of η_{ab} must be the same as that of g_{ij}, so if we are working in general relativity with Minkowskian signature we will have

$$\eta_{ab} = \mathrm{diag}\,(-1,1,1,\ldots,1) \ . \tag{2.3}$$

If, on the other hand, we are working in a space with Euclidean signature $(+,+,\ldots,+)$, then η_{ab} will just equal the Kronecker delta, $\eta_{ab} = \delta_{ab}$, or in other words

$$\eta_{ab} = \mathrm{diag}\,(1,1,1,\ldots,1) \ . \tag{2.4}$$

Of course the choice of vielbeins e^a as the square root of the metric in (2.2) is to some extent arbitrary. Specifically, we could, given a particular choice of vielbein e^a, perform an orthogonal-type transformation to get another equally-valid vielbein e'^a, given by

$$e'^a = \Lambda^a{}_b\, e^b \ , \tag{2.5}$$

where $\Lambda^a{}_b$ is a matrix satisfying the (pseudo)orthogonality condition

$$\eta_{ab}\, \Lambda^a{}_c\, \Lambda^b{}_d = \eta_{cd} \ . \tag{2.6}$$

Note that $\Lambda^a{}_b$ can be coordinate dependent. If the n-dimensional manifold has a Euclidean-signature metric then $\eta = \mathbb{1}$ and (2.6) is literally the orthogonality condition $\Lambda^T \Lambda = \mathbb{1}$. Thus in this case the arbitrariness in the choice of vielbein is precisely the freedom to make local $O(n)$ rotations in the tangent space. If the metric signature is Minkowskian, then instead (2.6) is the condition for Λ to be an $O(1, n-1)$ matrix; in other words, one then has the freedom to perform local Lorentz transformations in the tangent space. We shall typically use the words "local Lorentz transformation" regardless of whether we are working with metrics of Minkowskian or Euclidean signature.

Briefly reviewing the next steps, we introduce the spin connection, or connection 1-forms, $\omega^a{}_b = \omega^a{}_{b\,i}\, dxi$, and the torsion 2-forms $T^a = \frac{1}{2} T^a_{ij}\, dx^i \wedge dx^j$, defining

$$T^a = de^a + \omega^a{}_b \wedge e^b \ . \tag{2.7}$$

Next, we define the curvature 2-forms $\Theta^a{}_b$, *via* the equation

$$\Theta^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b \ . \tag{2.8}$$

Note that if we adopt the obvious matrix notation where the local Lorentz transformation (2.5) is written as $e' = \Lambda\, e$, then we have the property that $\omega^a{}_b$, T^a and $\Theta^a{}_b$ transform as follows:

$$\begin{aligned} \omega' &= \Lambda\, \omega\, \Lambda^{-1} + \Lambda\, d\Lambda^{-1} \ , \\ T' &= \Lambda\, T \ , \qquad \Theta' = \Lambda\, \Theta\, \Lambda^{-1} \ . \end{aligned} \tag{2.9}$$

Thus the torsion 2-forms T^a and the curvature 2-forms $\Theta^a{}_b$ both transform nicely, in a covariant way, under local Lorentz transformations, while the spin connection does not; it has an extra inhomogeneous term in its transformation rule. This is the characteristic way in which connections transform. Because of this, we can define a Lorentz-covariant exterior derivative D as follows:

$$DV^a{}_b \equiv dV^a{}_b + \omega^a{}_c \wedge V^c{}_b - \omega^c{}_b \wedge V^a{}_c \ , \tag{2.10}$$

where $V^a{}_b$ is some set of p-forms carrying tangent-space indices a and b. One can easily check that if $V^a{}_b$ itself transforms covariantly under local Lorentz transformations, then so does $DV^a{}_b$. In other words, the potentially-troublesome terms where the exterior derivative lands on the transformation matrix Λ are cancelled out by the contributions from the inhomgeneous second term in the transformation rule for $\omega^a{}_b$ in (2.9). We have taken the example of $V^a{}_b$ with one upstairs and one downstairs tangent space index for simplicity, but the generalisation to arbitrary numbers of indices is immediate. There is one term like the second term on the right-hand side of (2.10) for each upstairs index, and a term like the third term on the right-hand side of (2.10) for each downstairs index.

The covariant exterior derivative D will commute nicely with the process of contracting tangent-space indices with η_{ab}, provided that we require

$$D\,\eta_{ab} \equiv d\eta_{ab} - \omega^c{}_a\,\eta_{cb} - \omega^c{}_b\,\eta_{ac} = 0\ . \tag{2.11}$$

Since we are taking the components of η_{ab} to be literally constants, it follows from this equation, which is known as the equation of *metric compatibility*, that

$$\omega_{ab} = -\omega_{ba}\ , \tag{2.12}$$

where ω_{ab} is, by definition, $\omega^a{}_b$ with the upper index lowered using η_{ab}: $\omega_{ab} \equiv \eta_{ac}\,\omega^c{}_b$. With this imposed, it is now the case that we can take covariant exterior derivatives of products, and freely move the local-Lorentz metric tensor η_{ab} through the derivative. This means that we get the same answer if we differentiate the product and then contract some indices, or if instead we contract the indices and then differentiate.

In addition to the requirement of metric compatibiilty we usually also choose a *torsion-free* spin-connection, meaning that we demand that the torsion 2-forms T^a defined by (2.7) vanish. In practice, we shall now assume this in everything that follows. In fact equation (2.7), together with the metric-compatibility condition (2.12), now determine $\omega^a{}_b$ uniquely. In other words, the two conditions

$$de^a = -\omega^a{}_b \wedge e^b\ , \qquad \omega_{ab} = -\omega_{ba} \tag{2.13}$$

have a unique solution. It can be given as follows. Let us say that, by definition, the exterior derivatives of the vielbeins e^a are given by

$$de^a = -\tfrac{1}{2} c_{bc}{}^a\, e^b \wedge e^c\ , \tag{2.14}$$

where the structure functions $c_{bc}{}^a$ are, by definition, antisymmetric in bc. Then the solution for ω_{ab} is given by

$$\omega_{ab} = \tfrac{1}{2}(c_{abc} + c_{acb} - c_{bca})\, e^c\ , \tag{2.15}$$

where $c_{abc} \equiv \eta_{cd}\, c_{ab}{}^d$. It is easy to check by direct substitution that this indeed solves the two conditions (2.13).

The procedure, then, for calculating the curvature 2-forms for a metric $g_{\mu\nu}$ with vielebeins e^a is the following. We write down a choice of vielbein, and by taking the exterior derivative we read off the coefficients $c_{bc}{}^a$ in (2.14). Using these, we calculate the spin connection using (2.15). Then, we substitute into (2.8), to calculate the curvature 2-forms.

Each curvature 2-form $\Theta^a{}_b$ has, as its components, a tensor that is antisymmetric in two coordinate indices. This is the Riemann tensor, defined by

$$\Theta^a{}_b = \tfrac{1}{2} R^a{}_{b\,ij}\, dx^i \wedge dx^j \ . \tag{2.16}$$

We may always use the vielbein e^a_i, which is a non-degenerate $n \times n$ matrix in n dimensions, to convert between coordinate indices i and tangent-space indices a. For this purpose we also need the inverse of the vielbein, denoted by E^i_a, and satisfying the defining properties

$$E^i_a\, e^a a_j = \delta^i_j \ , \qquad E^i_a\, e^b_i = \delta^a_b \ . \tag{2.17}$$

Then we may define Riemann tensor components entirely within the tangent-frame basis, as follows:

$$R^a{}_{bcd} \equiv E^i_c\, E^j_d\, R^a{}_{b\,ij} \ . \tag{2.18}$$

Note that we use the same symbol for the tensors, and distinguish them simply by the kinds of indices that they carry. (This requires that one pay careful attention to establishing unambiguous notations, which keep track of which are coordinate indices, and which are tangent-spave indices!) In terms of $R^a{}_{bcd}$, it is easily seen from the various definitions that we have

$$\Theta^a{}_b = \tfrac{1}{2} R^a{}_{bcd}\, e^c \wedge e^d \ . \tag{2.19}$$

From the Riemann tensor two further quantities can be defined; the Ricci tensor R_{ab} and the Ricci scalar R:

$$R_{ab} = R^c{}_{acb} \ , \qquad R = \eta^{ab}\, R_{ab} \ . \tag{2.20}$$

Note that the Riemann tensor and Ricci tensor have the following symmetries, which can be proved straightforwardly from the definitions:

$$\begin{aligned} &R_{abcd} = -R_{bacd} = -R_{abdc} = R_{cdab} \ , \\ &R_{abcd} + R_{acdb} + R_{adbc} = 0 \ , \\ &R_{ab} = R_{ba} \ . \end{aligned} \tag{2.21}$$

2.2 Curvature in coordinate basis

For those more familiar with the "traditional" treatment of Riemannian geometry, we can give a "dictionary" for translating between the two formalisms. In the traditional approach, we construct the Christoffel connection, Γ^i_{jk} from the metric tensor, using the expression

$$\Gamma^i{}_{jk} = \tfrac{1}{2} g^{i\ell} \Big(\partial_j\, g_{\ell k} + \partial_k\, g_{j\ell} - \partial_\ell\, g_{jk}\Big)\ . \tag{2.22}$$

This is used in order to construct the covariant derivative, ∇_i. Its action on tensors with upstairs indices is defined by

$$\nabla_i\, V^j = \partial_i\, V^j + \Gamma^i{}_{jk}\, V^k\ , \tag{2.23}$$

while for downstairs indices it is

$$\nabla_i\, V_j = \partial_i\, V_j - \Gamma^k{}_{ij}\, V_k\ . \tag{2.24}$$

Acting on tensors with multiple indices, there will be one Γ term of the appropriate type for each upstairs or downstairs index. The expression (2.22) for the Christoffel connection is in fact determined by the requirement of metric compatibility, namely $\nabla_i\, g_{jk} = 0$. The covariant derivative has the property that acting on any tensor, it gives another tensor. In other words, the object constructed by acting with the covariant derivative will transform under general coordinate transformations according to the rule given in (1.6) and (1.12) for upstairs and downstairs indices.

From the Christoffel connection we construct the Riemman tensor, given by

$$R^i{}_{jk\ell} = \partial_k\, \Gamma^i{}_{\ell j} - \partial_\ell\, \Gamma^i{}_{kj} + \Gamma^i{}_{km}\, \Gamma^m{}_{\ell j} - \Gamma^i{}_{\ell m}\, \Gamma^m{}_{kj}\ . \tag{2.25}$$

Although it is not immediately obvious, in view of the fact that Γ^i_{jk} is *not* itself a tensor, the quantity $R^i_{jk\ell}$ does in fact transform tensorially. This can be shown from the previous definitions by a straightforward calculation.

To make contact with the curvature computations using differential forms in the previous section, we note that the Riemman tensor calculated here is the same as the one in the previous section, after converting the indices using the vielbein or inverse vielbein:

$$R^i{}_{jk\ell} = E^i_a\, e^b_j\, R^a{}_{bk\ell}\ . \tag{2.26}$$

The coordinate components of the Ricci tensor, and the Ricci scalar, are given by

$$R_{ij} = R^k{}_{ikj}\ , \qquad R = g^{ij}\, R_{ij}. \tag{2.27}$$

As usual, we can relate the tensors with tangent-space and coordinate indices by means of the vielbein, so that we have $R_{ij} = e^a_i\, e^b_j\, R_{ab}$.

One further identity, easily proven from the definitions in this section, is that

$$\nabla_{[m}\, R^i{}_{|j|k\ell]} = 0 \;, \tag{2.28}$$

where the vertical lines enclosing an index or set of indices indicate that they are excluded from the antisymmetrisation. An appropriate contraction of indices in the Bianchi identity (2.28) leads to the result that

$$\nabla^i\, R_{ij} = \tfrac{1}{2}\partial_j\, R \;. \tag{2.29}$$

A consequence of this is that if we define the so-called *Einstein tensor*

$$G_{ij} \equiv R_{ij} - \tfrac{1}{2} R\, g_{ij} \;, \tag{2.30}$$

then it is conserved, i.e. $\nabla^i\, G_{ij} = 0$.

3 Complex Manifolds and Calabi-Yau Spaces

3.1 Introduction

The material in this chapter draws heavily upon some lecture notes written by Philip Candelas, which appeared in the proceedings of the 1987 Trieste school "Superstrings 87."

A n-dimensional manifold M is a topological space together with an *atlas*, i.e. a collection of *charts* $(U_{(a)}, x_{(a)})$ where $U_{(a)}$ are open subsets of M and the $x_{(a)}$ are one-to-one maps of the corresponding $U_{(a)}$ to open subsets of $\mathbb{R}^n$. In other words, $x_{(a)}$ represents a set of coordinates $x^i_{(a)}$, $1 \le i \le n$, which covers the open region $U_{(a)}$ in M. The compete atlas of charts covers the entire manifold M, but in general, no single chart can cover all of M. If two of the regions $U_{(a)}$ and $U_{(b)}$ have an overlap, then the map obtained by composing $x_{(a)} \cdot x^{-1}_{(b)}$ takes us from one copy of $\mathbb{R}^n$ to the another. Put another way, this means that in the overlap region, one can view the coordinates $x^i_{(a)}$ as functions of the $x^j_{(b)}$, i.e. $x^i_{(a)} = f^i_{(a)(b)}(x^j_{(b)})$. The manifold is said to be C^r if the transition functions are r-times differentiable. Normally, one considers manifolds that are C^∞.

A *complex* n-manifold is a topological space M of complex dimension n with a holomorphic, or complex-analytic, atlas. Thus one now has a collection of charts $(U_{(a)}, z_{(a)})$, where in every non-empty intersection the maps $z_{(a)} \cdot z^{-1}_{(b)}$ are holomorphic. Of course the $z_{(a)}$ are now maps into C^n. Thus the transition functions are now required to be holomorphic functions of the complex coordinates in the two overlapping charts: $z^i_{(a)} = f^i_{(a)(b)}(z^j_{(b)})$, rather than being C^∞. Thus the $z^i_{(a)}$ are functions of $z^j_{(b)}$, but not of $\bar{z}^j_{(b)}$. Since C^n can be thought of as $\mathbb{R}^{2n}$, it follows that every complex n-manifold is also a real $(2n)$-manifold.[1]

Not every real $(2n)$-manifold is a complex n-manifold, however. A simple non-trivial example that *is* a complex manifold is the 2-sphere. Imagine sandwiching a 2-sphere between two infinite parallel plates T_1 and T_2, which are tangent to the sphere at the south and the north poles S and N respectively. We may parameterise a point P on the sphere in terms of the (x, y) coordinates in the planes T_1 or T_2 of the points obtained by passing a straight line from N through P to T_1, or S through P to T_2, respectively. Call these coordinates (x_1, y_1) and (x_2, y_2) respectively. Obviously, for a generic point P on the sphere, there are corresponding well-defined points (x_1, y_1) and (x_2, y_2) in the planes T_1 and T_2, and there is a one-to-one map between the two descriptions. This will break down only if $P = N$ or

[1] Do not confuse the use of C^r to mean an r-times differentiable function with C^n to mean complex n-dimensional space! Amost all the time, we mean the complex n space when the symbol C^n appears. It should be clear from the context, and so there should be no confusion.

$P = S$, since then (x_1, y_1) or (x_2, y_2) respectively will go to infinity. For generic points P, simple geometry shows that the relation between the coordinates in the two charts is

$$x_1 = \frac{x_2}{x_2^2 + y_2^2} \ , \qquad y_1 = -\frac{y_2}{x_2^2 + y_2^2} \ . \tag{3.1}$$

Clearly these functions are C^∞ for generic points where the two charts overlap, i.e. provided the north and south poles are excluded. To see that the 2-sphere is a complex manifold, we now introduce the complex coordinate $z_1 = x_1 + \mathrm{i}\, y_1$ on T_1, and likewise $z_2 = x_2 + \mathrm{i}\, y_2$ on T_2. It is easy to see that the real C^∞ transition functions defined by (3.1) can be re-expressed in terms of z as

$$z_1 = \frac{1}{z_2} \ . \tag{3.2}$$

This is holomorphic, or complex analytic, in the overlap region (i.e. for $z_2 \neq 0, \ \infty$), thus demonstrating that S^2 is a complex manifold.

3.2 Almost Complex Structures and Complex Structures

Suppose that M is a complex n-manifold, with coordinates z^μ in some neighbourhood U. We define the 2-index mixed tensor J, by

$$J = \mathrm{i}\, dz^\mu \otimes \frac{\partial}{\partial z^\mu} - \mathrm{i}\, dz^{\bar\mu} \otimes \frac{\partial}{\partial z^{\bar\mu}} \ , \tag{3.3}$$

where we use the notation $z^{\bar\mu}$ to stand for $\bar z^\mu$. In terms of components, we see that

$$J_\mu{}^\nu = \mathrm{i}\, \delta_\mu{}^\nu \ , \qquad J_{\bar\mu}{}^{\bar\nu} = -\mathrm{i}\, \delta_{\bar\mu}{}^{\bar\nu} \ , \qquad J_{\bar\mu}{}^\nu = 0 \ , \qquad J_\mu{}^{\bar\nu} = 0 \ . \tag{3.4}$$

J is called an *Almost Complex Structure.*

Note that J as defined is indeed a tensor, since it is independent of the choice of complex coordinates. The crucial point here is that we allow only *holomorphic* coordinate transformations, and so the first and the second terms in (3.3) are separately unchanged under such transformations. Thus if $z^\mu = z^\mu(w^\nu)$, then

$$dz^\mu \otimes \frac{\partial}{\partial z^\mu} = \frac{\partial z^\mu}{\partial w^\nu} \frac{\partial w^\rho}{\partial z^\mu} dw^\nu \otimes \frac{\partial}{\partial w^\rho} = dw^\mu \otimes \frac{\partial}{\partial w^\mu} \ , \tag{3.5}$$

with an analogous result for the complex conjugate. It is also evident that J is itself a real tensor.

As we remarked previously, we may think of the complex n-manifold as being also a real $(2n)$-manifold. Suppose that in the local neighbourhood U we have real coordinates x^i, with $1 \le i \le 2n$. It can also be useful to think of the real coordinates as occurring in pairs, so that $z^\mu = x^\mu + \mathrm{i}\, y^\mu$, for $1 \le \mu \le n$. We then see that the definition (3.3) for J becomes

$$J = dx^\mu \otimes \frac{\partial}{\partial y^\mu} - dy^\mu \otimes \frac{\partial}{\partial x^\mu} \ . \tag{3.6}$$

We may therefore represent the tensor J as a real $(2n) \times (2n)$ matrix in $n \times n$ blocks as

$$J = \begin{pmatrix} 0_n & \mathbb{1}_n \\ -\mathbb{1}_n & 0_n \end{pmatrix} \ , \tag{3.7}$$

where O_n denotes the zero $n \times n$ matrix, and $\mathbb{1}_n$ dentotes the unit $n \times n$ matrix. Evidently, therefore, the tensor J squares to minus 1:

$$J_i{}^j\, J_j{}^k = -\delta_i{}^k \ . \tag{3.8}$$

It is a theorem that every complex manifold admits a *globally defined* almost complex structure. The emphasis here is on the fact that it is globally defined, since obviously *any* real $(2n)$-manifold, since it looks locally like $\mathbb{R}^{2n}$, must look locally like C^n. The converse, however, is not true: not every manifold that admits an almost complex structure is a complex manifold. Rather, such a manifold is, by definition an *Almost Complex Manifold.* In the special case where an almost complex manifold is actually a complex manifold, the almost complex structure is called a complex structure. To see why it is the case that not every almost complex manifold is a complex manifold, we need to delve a little deeper into the properties of the almost complex structure tensor.

From $J_i{}^j$, in a complex manifold we can define projection operators $P_i{}^j$ and $Q_i{}^j$:

$$P_i{}^j = \tfrac{1}{2}(\delta_i{}^j - \mathrm{i}\, J_i{}^j) \ , \qquad Q_i{}^j = \tfrac{1}{2}(\delta_i{}^j + \mathrm{i}\, J_i{}^j) \ , \tag{3.9}$$

which clearly satisfy the relations

$$P^2 = P \ , \qquad Q^2 = Q \ , \qquad P\,Q = Q\,P = 0 \ , \qquad P + Q = \mathbb{1} \ , \tag{3.10}$$

in the obvious matrix notation. These operators project into the holomorphic and anti-holomorphic components of tensors. Thus, for example,

$$V_i\, dx^i = (P_i{}^j + Q_i{}^j)\, V_j\, dx^i = V_\mu\, dz^\mu + V_{\bar\mu}\, dz^{\bar\mu} \ , \tag{3.11}$$

where

$$P_i{}^j\, V_j\, dx^i = V_\mu\, dz^\mu \ , \qquad Q_i{}^j\, V_j\, dx^i = V_{\bar\mu}\, dz^{\bar\mu} \ . \tag{3.12}$$

These 1-forms are called $(1,0)$-forms and $(0,1)$-forms respectively. Generally, it is clear that the existence of an almost complex structure allows the refinement of the notion of n-forms, into subsets of (p,q)-forms, where $p + q = n$.

The question now is the following. What further conditions are necessary in order for an almost complex structure J to be a complex structure? In other words, what are the conditions for the almost complex manifold, with almost complex structure J, to be a complex manifold?

First of all, note that we can still define the projection operators $P_i{}^j$ and $Q_i{}^j$ as in (3.9) whenever we have an almost complex structure, although we should not yet think of them as projections into holomorphic and anti-holomorphic subspaces of forms. To have a complex manifold, we must be able to introduce complex coordinates z^μ. Thus, consider a neighbourhood U in the almost complex manifold M, with real coordinates x^i. We wish to see if we can find complex coordinates $z^\mu(x^i)$. Thus we can write

$$dz^\mu = \frac{\partial z^\mu}{\partial x^i}\, dx^i \ , \tag{3.13}$$

which can be expressed, by inserting $\delta_i{}^j = P_i{}^j + Q_i{}^j$, as

$$dz^\mu = \partial_j z^\mu\, P_i{}^j\, dx^i + \partial_j z^\mu\, Q_i{}^j\, dx^i \ . \tag{3.14}$$

Now, we saw previously that the two terms on the right-hand side are respectively $(1,0)$ and $(0,1)$ forms, while the left-hand side is manifestly what we should call a $(1,0)$ form if the complex coordinates do indeed exist. Consequently, is must be that

$$\partial_j z^\mu\, Q_i{}^j = 0 \ . \tag{3.15}$$

This can be viewed as a system of differential equations for the complex coordinates z^μ. If the equations are satisfied, then we can act with $Q_k{}^\ell\, \partial_\ell$ to get

$$\partial_\ell \partial_j z^\mu\, Q_i{}^j\, Q_k{}^\ell + \partial_j z^\mu\, \partial_\ell Q_i{}^j\, Q_k{}^\ell = 0 \ . \tag{3.16}$$

Taking the projection of this equation that is skew-symmetric in i and k, we therefore obtain the integrability condition

$$\partial_j z^\mu\, \partial_\ell Q_{[i}{}^j\, Q_{k]}{}^\ell = 0 \ . \tag{3.17}$$

We can now insert $P_m{}^j + Q_m{}^j = \delta_m{}^j$, and make use of (3.15), to re-express the integrability condition as

$$\partial_j z^\mu\, P_m{}^j\, \partial_\ell Q_{[i}{}^m\, Q_{k]}{}^\ell = 0 \ . \tag{3.18}$$

In order for the derivatives $\partial z^\mu / \partial x_j$, which must already satisfy (3.15), not to be overconstrained, it must be that

$$P_m{}^j\, \partial_\ell Q_{[i}{}^m\, Q_{k]}{}^\ell = 0 \ . \tag{3.19}$$

By taking the real and imaginary parts of this equation, one can easily show that each is equivalent to the statement that the following real tensor must vanish:

$$N_{ij}{}^k \equiv \partial_{[j} J_{i]}{}^k - J_{[i}{}^\ell\, J_{j]}{}^m\, \partial_m J_\ell{}^k \ . \tag{3.20}$$

This is known as the Nijenhuis tensor. (Note that it is indeed a tensor, even though it is defined using partial derivatives. This can be verified by direct calculation of its behaviour under general coordinate transformations. Alternatively, one can consider the manifestly tensorial object defined by replacing the partial derivatives by covariant derivatives, and then verify that all the connection terms cancel out by virtue of the antisymmetrisations.)

To summarise, then, we have seen that the vanishing of the Nijenhuis tensor is an integrability condition for the existence of a complex coordinate system in an almost complex manifold. As usual, establishing a completely watertight "if and only if" theorem is something best left to the hard-core mathemeticians. The bottom line, in any case, is that an almost complex manifold can be shown to be a complex manifold if and only if the Nijenhuis tensor vanishes.

Also, for future reference, let us establish some further notation and terminology for differential forms on almost complex and complex manifolds. We have seen that the tensors $P_i{}^j$ and $Q_i{}^j$ project 1-forms into two subspaces, which we are denoting by $(1,0)$ and $(0,1)$ forms. More generally, given any n form ω , we can make projections into $(n+1)$ subspaces, of (p,q)-forms where $p+q=n$, as follows:

$$\omega^{(p,q)}_{i_1\cdots i_p j_1\cdots j_q} = P_{i_1}{}^{k_1} \cdots P_{i_p}{}^{k_p}\, Q_{j_1}{}^{\ell_1} \cdots Q_{j_q}{}^{\ell_q}\, \omega_{k_1\cdots k_p \ell_1\cdots \ell_q} \ . \tag{3.21}$$

It is evident from the properties of P and Q as projection operators that the sum over these various (p,q)-forms gives back the original n-form:

$$\omega = \sum_{p+q=n} \omega^{(p,q)} \ . \tag{3.22}$$

Now consider the action of the exterior derivative d. It is easy to see from the definitions that if we apply d to a (p,q)-form in an almost complex manifold, we will obtain a $(p+q+1)$-form that is expressible in general as the sum of four distinct terms, namely

$$d\omega^{(p,q)} = (d\omega)^{(p,q+1)} + (d\omega)^{(p+1,q)} + (d\omega)^{(p+2,q-1)} + (d\omega)^{(p-1,q+2)} \ . \tag{3.23}$$

If J is in fact a *complex structure*, then the last two terms in this decomposition are absent. To see how this works, consider, for simplicity, a $(1,0)$ form A, which we may construct from a generic 1-form ω by defining $A = P_i{}^j\,\omega_j\,dx^i$. We now calculate dA, and then insert $1 = P + Q$ judicously in all necessary places so as to project out the various structures:

$$\begin{aligned} dA &= (\partial_k P_i{}^j\,\omega_j + P_i{}^j\,\partial_k\omega_j)dx^k\wedge dx^i \\ &= (\partial_\ell P_m{}^j\,\omega_j\,P_k{}^\ell\,P_i{}^m + \partial_\ell P_m{}^j\,\omega_j\,P_k{}^\ell\,Q_i{}^m + \partial_\ell P_m{}^j\,\omega_j\,Q_k{}^\ell\,P_i{}^m + \partial_\ell P_m{}^j\,\omega_j\,Q_k{}^\ell\,Q_i{}^m \\ &\quad + P_i{}^j\,P_k{}^\ell\,\partial_\ell\omega_j + P_i{}^j\,Q_k{}^\ell\,\partial_\ell\omega_j)dx^k\wedge dx^i\ . \end{aligned} \tag{3.24}$$

It is manifest that these six terms are of types $(2,0)$, $(1,1)$, $(1,1)$, $(0,2)$, $(2,0)$ and $(1,1)$ respectively. If the almost complex structure is in fact a complex structure, we expect that dA should have only $(2,0)$ and $(1,1)$ components, and so it would then have to be that the $(0,2)$ component were zero. This would imply that we need

$$\partial_\ell P_m{}^j\,Q_{[k}{}^\ell\,Q_{i]}{}^m = 0\ . \tag{3.25}$$

Now, since the projection operators satisfy $P_m{}^j\,Q_i{}^m = 0$, it follows that $\partial_\ell P_m{}^j\,Q_i{}^m + P_m{}^j\,\partial_\ell Q_i{}^m = 0$, and using this, we see that (3.25) reduces to (3.19). Thus we see that indeed the exterior derivative of a $(1,0)$-form gives only a $(2,0)$ and a $(1,1)$ form, but no $(0,2)$ form, provided that the Nijenhuis tensor vanishes, implying that the almost complex structure is a complex structure.

If we do have a complex structure, so that $d\omega^{(p,q)} = (d\omega)^{(p+1,q)} + (d\omega)^{(p,q+1)}$, we may then define holomorphic and antiholomorphic exterior derivative operators ∂ and $\bar\partial$, where

$$\begin{aligned} &d = \partial + \bar\partial\ , \\ &\partial\omega^{(p,q)} = (d\omega)^{(p+1,q)}\ , \qquad \bar\partial\omega^{(p,q)} = (d\omega)^{(p,q+1)}\ . \end{aligned} \tag{3.26}$$

Thus $\partial f(z,\bar z) = \partial f/\partial z^\mu\,dz^\mu$, and $\bar\partial f(z,\bar z) = \partial f/\partial z^{\bar\mu}\,dz^{\bar\mu}$, *etc.* Note that we have $d^2 = (\partial+\bar\partial)^2 = \partial^2 + \bar\partial^2 + \partial\bar\partial + \bar\partial\partial$, and hence in a complex manifold we have

$$\partial^2 = 0\ , \qquad \bar\partial^2 = 0\ , \qquad \partial\bar\partial + \bar\partial\partial = 0\ , \tag{3.27}$$

since these three parts of d^2 have different holomorphic degrees.

A further consequence of the vanishing of the Nijenuis tensor is that there exists a holomorphic atlas with respect to which the components of the complex structure are given by

$$J_\mu{}^\nu = \mathrm{i}\,\delta_\mu{}^\nu\ , \qquad J_{\bar\mu}{}^{\bar\nu} = -\mathrm{i}\,\delta_{\bar\mu}{}^{\bar\nu}\ , \qquad J_{\bar\mu}{}^\nu = 0\ , \qquad J_\mu{}^{\bar\nu} = 0\ . \tag{3.28}$$

To see this, note that we can write (3.15) as

$$\partial_j\,z^\mu + \mathrm{i}\,J_j{}^k\partial_k z^\mu = 0\ . \tag{3.29}$$

In fact this is precisely the n-dimensional generalisation of the Cauchy-Riemann equations. Contracting with the basis $dx^j \otimes \partial/\partial z^\mu$, we have

$$J_j{}^k\, dx^j\, \partial_k z^\mu \otimes \frac{\partial}{\partial z^\mu} = \mathrm{i}\, dz^\mu \otimes \frac{\partial}{\partial z^\mu}\ . \tag{3.30}$$

If we add the complex conjugate to this equation, we get

$$J_j{}^k \Big(dx^j\, \partial_k z^\mu \otimes \frac{\partial}{\partial z^\mu} + dx^j\, \partial_k z^{\bar\mu} \otimes \frac{\partial}{\partial z^{\bar\mu}}\Big) = \mathrm{i}\, dz^\mu \otimes \frac{\partial}{\partial z^\mu} - \mathrm{i}\, dz^{\bar\mu} \otimes \frac{\partial}{\partial z^{\bar\mu}}\ , \tag{3.31}$$

which, by the chain rule, is nothing but

$$J_j{}^k\, dx^j \otimes \partial_k = \mathrm{i}\, dz^\mu \otimes \frac{\partial}{\partial z^\mu} - \mathrm{i}\, dz^{\bar\mu} \otimes \frac{\partial}{\partial z^{\bar\mu}}\ . \tag{3.32}$$

Thus the complex structure tensor $J \equiv J_j{}^k\, dx^j \otimes \partial_k$ does indeed have components, with respect to the complex coordinate basis, given by (3.28).

3.3 Metrics on Almost Complex and Complex Manifolds

So far in the discussion, our considerations have been entirely independent of any metric on the manifold. Suppose that an almost complex manifold M has a metric h_{ij}, i.e. a real, symmetric 2-index tensor with positive-definite eigenvalues. We can always construct from this a so-called *almost Hermitean metric* g_{ij}, defined as

$$g_{ij} = \tfrac{1}{2}(h_{ij} + J_i{}^k\, J_j{}^\ell\, h_{k\ell})\ , \tag{3.33}$$

which is also real, symmetric and positive-definite. It clearly satisfies the almost-hermiticity condition

$$g_{ij} = J_i{}^k\, J_j{}^\ell\, g_{k\ell}\ . \tag{3.34}$$

Another way of expressing this, by multiplying by $J_m{}^i$, and defining $J_m{}^i\, g_{ij} = J_{mj}$, is

$$J_{mj} = -J_{jm}\ . \tag{3.35}$$

Thus with respect to an almost Hermitean metric, the almost complex structure defines a natural 2-form.

If M is actually a complex manifold, then it is evident that an Hermitean metric has the property that

$$ds^2 = 2\, g_{\mu\bar\nu}\, dz^\mu\, dz^{\bar\nu}\ . \tag{3.36}$$

This should be contrasted with a generic real symmetric metric, for which we would have

$$ds^2 = 2\, g_{\mu\bar\nu} dz^\mu\, dz^{\bar\nu} + g_{\mu\nu}\, dz^\mu\, dz^\nu + g_{\bar\mu\bar\nu}\, dz^{\bar\mu}\, dz^{\bar\nu}\ . \tag{3.37}$$

A consequence of the structure (3.36) of the Hermitean metric is that when complex indices are raised or lowered, barred become unbarred, and *vice versa.*

Suppose now that M is an Hermitean manifold, meaning that it is a complex manifold endowed with an Hermitean metric. We can now introduce the notion of a covariant derivative. Normally, in real geometry, we define the unique Christofel connection $\Gamma^i{}_{jk}$ by two conditions, namely that the covariant derivative (defined using $\Gamma^i{}_{jk}$) of the metric be zero, and that $\Gamma^i{}_{jk}$ be symmetric in its two lower indices. More generally, we could consider a connection for which the metric is still covariantly constant, but where there is an antisymmetric part $\Gamma^i{}_{[jk]}$ also. This extra term is known as a torsion tensor.

For the Hermitean manifold M, we may define a unique connection as follows. We require that the covariant derivative both of the metric, and of the complex structure tensor, vanish. In addition, we require that the torsion $\Gamma^i{}_{[jk]}$ be *pure* in its lower indices. In other words, if we use complex coordinates, we require that $\Gamma^i{}_{[\mu\bar\nu]}$ be zero, where i represents either ρ or $\bar\rho$, while no requirement is placed on $\Gamma^i{}_{[\mu\nu]}$ or $\Gamma^i{}_{[\bar\mu\bar\nu]}$. To see what this leads to, we may consider taking the covariant derivative of $P_i{}^j$, which, by our requirements for the connection, must vanish.

First, let us write down the general expression for the covariant derivative:

$$\nabla_i\, P_j{}^k \equiv \partial_i\, P_j{}^k + \Gamma^k{}_{i\ell}\, P_j{}^\ell - \Gamma^\ell{}_{ij}\, P_\ell{}^k \ . \tag{3.38}$$

Now, noting that we may choose complex coordinates such that the complex structure has components given by (3.28), it follows, from (3.9), that the only non-vanishing components of $P_i{}^j$ are given by

$$P_\mu{}^\nu = \delta_\mu{}^\nu \ . \tag{3.39}$$

Thus if we consider the covariant-constancy condition $\nabla_i\, P_j{}^k = 0$, then the content of this equation is encompassed by taking (j,k) to be either (μ,ν), or else $(\mu,\bar\nu)$. From (3.38), the first case tells us nothing, since we get

$$0 = \nabla_i\, P_\mu{}^\nu = \Gamma^\nu{}_{i\rho}\, \delta_\mu{}^\rho - \Gamma^\rho{}_{i\mu}\, \delta_\rho{}^\nu \ . \tag{3.40}$$

On the other hand, we do learn something from taking $(i,j) = (\mu,\bar\nu)$, since then we get

$$0 = \nabla_i\, P_\mu{}^{\bar\nu} = \Gamma^{\bar\nu}{}_{k\rho}\, \delta_\mu{}^\rho \ , \tag{3.41}$$

and hence $\Gamma^{\bar\nu}{}_{k\mu} = 0$. Thus we have

$$\begin{aligned} &\Gamma^{\bar\nu}{}_{\rho\mu} = 0 \ , \qquad \Gamma^{\bar\nu}{}_{\bar\rho\mu} = 0 \ , \\ &\Gamma^{\nu}{}_{\bar\rho\bar\mu} = 0 \ , \qquad \Gamma^{\nu}{}_{\rho\bar\mu} = 0 \ , \end{aligned} \tag{3.42}$$

where the second line follows by complex conjugation of the first. Now, we also have the condition that the mixed components $\Gamma^i{}_{[\rho\bar\mu]}$ of the torsion vanish. Together with what we already have, this therefore implies that *all* mixed components of $\Gamma^i{}_{jk}$ vanish. In other words, the only non-vanishing components of the Hermitean connection are the pure ones,

$$\Gamma^\mu{}_{\nu\rho} \qquad \text{and} \qquad \Gamma^{\bar\mu}{}_{\bar\nu\bar\rho} \ . \tag{3.43}$$

Since the claim is that the Hermitean connection just defined is unique, we expect to be able to solve for it in terms of the Hermitean metric. This is indeed possible. Since the metric is covariantly constant, we have

$$\nabla_i\, g_{jk} \equiv \partial_i\, g_{jk} - \Gamma^\ell{}_{ij}\, g_{\ell k} - \Gamma^\ell{}_{ik}\, g_{j\ell} = 0 \ . \tag{3.44}$$

If we take $(i,j,k) = (\mu,\nu,\bar\rho)$, we get, in view of the previous results for the purity of the connection,

$$\partial_\mu\, g_{\nu\bar\rho} - \Gamma^\lambda{}_{\mu\nu}\, g_{\lambda\bar\rho} = 0 \ , \tag{3.45}$$

which can therefore be immediately solved to give

$$\Gamma^\lambda{}_{\mu\nu} = g^{\lambda\bar\rho}\, \partial_\mu\, g_{\nu\bar\rho} \ . \tag{3.46}$$

As a consequence of the purity of the Hermitean connection, it follows that the Riemann tensor has a simple structure also. To see this, let us first write down the general expression for the Riemann tensor, namely

$$R^i{}_{jk\ell} = \partial_k\, \Gamma^i{}_{\ell j} - \partial_\ell\, \Gamma^i{}_{kj} + \Gamma^i{}_{km}\, \Gamma^m{}_{\ell j} - \Gamma^i{}_{\ell m}\, \Gamma^m{}_{kj} \ . \tag{3.47}$$

Taking first $(i,j) = (\bar\mu,\nu)$, we see that

$$R^{\bar\mu}{}_{\nu k\ell} = \partial_k\, \Gamma^{\bar\mu}{}_{\ell\nu} - \partial_\ell\, \Gamma^{\bar\mu}{}_{k\nu} + \Gamma^{\bar\mu}{}_{km}\, \Gamma^m{}_{\ell\nu} - \Gamma^{\bar\mu}{}_{\ell m}\, \Gamma^m{}_{k\nu} \ , \tag{3.48}$$

and that all the terms here vanish by virtue of the purity of the connection coefficients. Thus lowering the $\bar\mu$ index, and recalling that the only non-vanishing metric components are of the form $g_{\mu\bar\nu}$, we see that

$$R_{\mu\nu k\ell} = 0 \ . \tag{3.49}$$

Similarly, one can see that the purity of Γ implies that the components $R_{\mu\bar\nu\rho\sigma}$ must vanish. The components $R_{\bar\mu\nu\rho\sigma}$ vanish for a different reason, namely because of the expression (3.46) for the connection coefficients in terms of the metric. The upshot is that the only non-vanishing components of the Riemann tensor are those given by

$$R^\mu{}_{\nu\bar\rho\sigma} = -R^\mu{}_{\nu\sigma\bar\rho} = \partial_{\bar\rho}\, \Gamma^\mu{}_{\sigma\nu} \ , \tag{3.50}$$

together with those following by complex conjugation. In other words, the only non-vanishing components are those which, if we lower the upper index, are mixed on both their first and second index pairs:

$$R_{\mu\bar\nu\rho\bar\sigma}\ , \qquad R_{\bar\nu\mu\rho\bar\sigma}\ , \qquad R_{\mu\bar\nu\bar\sigma\rho}\ , \qquad R_{\bar\nu\mu\bar\sigma\rho}\ . \tag{3.51}$$

Owing to the existence of the complex structure tensor J, it is possible to define from the Riemann tensor a 2-form $\mathcal{R}$, known as the *Ricci form*, as follows:

$$\mathcal{R} = \tfrac{1}{4} R^i{}_{jk\ell}\, J_i{}^j\, dx^k \wedge dx^\ell\ . \tag{3.52}$$

In terms of complex coordinates, it follows from (3.28), and the structure that we have learnt for the Riemann tensor, that we have

$$\mathcal{R} = \mathrm{i}\, R^\mu{}_{\mu\rho\bar\sigma}\, dz^\rho \wedge dz^{\bar\sigma}\ . \tag{3.53}$$

From (3.46) and (3.50), it now follows that we can express the Ricci form as

$$\mathcal{R} = \mathrm{i}\, \partial\bar\partial \log\sqrt{g}\ , \tag{3.54}$$

where g is the determinant of the metric. From the properties of ∂ and $\bar\partial$ given in (3.27), it follows that $\partial\bar\partial = -\frac{1}{2} d(\partial - \bar\partial)$, and hence we have that

$$d\mathcal{R} = 0\ . \tag{3.55}$$

Note, however, although the Ricci form is closed, it is not, in general, exact, since the determinant of the metric is not a coordinate scalar. In fact, the Ricci form defines a cohomology class, namely the first Chern class, of the complex manifold. This is a topological class, which is invariant under smooth deformations of the complex structure J, and the metric. In other words, under any such deformation, the Ricci form changes by an exact form, and thus its integral over any closed 2-cycle is unchanged. The first Chern class c_1 is defined as the equivalence class of all 2-forms related to a certain multiple of the Ricci-form by the addition of an exact form, and is written as

$$c_1 = \Big[\frac{1}{2\pi}\, \mathcal{R}\Big]\ . \tag{3.56}$$

It is easy to see that $\mathcal{R}$ changes by an exact form under infinitesimal deformations of the metric, since under $g_{ij} \longrightarrow g_{ij} + \delta g_{ij}$ we have $\delta\sqrt{g} = \frac{1}{2} g^{ij}\, \delta g_{ij}\, \sqrt{g}$, and hence

$$\delta\mathcal{R} = \mathrm{i}\, \partial\bar\partial (g^{\mu\bar\nu}\, \delta g_{\mu\bar\nu}) = -\frac{\mathrm{i}}{2}\, d\Big((\partial - \bar\partial) g^{\mu\bar\nu}\, \delta g_{\mu\bar\nu}\Big)\ . \tag{3.57}$$

Since $g^{\mu\bar\nu}\, \delta g_{\mu\bar\nu}$ *is* a genuine general-coordinate scalar, even though $\det(g_{\mu\bar\nu})$ is not, it follows that $\mathcal{R}$ changes by an exact form, and thus c_1 is unaltered.

3.4 Kähler Manifolds

An Hermitean manifold M has, as we have seen, a natural 2-form $J = \frac{1}{2} J_{ij}\, dx^i \wedge dx^j$ that is obtained by lowering the upper index on the complex structure tensor $J_i{}^j$. We can now impose one further level of structure on the Hermitean manifold, by requiring that the 2-form be closed,

$$dJ = 0 \; . \tag{3.58}$$

An Hermitean manifold that satisfies this condition is called a Kähler manifold.[2] The 2-form J is then called the Kähler form. Note that all manifolds of complex dimension 1 are necessarily Kähler, since the exterior derivative of the 2-form J is a 3-form, which exceeds the real dimension of the manifold.

Note that from the pattern of the non-vanishing components of $J_i{}^j$ given in (3.28), it follows that the Kähler form can be written as

$$J = \mathrm{i}\, g_{\mu\bar\nu}\, dz^\mu \wedge dz^{\bar\nu} \; . \tag{3.59}$$

It is therefore a $(1,1)$-form. In terms of components, we have

$$J_{\mu\bar\nu} = \mathrm{i}\, g_{\mu\bar\nu} \, . \tag{3.60}$$

Of course J_{ij} is antisymmetric, unlike g_{ij} which is symmetric, and so $J_{\bar\nu\mu} = -\mathrm{i} g_{\mu\bar\nu} = -\mathrm{i} g_{\bar\nu\mu}$.

Writing dJ as $\partial J + \bar\partial J = 0$, we may note that these two pieces must vanish separately, since they are forms of different types, namely $(2,1)$ and $(1,2)$. Thus we have

$$dJ = \mathrm{i}\, \partial_\rho\, g_{\mu\bar\nu}\, dz^\rho \wedge dz^\mu \wedge dz^{\bar\nu} + \mathrm{i}\, \partial_{\bar\rho}\, g_{\mu\bar\nu}\, dz^{\bar\rho} \wedge dz^\mu \wedge dz^{\bar\nu} = 0 \; , \tag{3.61}$$

and hence

$$\partial_\rho\, g_{\mu\bar\nu} - \partial_\mu\, g_{\rho\bar\nu} = 0 \; , \qquad \partial_{\bar\rho}\, g_{\mu\bar\nu} - \partial_{\bar\nu}\, g_{\mu\bar\rho} = 0 \; . \tag{3.62}$$

These equations imply that locally we must be able to express the Kähler metric in the form

$$g_{\mu\bar\nu} = \partial_\mu\, \partial_{\bar\nu}\, K \; , \tag{3.63}$$

where $K = K(z,\bar z)$ is a real function of the complex coordinates and their complex conjugates. This implies that we have

$$J = \mathrm{i}\, \partial\, \bar\partial\, K \; . \tag{3.64}$$

[2] By now, one might almost suspect that there would exist also the notion of an "almost Kähler manifold," for which the 2-form J in an almost Hermitean manifold would be closed. In fact, it can be shown that an almost Kähler manifold is actually Kähler. It was some while before this was appreciated, and so in some older literature one can find a distinction between the two concepts.

Note that the equations (3.62) imply, from (3.46), that the non-vanishing connection coefficients are now symmetric in their lower indices:

$$\Gamma^\mu{}_{\nu\rho} = \Gamma^\mu{}_{\rho\nu}\,, \qquad \Gamma^{\bar\mu}{}_{\bar\nu\bar\rho} = \Gamma^{\bar\mu}{}_{\bar\rho\bar\nu}\,. \tag{3.65}$$

Thus the Hermitean connection, which generically has torsion in a Hermitean manifold, becomes equal to the standard Christoffel connection without torsion in the case of a Kähler manifold.

The function K is called the Kähler function. However, it should be emphasised that it is not, in general, a general-coordinate scalar. To see this, consider the n-fold wedge product $J^n = J\wedge J\wedge\cdots\wedge J$ on the complex n-manifold. From (3.59), it is evident that

$$\begin{aligned} J^n &= \mathrm{i}^n\, g_{\mu_1\bar\nu_1}\,\cdots g_{\mu_n\bar\nu_n}\, dz^{\mu_1}\, dz^{\bar\nu_1}\cdots dz^{\mu_n}\, dz^{\bar z_n}\ , \\ &= \mathrm{i}^n\, \epsilon^{\mu^1\cdots\mu_n}\, \epsilon^{\bar\nu_1\cdots\bar\nu_n}\, g_{\mu_1\bar\nu_1}\,\cdots g_{\mu_n\bar\nu_n}\, dz^1\, dz^{\bar 1}\cdots dz^n\, dz^{\bar n}\ , \\ &= \mathrm{i}^n\, n!\, \det(g_{\mu\bar\nu})\, dz^1\, dz^{\bar 1}\cdots dz^n\, dz^{\bar n}\ . \end{aligned} \tag{3.66}$$

Now clearly $\det(g_{\mu\bar\nu}) = \sqrt{\det(g_{ij})}$, in view of the off-diagonal Hermitean structure of g_{ij}, and so we have $J^n = c *1$, for some specific n-dependent constant c, where $*1$ is the volume $(2n)$-form on the manifold M. Thus on a compact manifold it must be that $\int_M J^n$ is a non-vanishing constant. But (3.64) can be rewritten as

$$J = -\tfrac12 d(\partial - \bar\partial)\, K\ . \tag{3.67}$$

Thus if K were a coordinate scalar then it would follow that $J = dA$ for some globally-defined 1-form A. However, we would then be able to write $\int_M J^n$ as $\int_M d(A\, J^{n-1}) = \int_{\partial M} A\, J^{n-1}$, and so if M had no boundary, we would have $\int_M J^n = 0$, in contradiction to the previous result. Therefore A is not globally defined, and so K is not a general-coordinate scalar.

In fact, if we consider Kähler functions K_1 and K_2 defined in open neighbourhoods U_1 and U_2 in M, with a non-trivial intersection, then they are related by

$$K_1 = K_2 + f(z) + \overline{f(z)}\ , \tag{3.68}$$

where $f(z)$ is an arbitrary holomorphic function of the coordinates. Clearly, these functions are by the $\partial\,\bar\partial$ derivatives that act on K, and so the Kähler form itself is well defined and transforms properly across the open neighbourhoods.

A couple of examples will be instructive at this point. First, let us consider the natural flat metric on C^n, namely $ds^2 = dz^\mu\, dz^{\bar\nu}\, \delta_{\mu\bar\nu} = |dz^1|^2 + |dz^2|^2 + \cdots + |dz^n|^2$. It is easy to see that if we define the Kähler function

$$K = z^\mu \, z^{\bar\nu} \, \delta_{\mu\bar\nu} \ , \tag{3.69}$$

then substituting into (3.63) we indeed find the desired metric

$$g_{\mu\bar\nu} = \partial_\mu \, \partial_{\bar\nu} \, (z^\rho \, z^{\bar\sigma} \, \delta_{\rho\bar\sigma}) = \delta_{\mu\bar\nu} \ . \tag{3.70}$$

Similarly, the Kähler form, given by (3.64), is

$$J = \mathrm{i} \, \partial \, \bar\partial \, (z^\mu \, z^{\bar\nu} \, \delta_{\mu\bar\nu}) = \mathrm{i} \, dz^\mu \wedge dz^{\bar\nu} \, \delta_{\mu\bar\nu} \ , \tag{3.71}$$

as expected. There is no issue of looking at overlaps between coordinate patches in this case, since this is the one example where a single coordinate patch covers the entire manifold.

For a less trivial example, consider the complex projective spaces CP^n. These are defined as follows. Begin by taking the flat metric on the complex manifold C^{n+1}, with coordinates Z^M for $1 \le M \le n+1$:

$$ds_{n+1}^2 = \sum_{M=1}^{n+1} |dZ^M|^2 \ . \tag{3.72}$$

Now impose the quadratic condition

$$\sum_{M=1}^{n+1} |Z^M|^2 = 1 \ . \tag{3.73}$$

It is evident that both (3.72) and (3.73) are invariant under $SU(n+1)$ transformations, acting by matrix multiplication on Z^M viewed as a column vector. Imposing the constraint (3.73) clearly places the Z^M coordinates on the surface of a unit-radius sphere S^{2n+1}.

Now introduce the so-called inhomogeneous coordinates ζ^i, defined by

$$\zeta^i = Z^i / Z^{n+1} \ , \qquad 1 \le i \le n \ . \tag{3.74}$$

Actually, this is just one choice for the definition, where Z^{n+1} among the original homogeneous coordinates Z^M is singled out for special treatment. We could, and indeed later will, consider a different choice where one of the other Z^M is singled out as the special one.

Proceeding with the choice (3.74) for now, we may now express the Z^i in terms of ζ^i and Z^{n+1} using (3.74), and express $|Z^{n+1}|^2$ in terms of $|\zeta^i|^2$ using (3.73). Substituting into the metric (3.72), we therefore find

$$ds^2 = F^{-1} \, d\zeta^i \, d\bar\zeta^i + \frac{|dZ^{n+1}|^2}{|Z^{n+1}|^2} + (\bar\zeta^i \, d\zeta^i \, Z^{n+1} \, d\bar Z^{n+1} + \zeta^i \, d\bar\zeta^i \, \bar Z^{n+1} \, dZ^{n+1}) \ , \tag{3.75}$$

where

$$F \equiv 1 + \sum_i |\zeta^i|^2 = |Z^{n+1}|^{-2} \ . \tag{3.76}$$

The metric can be re-expressed in the following form, by completing the square in the terms involving dZ^{n+1} and $d\bar{Z}^{n+1}$:

$$ds^2 = \Big|\frac{dZ^{n+1}}{Z^{n+1}} + F^{-1}\,\zeta^i\,d\bar{\zeta}^i\Big|^2 + F^{-1}\,d\zeta^i\,d\bar{\zeta}^i - F^{-2}\,\bar{\zeta}^i\,\zeta^j\,d\zeta^i\,d\bar{\zeta}^j \ . \tag{3.77}$$

If we now parameterise the coordinate Z^{n+1} as $Z^{n+1} = e^{\mathrm{i}\,\psi}\,F^{-1/2}$, we see that the metric becomes

$$ds^2 = (d\psi + A)^2 + F^{-1}\,d\zeta^i\,d\bar{\zeta}^i - F^{-2}\,\bar{\zeta}^i\,\zeta^j\,d\zeta^i\,d\bar{\zeta}^j \ , \tag{3.78}$$

where

$$A = \tfrac{\mathrm{i}}{2}\,F^{-1}\,(\bar{\zeta}^i\,d\zeta^i - \zeta^i\,d\bar{\zeta}^i) \ . \tag{3.79}$$

It will be recalled that (3.78) is still a metric on the unit sphere S^{2n+1}, since we have really done nothing more than reparameterise the metric we had at the beginning of the construction. Now, let us project the metric orthogonally to the orbits of the Killing vector $\partial/\partial\psi$. This is achieved by simply dropping the first term in (3.78), leading to the $(2n)$-dimensional metric

$$ds^2 = F^{-1}\,d\zeta^i\,d\bar{\zeta}^i - F^{-2}\,\bar{\zeta}^i\,\zeta^j\,d\zeta^i\,d\bar{\zeta}^j \ . \tag{3.80}$$

It will be recognised that what we are doing here is really a Kaluza-Klein dimensional reduction from $D = 2n+1$ to $D = 2n$, with ψ being the coordinate on the internal circle, and A the Kaluza-Klein vector. The metric that we have thus obtained in (3.80) is a metric on CP^n, or complex projective n-space. It is know as the Fubini-Study[3] metric on CP^n.

The CP^n manifold is a complex n-manifold. This can be seen from the fact that the complex coordinates ζ^i, defined in (3.74), are valid in the open neighbourhood where $Z^{n+1} \neq 0$. A different open neighbourhood can be covered by sigling out a different one of the $(n+1)$ homogeneous coordinates Z^M, say Z^A, for some specific value of A chosen from the range $1 \le A \le n+1$. Then we can define inhomogeneous coordinates ζ_A^i, valid in the open neighbourhood U_A defined by $Z^A \neq 0$, by

$$\zeta_A^i = Z^i/Z^A \ , \qquad i \neq A \ . \tag{3.81}$$

The construction of CP^n proceeds analogously in the neighbourhood U_A. To see that CP^n is a complex manifold we just have to look at the transition functions relating the

[3]Following in the tradition of mathematicians with misleading names, we may now add Study to the list that includes also Killing and Lie.

coordinates ζ^i_A in region U_A to the coordinates ζ^i_B in region U_B, in their intersection, which comprises all points for which $Z^A \neq 0$ and $Z^B \neq 0$. Then we have:

$$\begin{aligned} \zeta^i_A &= Z^i/Z^A = (Z^i/Z^B)(Z^B/Z^A) \,, \\ &= \zeta^i_B/\zeta^A_B \,. \end{aligned} \tag{3.82}$$

This shows that the complex coordinates of different open neighbourhoods are related holomorphically in their overlap regions, thus establishing that CP^n is a complex manifold.

Having seen that CP^n is a complex manifold, let us now show that it is a Kähler manifold. To do this, let us go back to the specific choice of the open neighbourhood U_{n+1}, for which the inhomogeneous coordinates are given by (3.74). Let K be the function

$$K = \log F \,. \tag{3.83}$$

To adjust our notation to fit better with the previous general discussion of Kähler manfolds, let us change the labelling for the homogeneous coordinates ζ^i to z^μ, so that $F = 1 + z^\mu\, z^{\bar\nu}\, \delta_{\mu\bar\nu}$. If we take K as the Kähler function, then from (3.63) we will have that

$$g_{\mu\bar\nu} = \partial_\mu\, \partial_{\bar\nu}\, K = F^{-1}\, \delta_{\mu\bar\nu} - F^{-2}\, z^{\bar\mu}\, z^\nu \,, \tag{3.84}$$

which is easily seen to be equivalent to the Fubini-Study metric (3.80) on CP^n that we derived previously. The Kähler form $J = \mathrm{i}\, \partial\, \bar\partial\, K$ is likewise easily calculated, and comes out to be

$$J = \mathrm{i}\, F^{-1}\, dz^\mu \wedge dz^{\bar\mu} - \mathrm{i}\, F^{-2}\, z^{\bar\mu}\, z^\nu\, dz^\mu \wedge dz^{\bar\nu} \,. \tag{3.85}$$

Now we are in a position to check how the Kähler form transforms under the change between coordinate systems in overlapping patches. Using (3.82), we see that the Kähler function K_A in region U_A is related in the overlap to the Kähler function K_B in region U_B by

$$\begin{aligned} K_A &= \log\Big(1 + \sum_{i\neq A} |\zeta^i_A|^2\Big) = \log\Big(1 + |\zeta^A_B|^{-2} \sum_{i\neq A} |\zeta^i_B|^2\Big) \\ &= -\log|\zeta^A_B|^2 + \log\Big(1 + \sum_{i\neq B} |\zeta^i_B|^2\Big) \\ &= K_B - \log\zeta^A_B - \log\bar\zeta^A_B \,. \end{aligned} \tag{3.86}$$

Thus, as we saw in general in (3.68), the Kähler function transforms by the addition of purely holomorphic and anti-holomorphic functions under a change of coordinates.

Let us now return to a general Kähler manifold. Recall that we found in the previous subsection that on any Hermitean manifold the uniquely-defined Hermitean connection is given by (3.46) together with its complex conjugate. Thus the Hermitean connection is always pure in its indices. However, in general it has torsion, reflected in the fact that $\Gamma^\mu{}_{\nu\rho}$ can have a part that is antisymmetric in ν and ρ. However, in a Kähler manifold we saw that the Kähler metric can be written in terms of the Kähler function K, as given in (3.63). It is therefore immediately evident, on account of the commutativity of partial derivatives, that the Hermitean connection $\Gamma^\mu{}_{\nu\rho}$ for a Kähler metric is in fact symmetric in ν and ρ. Thus the torsion vanishes, and so in fact the Hermitean connection for the Kähler metric coincides with the usual Christoffel connection. In particular, this means that all the usual additional symmetries of the Riemann tensor for a torsion-free connection hold, namely

$$R_{ijk\ell} = R_{k\ell ij}\ , \qquad R_{i[jk\ell]} = 0\ . \tag{3.87}$$

In terms of holomorphic and anti-holomorphic indices, this means that the Riemann tensor has the symmetries

$$R_{\mu\bar\nu\rho\bar\sigma} = R_{\rho\bar\nu\mu\bar\sigma} = R_{\mu\bar\sigma\rho\bar\nu}\ . \tag{3.88}$$

In other words, it is symmetric in its holomorphic indices, and asymmetric in its anti-holomorphic indices.

It then follows that the Ricci tensor is symmetric, and has only mixed components:

$$R_{\mu\bar\nu} = g^{\rho\bar\sigma}\, R_{\bar\sigma\mu\rho\bar\nu} = -g^{\rho\bar\sigma}\, R_{\mu\bar\nu\rho\bar\sigma}\ . \tag{3.89}$$

Comparing with the expression (3.53) for the Ricci form of an Hermitean manifold, we see that for a Kähler metric, the components $\mathcal{R}_{\mu\bar\nu}$ of the Ricci form are precisely given by the components $-R_{\mu\bar\nu}$ of the Ricci tensor. Of course since the former is antisymmetric, while the latter is symmetric, we have also that $\mathcal{R}_{\bar\nu\mu} = R_{\bar\nu\mu}$.

3.5 The Monge-Ampère Equation

As we saw in (3.53), the Ricci form can be expressed very simply in terms of holomorphic and antiholomorphic derivatives of the metric. Furthermore, in a Kähler manifold we have the metric written very simply in terms of holomorphic and antiholomorphic derivatives of the Kähler function. Suppose now that we wish to find a Kähler solution of the vacuum Einstein equations (in Euclidean signature), i.e. we wish to find a Ricci-flat Kähler metric. Since in a Kähler manifold the Ricci form really is just the Ricci tensor, in that $\mathcal{R}_{\mu\bar\nu} = -R_{\mu\bar\nu}$, it follows from (3.53) that Ricci-flatness means that locally we have

$$\log g = f(z) + \overline{f(z)}\ , \tag{3.90}$$

where g is the determinant of the metric, and f is an arbitrary holomorphic function. Equivalently, we may say that $g = |h(z)|^2$, where $h(z)$ is an arbitrary holomorphic function. Now, under a holomorphic general coordinate transformation, the determinant g will change by a multiplicative Jacobian factor, which itself is the modulus-squared of the holomorphic Jacobian. Thus we may use this coordinate transformation freedom to choose a coordinate frame where we simply have $g = 1$. Now, from (3.63) we therefore find that the condition of Ricci-flatness on a Kähler manifold can be expressed simply as

$$\det\Big(\partial_\mu\, \partial_{\bar\nu}\, K\Big) = 1 \ , \tag{3.91}$$

where the determinant is taken over the μ and $\bar\nu$ indices. This very simple re-expression of the vacuum Einstein equations is a special case of the *Monge-Ampère equation*.

More generally, we can look for Kähler metrics that are not Ricci flat, but whose Ricci tensor is proprtional to the metric; this condition on a metric defines what is known as an *Einstein metric*:

$$R_{ij} = \Lambda\, g_{ij} \ . \tag{3.92}$$

The factor Λ is necessarily constant, as can be seen from the Bianchi identity for the curvature. In physical terms, when the metric signature is Lorentzian, these are solutions of the vacuum Einstein equations with a cosmological constant Λ. For this more general case, the condition $R_{\mu\bar\nu} = \Lambda\, g_{\mu\bar\nu}$ for an Einstein-Kähler metric can be expressed as

$$\partial_\mu\, \partial_{\bar\nu}\, \log g^{1/2} = -\Lambda\, \partial_\mu\, \partial_{\bar\nu}\, K \ , \tag{3.93}$$

and, exploiting the various reparameterisation freedoms as before, we can without loss of generality reduce this to the condition

$$\det\Big(\partial_\mu\, \partial_{\bar\nu}\, K\Big) = e^{-\Lambda\, K} \ . \tag{3.94}$$

This is the general case of the Monge-Ampère equation. It can provide a useful way of solving for Einstein-Kähler metrics.

For example, suppose we make the ansatz that the Kähler function K on a complex n-manifold will depend on the complex coordinates z^μ only through the isotropic quantity $x \equiv \sum_\mu |z^\mu|^2$. This is, of course, a great specialisation, but it does allow one to obtain a rather simple result. Since $\partial_\mu\, x = z^{\bar\mu}$ and $\partial_{\bar\mu}\, x = z^\mu$, we see that

$$\partial_\mu\, \partial_{\bar\nu}\, K(x) = K'\, \delta_{\mu\nu} + K''\, z^{\bar\mu}\, z^\nu \ , \tag{3.95}$$

where $K' = \partial K/\partial x$, *etc.* After a little matrix algebra, it is easy to see that this implies that

$$\det\Big(\partial_\mu\,\partial_{\bar\nu}\,K\Big) = K'^{n-1}\,(K' + x\,K'')\ , \tag{3.96}$$

and consequently the Monge-Ampère equation becomes

$$(K')^{n-1}\,(K' + x\,K'') = e^{\Lambda\,K}\ . \tag{3.97}$$

Thus for this particular isotropic ansatz, the Einstein equation is reduced to an ordinary differential equation for K.

A particular solution to (3.97) can be obtained by taking $K = \log(1 + x)$. Substituting into (3.97), we see that it is satisfied if $\Lambda = n + 1$. Comparing with (3.76) and (3.83), we see that the solution $K = \log(1 + x)$ is nothing but the Kähler function for CP^n. Our calculation has therefore shown that the Fubini-Study metric on CP^n is an Einstein-Kähler metric. An equivalent way to express this is that the Ricci-form is proportional to the Kähler form; in fact, in this CP^n case we have

$$\mathcal{R} = -(n + 1)\,J\ . \tag{3.98}$$

Recall from previously that we saw that the equivalence class (3.56) of all 2-forms related to $\mathcal{R}/(2\pi)$ by the additional of an arbitrary exact 2-form defines the topological class c_1 known as the first Chern class. We have also seen that in a compact manifold M the Kähler form J is topologically non-trivial, since J^n integrates over M to give a non-zero constant. Thus J is closed, but not exact; it is *harmonic*. The expression (3.98) therefore shows that the first Chern class of CP^n is non-trivial. A consequence of this is that it is not possible to find a Ricci-flat metric on CP^n. Of course we have already seen that the Fubini-Study metric is not Ricci flat, but this, in itself, would not rule out the logical possibility that one might find a different metric that was Ricci flat. But since we know that c_1 is non-trivial, that means that we are guaranteed that no metric deformation could take us to a new metric for which the Ricci form vanished, since if it could, this would mean that c_1 would then be zero, contradicting the fact that it is a topological invariant.

Thus we have the result that a *necessary* condition for having a Ricci-flat Kähler metric is that the first Chern class c_1 must vanish. In the 1950's it was conjectured by Calabi that this is the *only* obstruction to the existence of a Ricci-flat Kähler metric on a Kähler manifold. It took until the 1970's before the Calabi conjecture was proved by Yau. The precise statement of Yau's result is the following:

Given a complex manifold M with $c_1 = 0$, and any Kähler metric g_{ij} on M with Kähler form J, then there exists a unique Ricci-flat metric g'_{ij} whose Kähler form J' is in the same cohomology class as J.

Put more plainly, the claim is that one can find a Ricci-flat Kähler metric on any Kähler manifold with vanishing first Chern class. The metric is known as a Calabi-Yau metric. The proof is highly intricate and involved, and essentially consists of an "epsilon and delta" analysis of the Monge-Ampère equation.

3.6 Holonomy and Calabi-Yau Manifolds

An important concept in any manifold with curvature is the notion of *holonomy*. This is the characterisation of the way in which a vector is rotated after being parallely transported around a closed curve, and it is a way in which inhabitants of a curved world can "detect" the curvature. A classic example is the explorer on the earth who, like superman, starts at the north pole and then walks south. At the equator he turns through 90 degrees, walks along it for a while, and then turns a further 90 degrees and returns to the north pole. All the while, he carefully follows the rules of parallel transport for his vector that he carries with him. He finds that it is pointing in a different direction from that of the original vector before he started the trip. In fact, it is rotated through an angle ϕ, where ϕ is the azimuthal angle that he has traversed while marching along the line of latitude. This $SO(2)$ rotation is an element of the *holonomy group* of the manifold S^2. Any rotation angle ϕ can be achieved, by walking the appropriate distance along the equator. Since the manifold in this example is two-dimensional, this in fact means that the most general possible rotation of a vector can be achieved by parallel transport around an appropriate closed curve. More generally, an explorer on an m-sphere would find that he could achieve any desired $SO(m)$ rotation of a vector, by parallely transporting it appropriately. Again, this would be the most general possible rotation that a vector in m dimensions could undergo.

It is not necessary to take such long walks in order to see the holonomy of the manifold. Parallel transprt around a small closed path will also reveal the presence of curvature, although now the rotation will correspondingly be only a small one. But still, on a sphere, for example, one would be able to achieve *any* desired small rotation, by choosing the path appropriately. An infinitesimal closed path can be characterised by an infinitesimal 2-form $d\Sigma_{ij}$, which defines the 2-surface spanning the closed curve. It is a straightforward result from differential geometry that a vector V^i parallely-propagated around this curve will suffer a rotation

$$\delta V^i = V^j\, R^i{}_{jk\ell}\, d\Sigma^{k\ell}\ . \tag{3.99}$$

The fact that it is a pure rotation, with no change in length, is assured by the fact that the Riemann tensor is antisymmetric in its first two indices; $\delta(V^i\, V_i) = 2V_i\, \delta V^i = 2V_i\, V^j\, R^i{}_{jk\ell}\, d\Sigma^{k\ell} = 0$. In fact, we can think of the infinitesimal rotation as being

$$\delta V^i = \Lambda^i{}_j\, V^j\ , \tag{3.100}$$

where $\Lambda_{ij} = -\Lambda_{ji}$ is an infinitesimal generaor of the holonomy group, given by $\Lambda_{ij} = R_{ijk\ell}\, d\Sigma^{k\ell}$.

In a generic manifold, and for these purposes the n-sphere is an example of such, the generators $\Lambda^i{}_j$ fill out the entire set of $SO(m)$ Lie algebra generators, in m dimensions. In fact, for the sphere with its standard unit-radius metric we have

$$R_{ijk\ell} = g_{ik}\, g_{j\ell} - g_{i\ell}\, g_{jk}\ , \tag{3.101}$$

and so we have $\Lambda_{ij} = 2d\Sigma_{ij}$. Thus we indeed see that we can achieve *any* desired infinitesimal Λ_{ij}, by choosing our closed curve appropriately.

A Kähler manifold, however, is *not* a generic manifold. It has, as we have seen, a very special kind of curvature where, in terms of complex components, only the mixed-index components $R_{\mu\bar\nu\rho\bar\sigma}$, and those related by the usual Riemann-tensor symmetries, are non-zero. If we raise the first index, we have that $R^\mu{}_{\nu\rho\bar\sigma} = -R^\mu{}_{\nu\bar\sigma\rho}$ and $R^{\bar\mu}{}_{\bar\nu\rho\bar\sigma} = -R^{\bar\mu}{}_{\bar\nu\bar\sigma\rho}$ can be non-zero, while the components with mixed indices on the first pair must vanish. From the general expression (3.99) for infinitesimal parallel transport, we see that a holomorphic vector V^μ can suffer only holomorphic rotations, while an antiholomorphic vector $V^{\bar\mu}$ can suffer only antiholomorphic ones. In other words, instead of being infinitesimal rotations of the generic $SO(2n)$ holonomy group that one would expect in a generic real $(2n)$-manifold, the rotations here are elements of $U(n)$. Thus the holonomy group of a Kähler metric on a complex n-manifold is $U(n)$. This is, of course, a subgroup of $SO(2n)$.

There is a slight further specialisation of the holonomy group that occurs if the Kähler metric is Ricci flat. It is clear from the form of the rotation of a holomorphic vector,

$$\delta V^\mu = V^\nu\, R^\mu{}_{\nu k\ell}\, d\Sigma^{k\ell} \equiv \Lambda^\mu{}_\nu\, V^\nu\ , \tag{3.102}$$

that the $U(n)$ rotation-group element will have unit determinant if the generator $\Lambda^\mu{}_\nu$ is traceless. But from (3.89), and the symmetries of the Riemann tensor, this is exactly what happens if the Kähler metric is Ricci-flat. Thus we arrive at the conclusion that a Ricci-flat Kähler metric on a complex n-manifold has $SU(n)$ holonomy.

4 Parallel Spinors and Special Holonommy

If a manifold admits paralle spinors, it provides a potentially interesting background geometry for the extra "internal" dimensions in a compactifocation of a higher-dimensional theory such as supergravity, superstring theory or M-theory. The existence of the parallel spionr implies that the background will be supersymmetric.

4.1 Ricci-flat Kähler Metrics

We have seen that in a Kähler manifold the only non-vanishing components of the Riemann tensor, using complex indices, are $R_{\mu\bar{\nu}\rho\bar{\sigma}}$ and those related by the standard symmetries of the Riemann tensor. In particular, the holonomy group (the group of rotations suffered by a vector that is parallel transported around a closed loop) is contained within $U(n) \subset SO(2n)$, where $2n$ is the real dimension of the manifold. These rotations take the form

$$\Delta V^\mu = R^\mu{}_{\nu ij}\, \Delta\Sigma^{ij}\, V^\nu \tag{4.1}$$

and the complex conjugate expression for $V^{\bar{\mu}}$. These $U(n)$ rotations reduce to $SU(n)$ rotations if $R^\mu{}_{\nu ij}$ is trace-free in μ and ν; i.e. if the mateic is Kähler and Ricci flat.

If the metric is Ricci flat and Kähler then there exists a pair of spinors that are *covariantly constant*. That is, such a spinor η satisfies

$$\nabla_i \eta = 0\,, \tag{4.2}$$

where ∇_i is the standard covariant derivative acting on spinors:

$$\nabla_i \eta = \partial_i \eta + \tfrac{1}{4}\omega_i^{ab}\, \Gamma_{ab}\eta\,. \tag{4.3}$$

Here ωab is the spin connection, Γ_a are the Dirac matrices, and $\Gamma_{ab} \equiv \frac{1}{2}[\Gamma_a, \Gamma_b]$. In the mathematics literature, a spinor satisfying the covariant constancy condition (4.2) is known as a *parallel spinor*.

One way to see the existence of the parallel spinors is by looking at how the basic spinor representations of the $SO(2n)$ tangent-space group of the manifold decompose under the $SU(n)$ holonomy subgroup of the Ricci-flat Kähler metric. There is a difference between the cases where n is odd and where n is even. For n is odd, such as $n = 3$, the right-handed and left-handed spinors decompose as:

$$SO(6) \to SU(3): \qquad 4 \to 3 + 1\,, \quad \bar{4} \to \bar{3} + 1\,. \tag{4.4}$$

When n is even, such as $n = 4$, we have

$$SO(8) \to SU(4): \qquad 8_R \to 6 + 1 + 1\,, \quad 8_L \to 4 + \bar{4}\,. \tag{4.5}$$

The decompositions go in a similar kind of way for all n. Namely, when n is odd there are two singlets under $SU(n)$, one of each chirality, corresponding to one right-handed paralllel spinor and one -left-handed parallel spinor. When n is even, there are again two singlets, but this time of the same chirality. Thus there are again two parallel spinors, which now have the same chirality.

The relation between the existence of covariantly-constant spinors and singlets in the decomppposition of the spinor reps of the tangent-space group follows by considering the integrability condition for the equation (4.2). We have

$${\scriptstyle (0)} = [\nabla_i, \nabla_j]\eta = \tfrac{1}{4} R_{k\ell ij}\, \Gamma^{k\ell} \eta\,. \tag{4.6}$$

Now, the parallel transport of a spinor ψ around a closed loop transforms it according to

$$\Delta\psi = \tfrac{1}{4} R_{k\ell ij} \Delta\Sigma^{ij}\, \Gamma^{k\ell}\, \psi\,, \tag{4.7}$$

and so if a spinor η is a singlet under the holonomy group it follows that

$$R_{k\ell ij}\, \Gamma^{k\ell} \eta = 0\,, \tag{4.8}$$

and so by the integrability condition (4.6) it must satisfy $[\nabla_i, \nabla_j]\eta = 0$. As its name implies, if this integrability condition is satisfied then one can integrate up and obtain a solution to the equation $\nabla_i \eta = 0$. Thus, for every singlet in the decomposition of the spinor representations of the tangent-space group under the holonomy group, there exists a parallel spinor.

In all the cases, whether n is odd or even, we can group the two parallel spinors together into a single complex spinor. For example, in the case $n = 4$ the two (real) right-handed parallel spinors can be combined to make a single complex right-handed spinor. In the case $n = 3$, the left-handed and right-handed parallel spinors togther comprise an 8-component spinor with chiral and anti-chiral parts. We shall typically just denote the composite spinor by η in what follows. Since it follows from (4.2) that $\bar{\eta}\eta$ is constant, we may normalise η so that

$$\bar{\eta}\eta = 1\,. \tag{4.9}$$

The covariantly-constant spinors are extremely useful for a variety of constructions. For example, consider the quantity

$$\mathrm{i}\, \bar{\eta}\Gamma_{ij}\eta\,. \tag{4.10}$$

Clearly, it follows from (4.2) that this 2-index tensor is covariantly constant. It can easily be seen, using Fierz identities, that it squares to minus one, and in fact we have

$$J_{ij} = -\mathrm{i}\, \bar\eta \Gamma_{ij} \eta\,, \tag{4.11}$$

where J_{ij} is the Kähler form.

The Dirac matrices obey the Clifford algebra

$$\{\Gamma^i, \Gamma^j\} = 2g^{ij}\,. \tag{4.12}$$

In the case of a Kähler metric we have, as we have seen, that $g^{\mu\nu} = 0 = g^{\bar\mu\bar\nu}$, and so

$$\{\Gamma^\mu, \Gamma^\nu\} = 0\,, \qquad \{\Gamma^{\bar\mu}, \Gamma^{\bar\nu}\} = 0\,, \qquad \{\Gamma^\mu, \Gamma^{\bar\nu}\} = 2g^{\mu\bar\nu}\,. \tag{4.13}$$

This is the same as the algebra of fermionic creation and annihilation operators. Since we have $J_{\mu\bar\nu} = \mathrm{i} g_{\mu\bar\nu}$ it follows from (4.11) that

$$\mathrm{i} g_{\mu\bar\nu} = -\mathrm{i}\, \bar\eta \Gamma_{\mu\bar\nu} \eta = -\mathrm{i}\, \bar\eta\, (\Gamma_\mu \Gamma_{\bar\nu} - g_{\mu\bar\nu}) \eta\,, \tag{4.14}$$

and hence $\bar\eta \Gamma_\mu \Gamma_{\bar\nu} \eta = 0$. We can write this as

$$(\Gamma_{\bar\mu} \eta)^\dagger\, (\Gamma_{\bar\nu} \eta) = 0\,, \tag{4.15}$$

and hence by taking $\bar\mu$ equal to $\bar\nu$ we conclude that

$$\Gamma_{\bar\mu} \eta = 0\,, \qquad \Rightarrow \qquad \Gamma^\mu \eta = 0\,. \tag{4.16}$$

Thus η is like the "highest-weight state" in the algebra of fermionic creation and annihilation operators.

We can also form the n-index antisymmetric tensor

$$\Omega_{i_1 \cdots i_n} = \mathrm{i}\, \bar\eta \Gamma_{i_1 \cdots i_n} \eta\,. \tag{4.17}$$

This too is clearly covariantly constant. It can be seen from this expression and (4.11), after using some Fierz rearranging, that

$$J_i{}^{j_1}\, \Omega_{j_1 \cdots j_n} = \mathrm{i}\, \Omega_{i j_2 \cdots j_n}\,, \tag{4.18}$$

(and analogously, of course, if the J is hooked onto *any* index), and hence that Ω is an holormorphic $(n, 0)$ form. It can in fact be written as the form

$$\Omega = \frac{1}{n!}\, \epsilon_{\mu_1 \cdots \mu_n}\, dz^{\mu_1} \wedge \cdots \wedge dz^{\mu_n}\,, \tag{4.19}$$

where $\epsilon_{\mu_1\cdots\mu_n}$ is the totally-antisymmetric invariant holormorphic n-index tensor.

We remark, in closing our discussion of Ricci-flat Kähler metrics, by remarking that one can show that the conditions for a Hermitean manifold with Hermitean metric g_{ij} and complex structure tensor $J_i{}^j$ to be Kähler and Ricci-flat can be simply stated as

$$dJ = 0 \qquad \text{and} \qquad d\Omega = 0\,. \tag{4.20}$$

4.2 Berger Classification of Special Holonomy

We have seen that a real manifold of dimension $2n$ equipped with a Kähler metric has the special holonomy $U(n) \subset SO(2n)$. If in addition the metric is Ricci flat, then the holonomy is reduced further to $SU(n)$. It therefore becomes of interest to ask what are the special holonomies that can arise in general. This question was answered by the mathematician Berger, and his list of possible irreducible holonomies is as follows:

Dim(M)	Holonomy	Type of manifold
n	$SO(n)$	Orientable manifold
$2n$	$U(n)$	Kähler manifold
$2n$	$SU(n)$	Ricci-flat Kähler
$4n$	$Sp(n)\cdot Sp(1)$	Quaternionic Kähler manifold
$4n$	$Sp(n)$	Hyperkähler manifold
7	G_2	G_2 manifold
8	$Spin(7)$	$Spin(7)$ manifold

Table 1. *Berger's list of irreducible manifolds of special holonomy.*

The term "irreducible" above serves the purpose of excluding "trivial" possibilities like the direct product of irreducible cases mentioned above. The first case, with $SO(n)$ holonomy, is really just the general case with no reduction in holonomy at all. (Except for the assumption of orientability, which reduces $O(n)$ to $SO(n)$.)

The group $Sp(n)$ denotes the group of $n\times n$ quaternionic unitary matrices, and is often written as $USp(2n)$, which is the intersection $U(2n)\cap Sp(2n,C)$. Low-dimensional isomorphisms are $Sp(1) = SU(2)$ and $Sp(2) = SO(5)$ (locally, at least).

The dot in $Sp(n)\dot{S}p(1)$ is $Sp(n)\times Sp(1)/Z_2$. The quaternionic Kähler manifolds are like quaternionic analogues of the complex Kähler manifolds. They are not (in general)

Ricci flat. The hyperkähler specialisation, where the $Sp(1)$ factor in the holonomy group is absent, is the analogue of the Ricci-flat Kähler specialisation of Kähler metrics, where the $U(1)$ factor in $U(n) = SU(n) \times U(1)$ (locally) is absent.

The last two cases in the table above, G_2 and $Spin(7)$, are isolated examples, in that unlike the previous sequences that exist for all even dimensions or multiples of 4, they occur only in the dimensions 7 and 8 respectively. They are known as *exceptional holonomy* manifolds for this reason.

All of the special-holonomy cases in the table below are necessarily Ricci flat, and they all admit parallel spinors (right-handed,left-handed), as indicated:

Dim(M)	Holonomy	Type of manifold	Number of parallel spinors
$4n$	$SU(2n)$	Ricci-flat Kähler	$(2,0)$
$4n+2$	$SU(n+1)$	Ricci-flat Kähler	$(1,1)$
$4n$	$Sp(n)$	Hyperkähler manifold	$(n+1,0)$
7	G_2	G_2 manifold	1
8	$Spin(7)$	$Spin(7)$ manifold	$(1,0)$

Table 2. *Manifolds of special holonomy admitting parallel spinors.*

Of course in the G_2 manifold, being seven dimensional, there is no spinorial chirality.

As in the Ricci-flat Kähler case we discussed previously, we can again see how the parallel spinors arise in these other examples, by looking for singlets in the decomposition of the spinor representations of the tangent-space group under the holonomy subgroups. Let us focus here on the two exceptional cases, of G_2 holonomy in seven dimensions, and $Spin(7)$ holonomy in eight dimensions. The way the spinors decompose in these cases are as follows:

$$\begin{aligned} SO(7) \to G_2 : &\qquad 8 \to 7+1\,, \\ SO(8) \to Spin(7) : &\qquad 8_R \to 7+1\,, \qquad 8_L \to 8\,. \end{aligned} \tag{4.21}$$

Thus we see there will be one (real, i.e. Majorana) parallel spinor in the G_2 holonomy manifold, and one real and chiral parallel spinor in the $Spin(7)$ holonomy manifold.

The existence of any parallel spinor autmatically implies that the metric must be Ricci flat. To see this, we take the integrability condition

$$R_{k\ell ij}\,\Gamma^{k\ell}\eta = 0 \tag{4.22}$$

and multiply on the left by Γ^i. Using the Dirac algebra result that

$$\Gamma^i\,\Gamma^{k\ell} = \Gamma^{ik\ell} + g^{ik}\,\Gamma^\ell - g^{i\ell}\,\Gamma^k\,, \tag{4.23}$$

where $\Gamma^{ik\ell} = \Gamma^{[i}\Gamma^k\Gamma^{\ell]}$, and noting that $\Gamma^{ik\ell}\,R_{k\ell ij}$ by virtue of the cyclic identity $R_{[k\ell i]j} = 0$ of the Riemann tensor, we see that

$$R_{ij}\,\Gamma^j\eta = 0\,. \tag{4.24}$$

From this it follows that we must have $R_{ij} = 0$. Thus all the manifolds of special holonomy that are listed in table 2 above have Ricci-flat metrics.

4.3 Manifolds of G_2 holonomy

Here, we look in a bit more detail at the case of G_2 holonomy manifolds. Using the Majorana parallel spinor η, which we normalise so that $\bar\eta\eta = 1$, we can construct the 3-index antisymmetric tensor

$$\Phi_{ijk} = \mathrm{i}\,\bar\eta\Gamma_{ijk}\eta\,, \tag{4.25}$$

which is, of course, covariantly constant since $\nabla_i\eta = 0$. (Note that we cannot make a 2-index tensor $\bar\eta\Gamma_{ij}\eta$ here, because η is a Majorana spinor, and in the Majorana basis the Γ_{ij} matrices are antisymmetric in their two spinor indices.) Using Fierz identities, one can easily establish that Φ satisfies the identity

$$\Phi_{ijm}\,\Phi^{k\ell m} = \delta_i^k\delta_j^\ell - \delta_j^k\delta_i^\ell - \Phi_{ij}{}^{k\ell}\,, \tag{4.26}$$

where

$$\Phi^{ijk\ell} \equiv \tfrac{1}{6}\epsilon^{ijk\ell mnp}\,\Phi_{mnp} = \bar\eta\Gamma^{ijk\ell}\eta\,. \tag{4.27}$$

The antisymmetric tensor Φ_{ijk} defines a 3-form

$$\Phi_{(3)} = \tfrac{1}{6}\Phi_{ijk}dx^i\wedge dx^j\wedge dx^k \tag{4.28}$$

that is known as the *associative 3-form* of the G_2 manifold. If one calculates its components Φ_{abc} in a vielbein basis, i.e.

$$\Phi_{abc} = e_a^i\,e_b^j\,e_c^k\,\Phi_{ijk}\,, \tag{4.29}$$

then it actually gives the multiplication table of the 7 imaginary unit octonions o_a:

$$o_a\,o_b = -\delta_{ab} + \Phi_{abc}\,o_c\,. \tag{4.30}$$

For example, in a suitable basis one finds that the non-vanishing components of Φ_{abc} are given by

$$\Phi_{123} = 1\,, \quad \Phi_{147} = \Phi_{257} = \Phi_{367} = \Phi_{156} = \Phi_{264} = \Phi_{345} = -1\,. \tag{4.31}$$

(Together with those following from the antisymmetry of Φ_{abc}.)

It can be shown that the definition of a manifold of G_2 holonomy that we gave above (i.e. a 7-manifold admitting a covariantly-constant Majorana spinor) is completely eqivalent to the statement that there is an associative 3-form $\Phi_{(3)}$ satisfying

$$d\Phi_{(3)} = 0\,, \qquad d{*\Phi_{(3)}} = 0\,, \tag{4.32}$$

where $*$ is the seven-dimensional Hodge dual. (Equivalently, the second equation here can be written as $d\Phi_{(4)} = 0$, where $\Phi_{(4)} = \frac{1}{4!}\,\Phi_{ijk\ell}\,dx^i \wedge dx^j \wedge dx^k \wedge dx^\ell$.)

4.4 Manifolds with $Spin(7)$ holonomy

An eight-dimensional manifold of $Spin(7)$ holonomy may be defined as one that admits a covariantly-constant Majorana-Weyl spinor η. As in the previous cases, we may normalise η so that $\bar\eta\eta = 1$. Because η is chiral, any antisymmetric tensor $\bar\eta\Gamma_{i_1\cdots i_p}\eta$ with an odd number of indices will be identically zero. Also, because η is Majorana, the 2-index tensor and 6-ndex tensor vanish also, since Γ_{ij} and $\Gamma_{ijk\ell mn}$ are antisymmetric in a Majorana basis. The only non-trivial tensor that can be built from η is the 4-index one

$$\Psi_{ijk\ell} = \bar\eta\Gamma_{ijk\ell}\eta\,. \tag{4.33}$$

This is, of course, covariantly constant. It defines a 4-form

$$\Psi_{(4)} = \tfrac{1}{4!}\,\Psi_{ijk\ell}\,dx^i \wedge dx^j \wedge dx^k \wedge dx^\ell\,, \tag{4.34}$$

which is known as the *calibrating 4-form* on the $Spin(7)$ manifold.

There are various algebraic identities satisfied by contracted products of Ψ tensors, all of which can be straightforwardly derived by means of Fierz identities.