USEFUL FORMULAE IN DIFFERENTIAL GEOMETRY

Differential forms:

$$\alpha = \frac{1}{p!}\alpha_{\mu_1\ldots\mu_p} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}; \qquad \alpha \in \wedge^p. \tag{1}$$

$$\alpha \wedge \beta = (-)^{pq} \beta \wedge \alpha; \qquad \alpha \in \wedge^p, \quad \beta \in \wedge^q. \tag{2}$$

Exterior derivative, d:

$$d\alpha \equiv \frac{1}{p!}\, \partial_{[\nu} \alpha_{\mu_1\ldots\mu_p]}\, dx^v \wedge dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}. \tag{3}$$

d maps p-forms to $(p+1)$-forms:

$$d : \wedge^p \longrightarrow \wedge^{p+1}; \qquad d^2 = 0. \tag{4}$$

Defining the components of $d\alpha$, $(d\alpha)_{\mu_1\ldots\mu_{p+1}}$, by

$$d\alpha \equiv \frac{1}{(p+1)!}\, (d\alpha)_{\mu_1\ldots\mu_{p+1}}\, dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_{p+1}}, \tag{5}$$

we have

$$(d\alpha)_{\mu_1\ldots\mu_{p+1}} = (p+1)\partial_{[\mu_1}\, \alpha_{\mu_2\ldots\mu_{p+1}]}, \tag{6}$$

where

$$T_{[\mu_1\ldots\mu_q]} \equiv \frac{1}{q!}\Big(T_{\mu_1\ldots\mu_q} + \text{even perms} - \text{odd perms}\Big). \tag{7}$$

Leibnitz rule:

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-)^p \alpha \wedge d\beta, \qquad \alpha \in \wedge^p, \quad \beta \in \wedge^q. \tag{8}$$

Stokes' Theorem:

$$\int_M d\omega = \int_{\partial M} \omega, \tag{9}$$

where M is an n-manifold and $\omega \in \wedge^{n-1}$.

Epsilon tensors and densities:

$$\varepsilon_{\mu_1\ldots\mu_n} \equiv (+1, -1, 0) \tag{10}$$

if $\mu_1 \ldots \mu_n$ is an (even, odd, no) permutation of a lexical ordering of indices $(1\ldots n)$. It is a tensor density of weight $+1$. We may also define the quantity $\varepsilon^{\mu_1\cdots\mu_n}$, with components given numerically by

$$\varepsilon^{\mu_1\cdots\mu_n} \equiv (-1)^t\, \varepsilon_{\mu_1\cdots\mu_n},$$

where t is the number of timelike coordinates. NOTE: This is the *only* quantity where we do not raise and lower indices using the metric tensor. $\varepsilon^{\mu_1\cdots\mu_n}$ is a tensor density of weight -1. We define epsilon *tensors*:

$$\epsilon_{\mu_1\ldots\mu_n} = \sqrt{|g|}\,\varepsilon_{\mu_1\ldots\mu_n}, \qquad \epsilon^{\mu_1\ldots\mu_n} = \frac{1}{\sqrt{|g|}}\,\varepsilon^{\mu_1\ldots\mu_n}, \tag{11}$$

where $g \equiv \det(g_{\mu\nu})$ is the determinant of the metric tensor $g_{\mu\nu}$. Note that the tensor $\epsilon^{\mu_1\cdots\mu_n}$ *is* obtained from $\epsilon_{\mu_1\ldots\mu_n}$ by raising the indices using inverse metrics.

Epsilon-tensor identities:

$$\epsilon_{\mu_1\ldots\mu_n}\epsilon^{\nu_1\ldots\nu_n} = (-1)^t\, n!\, \delta^{\nu_1\ldots\nu_n}_{\mu_1\ldots\mu_n}\ . \tag{12a}$$

From this, contractions of indices lead to the special cases

$$\epsilon_{\mu_1\ldots\mu_r\mu_{r+1}\ldots\mu_n}\epsilon^{\mu_1\ldots\mu_r\nu_{r+1}\ldots\nu_n} = (-1)^t\, r!(n-r)!\, \delta^{\nu_{r+1}\ldots\nu_n}_{\mu_{r+1}\ldots\mu_n}\ , \tag{12b}$$

where again t denotes the number of timelike coordinates. The multi-index delta-functions have unit strength, and are defined by

$$\delta^{\nu_1\cdots\nu_p}_{\mu_1\cdots\mu_p} \equiv \delta^{[\nu_1}_{[\mu_1}\cdots\delta^{\nu_p]}_{\mu_p]}\ . \tag{13}$$

(Note that only one set of square brackets is actually needed here; but with our "unit-strength" normalisation convention (7), the second antisymmetrisation is harmless.) It is worth pointing out that a common occurrence of the multi-ndex delta-function is in an expression like $B_{\nu_1}\, A_{\nu_2\cdots\nu_p}\, \delta^{\nu_1\cdots\nu_p}_{\mu_1\cdots\mu_p}$, where $A_{\nu_2\cdots\nu_p}$ is totally antisymmetric in its $(p-1)$ indices. It is easy to see that this can be written out as the p terms

$$B_{\nu_1}\, A_{\nu_2\cdots\nu_p}\, \delta^{\nu_1\cdots\nu_p}_{\mu_1\cdots\mu_p} = \frac{1}{p}\Big(B_{\mu_1}\, A_{\mu_2\cdots\mu_p} + B_{\mu_2}\, A_{\mu_3\cdots\mu_p\mu_1} + B_{\mu_3}\, A_{\mu_4\cdots\mu_p\mu_1\mu_2} + \cdots + B_{\mu_p}\, A_{\mu_1\cdots\mu_{p-1}}\Big)$$

if p is odd. If instead p is even, the signs alternate and

$$B_{\nu_1}\, A_{\nu_2\cdots\nu_p}\, \delta^{\nu_1\cdots\nu_p}_{\mu_1\cdots\mu_p} = \frac{1}{p}\Big(B_{\mu_1}\, A_{\mu_2\cdots\mu_p} - B_{\mu_2}\, A_{\mu_3\cdots\mu_p\mu_1} + B_{\mu_3}\, A_{\mu_4\cdots\mu_p\mu_1\mu_2} - \cdots - B_{\mu_p}\, A_{\mu_1\cdots\mu_{p-1}}\Big)\ .$$

Hodge $*$ operator:

$$*(dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_p}) \equiv \frac{1}{(n-p)!}\,\epsilon_{\nu_1\ldots\nu_{n-p}}{}^{\mu_1\ldots\mu_p}\, dx^{\nu_1}\wedge\cdots\wedge dx^{\nu_{n-p}}. \tag{14}$$

The Hodge $*$, or dual, is thus a map from p-forms to $(n-p)$-forms:

$$*: \qquad \wedge^p \longrightarrow \wedge^{n-p}. \tag{15}$$

Note in particular that taking $p=0$ in (14) gives

$$*1 = \epsilon = \frac{1}{n!}\epsilon_{\mu_1\ldots\mu_n} dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_n} = \sqrt{|g|}\, dx^1\wedge\cdots\wedge dx^n. \tag{16}$$

This is the general-coordinate-invariant volume element $\sqrt{|g|}\, d^n x$ of Riemannian geometry. It should be emphasised that conversely, we have

$$dx^{\mu_1} \wedge dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_n} = (-1)^t\, \varepsilon^{\mu_1\mu_2\cdots\mu_n}\, d^n x = (-1)^t\, \epsilon^{\mu_1\mu_2\cdots\mu_n}\, \sqrt{|g|}\, d^n x\ .$$

This extra $(-1)^t$ factor is tiresome, but unavoidable if we want our definitions to be such that $*1$ is always the *positive* volume element.

From these definitions it follows that

$$*\alpha \wedge \beta = \frac{1}{p!}\, |\alpha \cdot \beta|\, \epsilon, \tag{17}$$

where α and β are p-forms and

$$|\alpha \cdot \beta| \equiv \alpha_{\mu_1 \cdots \mu_p} \beta^{\mu_1 \cdots \mu_p}. \tag{18}$$

Applying $*$ twice, we get

$$* * \omega = (-)^{p(n-p)+t}\, \omega, \qquad \omega \in \wedge^p. \tag{19}$$

In even dimensions, $n = 2m$, m-forms can be eigenstates of $*$, and hence can be self-dual or anti-self-dual, in cases where $** = +1$. From (19), we see that this occurs when m is even if t is even, and when m is odd if t is odd. In particular, we can have real self-duality and anti-self-duality in $n = 4k$ Euclidean-signature dimensions, and in $n = 4k + 2$ Lorentzian-signature dimensions.

Adjoint operator, δ:

First define the inner product

$$(\alpha, \beta) \equiv \int_M *\alpha \wedge \beta = \frac{1}{p!} \int_M |\alpha \cdot \beta| \epsilon = (\beta, \alpha), \tag{20}$$

where α and β are p-forms. Then δ, the adjoint of the exterior derivative d, is defined by

$$(\alpha, d\beta) \equiv (\delta\alpha, \beta), \tag{21}$$

where α is an arbitrary p-form and β is an arbitrary $(p-1)$-form. Hence

$$\delta\alpha = (-)^{np+t} {*d*}\alpha, \qquad \alpha \in \wedge^p. \tag{22}$$

(We assume that the boundary term arising from the integration by parts gives zero, either because M has no boundary, or because appropriate fall-off conditions are imposed on the fields.)

δ is a map from p-forms to $(p-1)$-forms:

$$\delta: \qquad \wedge^p \to \wedge^{p-1}; \qquad \delta^2 = 0. \tag{23}$$

Note that in Euclidean signature spaces, δ on p-forms is given by

$$\begin{aligned} \delta\alpha &= *d*\alpha \qquad \text{if at least one of } n \text{ and } p \text{ even}, \\ \delta\alpha &= -*d*\alpha, \qquad \text{if } n \text{ and } p \text{ both odd}. \end{aligned} \tag{24}$$

The signs are reversed in Lorentzian spacetimes.

In terms of components, the above definitions imply that for all spacetime signatures, we have

$$\delta\alpha = -\frac{1}{(p-1)!}(\nabla_\nu\, \alpha^\nu{}_{\mu_1\cdots\mu_{p-1}})\, dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_{p-1}}, \tag{25}$$

where

$$\nabla_\nu\, \alpha^{\nu\mu_1\cdots\mu_{p-1}} \equiv \frac{1}{\sqrt{g}}\partial_\nu\Big(\sqrt{g}\alpha^{\nu\mu_1\cdots\mu_{p-1}}\Big) \tag{26}$$

is the covariant divergence of α. Defining the components of $\delta\alpha$, $(\delta\alpha)_{\mu_1\cdots\mu_{p-1}}$, by

$$\delta\alpha \equiv \frac{1}{(p-1)!}(\delta\alpha)_{\mu_1\cdots\mu_{p-1}} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_{p-1}}, \tag{27}$$

we have

$$(\delta\alpha)_{\mu_1\cdots\mu_{p-1}} = -\nabla_\nu \alpha^\nu{}_{\mu_1\cdots\mu_{p-1}}. \tag{28}$$

Hodge-de Rham operator:

$$\Delta \equiv d\delta + \delta d = (d+\delta)^2. \tag{29}$$

Δ maps p-forms to p-forms:

$$\Delta: \qquad \wedge^p \to \wedge^p. \tag{30}$$

On 0-, 1-, and 2-forms, we have

$$\begin{aligned} &\text{0-forms:} \quad \Delta\phi = -\nabla_\lambda \nabla^\lambda \phi, \\ &\text{1-forms:} \quad \Delta\omega_\mu = -\nabla_\lambda \nabla^\lambda \omega_\mu + R_\mu{}^\nu \omega_\nu, \\ &\text{2-forms:} \quad \Delta\omega_{\mu\nu} = -\nabla_\lambda \nabla^\lambda \omega_{\mu\nu} - 2R_{\mu\rho\nu\sigma}\omega^{\rho\sigma} + R_\mu{}^\sigma \omega_{\sigma\nu} - R_\nu{}^\sigma \omega_{\sigma\mu}, \end{aligned} \tag{31}$$

where $R_{\mu\nu\rho\sigma}$ is the Riemann tensor and

$$R_{\mu\nu} \equiv R^\rho{}_{\mu\rho\nu} \tag{32}$$

is the Ricci tensor.

Hodge's theorem:

We can uniquely decompose an arbitrary p form ω as

$$\omega = d\alpha + \delta\beta + \omega_H, \tag{33}$$

where $\alpha \in \wedge^{p-1}$, $\beta \in \wedge^{p+1}$ and ω_H is harmonic, $\Delta\omega_H = 0$.

RIEMANNIAN GEOMETRY

For a metric $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$, we define a *vielbein* e_μ^a as a "square root" of $g_{\mu\nu}$:

$$g_{\mu\nu} = e_\mu^a\, e_\nu^b\, \eta_{ab}, \tag{34}$$

where η_{ab} is a local Lorentz metric. Usually, we work with positive-definite metric signature, so $\eta_{ab} = \delta_{ab}$. The inverse vielbein, which we denote by E_a^μ, satisfies

$$E_a^\mu\, e_\nu^a = \delta_\nu^\mu; \qquad E_a^\mu\, e_\mu^b = \delta_a^b. \tag{35}$$

The "solder forms" $e^a = e_\mu^a\, dx^\mu$ give an orthonormal basis for the cotangent space. Similarly, the vector fields $E_a^\mu\, \partial_\mu$ give an orthonormal basis for the tangent space.

Torsion and curvature

We define the *spin connection* $\omega^a{}_b = \omega_{\mu b}^a dx^\mu$, the *torsion* 2-form T^a and the *curvature* 2-form $\Theta^a{}_b$ by

$$T^a \equiv \tfrac{1}{2} T_{\mu\nu}^a dx^\mu \wedge dx^\nu = de^a + \omega^a{}_b \wedge e^b, \tag{36}$$

$$\Theta^a{}_b \equiv \tfrac{1}{2} R^a{}_{b\mu\nu} dx^\mu \wedge dx^\nu = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b. \tag{37}$$

Define a Lorentz-covariant and general-coordinate covariant derivative D_μ that acts on tensors with coordinate and Lorentz indices:

$$D_\mu\, V_{\rho b}^{\nu a} = \nabla_\mu\, V_{\rho b}^{\nu a} + \omega_{\mu c}^a V_{\rho b}^{\nu c} - \omega_{\mu b}^c V_{\rho c}^{\nu a}, \tag{38}$$

where ∇_μ is the usual general-coordinate covariant derivative:

$$\nabla_\mu\, V_\rho^\nu = \partial_\mu\, V_\rho^\nu + \Gamma_{\mu\sigma}^\nu V_\rho^\sigma - \Gamma_{\mu\rho}^\sigma V_\sigma^\nu, \tag{39}$$

and $\Gamma_{\nu\rho}^\mu$ is the Christoffel connection. Demanding *metricity* for $g_{\mu\nu}$, *i.e.* $D_\mu\, g_{\nu\rho} = 0$, implies

$$\Gamma_{\nu\rho}^\mu = \tfrac{1}{2} g^{\mu\sigma} \Big(\partial_\nu\, g_{\sigma\rho} + \partial_\rho\, g_{\nu\sigma} - \partial_\sigma\, g_{\nu\rho} \Big). \tag{40}$$

Demanding metricity for η_{ab}, *i.e.* $D_\mu\, \eta_{ab} = 0$, implies

$$\omega_{ab} = -\omega_{ba}, \tag{41}$$

where $\omega_{ab} \equiv \eta_{ac} \omega^c{}_b$.

Bianchi Identities

Taking exterior derivative of (36) and (37) gives

$$D\, T^a \equiv d\, T^a + \omega^a{}_b \wedge T^b = \Theta^a{}_b \wedge e^b, \tag{42}$$

$$D\, \Theta^a{}_b \equiv d\, \Theta^a{}_b + \omega^a{}_c \wedge \Theta^c{}_b - \Theta^a{}_c \wedge \omega^c{}_b = 0. \tag{43}$$

In general, on Lorentz-valued p forms such as $\alpha^a{}_b$, we define the Lorentz-covariant exterior derivative by

$$D\,\alpha^a{}_b \equiv d\,\alpha^a{}_b + \omega^a{}_c \wedge \alpha^c{}_b - \omega^c{}_b \wedge \alpha^a{}_c. \tag{44}$$

Torsion-free metric connection

With the metricity assumption, implying (41), and the assumption that the torsion vanishes, it follows that $\omega^a{}_b$ is then uniquely determined by (36) and (41);

$$d\,e^a = -\omega^a{}_b \wedge e^b; \qquad \omega_{ab} = -\omega_{ba}. \tag{45}$$

Defining $c_{ab}{}^c = -c_{ba}{}^c$ by

$$d\,e^a = -\tfrac{1}{2} c_{bc}{}^a\, e^b \wedge e^c, \tag{46}$$

it follows that ω_{ab} is given by

$$\omega_{ab} = \tfrac{1}{2}(c_{abc} + c_{acb} - c_{bca})e^c. \tag{47}$$

Note that the vielbein is constant with respect to the Lorentz- and general-coordinate covariant derivative defined by (38); $D_\mu\, e^a_\nu = 0$.

Symmetries of the Riemann tensor

It follows from its definition as 2-form (37) that it is always antisymmetric on the final index pair:

$$R_{ab\mu\nu} = -R_{ab\nu\mu}; \qquad R_{abcd} = -R_{abdc}, \tag{48}$$

where we can always freely convert coordinates indices to Lorentz indices, and *vice versa*, using the vielbein. Thus $R_{abcd} = E^\mu_c E^\nu_d R_{ab\mu\nu}$ and conversely $R_{ab\mu\nu} = e^c_\mu e^d_\nu R_{abcd}$. The metricity condition $D_\mu \eta_{ab} = 0$ implies $\omega_{ab} = -\omega_{ba}$, and hence $\Theta_{ab} = -\Theta_{ba}$. Thus

$$R_{abcd} = -R_{bacd}. \qquad\qquad \text{Metricity} \tag{49}$$

The torsion-free condition, using (42), implies that

$$R_{a[bcd]} = 0, \qquad\qquad \text{Torsion-free} \tag{50}$$

where $R_{a[bcd]} = \frac{1}{3}(R_{abcd} + R_{acdb} + R_{adbc})$. Together, (48), (49) and (50) imply

$$R_{abcd} = R_{cdab}. \tag{51}$$

The Ricci tensor and scalar, and Weyl tensor

We define the Ricci tensor R_{ab} and Ricci scalar R by

$$R_{ab} \equiv R^c{}_{acb}; \qquad R \equiv R_{ab}\eta^{ab}. \tag{52}$$

Note that (51) implies that the Ricci tensor is symmetric, $R_{ab} = R_{ba}$.

The Weyl tensor C_{abcd} is defined in n dimensions by

$$\begin{aligned} C_{abcd} \equiv R_{abcd} - \frac{1}{n-2}\big(R_{ac}\eta_{bd} - R_{ad}\eta_{bc} + R_{bd}\eta_{ac} - R_{bc}\eta_{ad}\big) \\ + \frac{1}{(n-1)(n-2)} R\big(\eta_{ac}\eta_{bd} - \eta_{ad}\eta_{bc}\big). \end{aligned} \tag{53}$$

It is the "traceless" part of the Riemann tensor, in the sense that $C^c{}_{acb} \equiv 0$. It has the same symmetries (48)-(51) as the Riemann tensor for torsion-free connection. One may define the Weyl 2-form Ω_{ab},

$$\begin{aligned} \Omega_{ab} &\equiv \tfrac{1}{2} C_{abcd} e^c \wedge e^d \\ &= \Theta_{ab} - \frac{1}{n-2}\big(R_{ac}\eta_{bd} - R_{bc}\eta_{ad}\big) e^c \wedge e^d + \frac{1}{(n-1)(n-2)} R \eta_{ac}\eta_{bd} e^c \wedge e^d. \end{aligned} \tag{54}$$

YANG-MILLS THEORY

If φ is a set of scalar fields in some representation R of a Lie group G, then we define φ', the gauge-transformed field, by

$$\varphi' = h^{-1}\varphi, \tag{55}$$

where $h = h(x)$ is a map from the base space M into the group G. The Yang-Mills covariant derivative D of φ is defined to be

$$D\varphi \equiv (d + A)\varphi, \tag{56}$$

where the Yang-Mills potential, or connection, A, taking its values in the adjoint representation of G, is defined to transform under gauge transformations as

$$A' \equiv h^{-1}Ah + h^{-1}dh. \tag{57}$$

It then follows that $D\varphi$ indeed transforms in the desired covariant manner, namely

$$(D\varphi)' \equiv D'\varphi' = h^{-1}D\varphi. \tag{58}$$

The Yang-Mills field strength, or curvature, F, is defined by

$$F \equiv dA + A \wedge A. \tag{59}$$

Under gauge transformations, it transforms covariantly, as

$$F' = h^{-1}Fh. \tag{60}$$

The infinitesimal forms of these transformations, when $h = 1 + \Lambda$, where Λ is infinitesimal, reduce to the results derived in the lectures.

Kaluza-Klein and O'Neill's formula

Given a base space M with metric ds^2, and a principal bundle with fibre group G defined over it, with connection (Yang-Mills potential) A, we may write down a 1-parameter family of natural metrics on the bundle as

$$d\tilde{s}^2 = \lambda^2(\Sigma_i - A^i)^2 + ds^2, \tag{61}$$

where λ is an arbitrary constant, and a summation over $i = 1, \ldots, \dim(G)$ is understood. The Σ_i are left-invariant 1-forms on the group G, which means that they satisfy

$$d\Sigma_i = -\tfrac{1}{2} f_{ijk} \Sigma_j \wedge \Sigma_k, \tag{62}$$

where $f_{ijk} = f_{[ijk]}$ are the structure constants of the group. Then the Riemann tensor for the metric $d\tilde{s}^2$ is given by

$$\begin{aligned}
\widetilde{R}_{\alpha\beta\gamma\delta} &= R_{\alpha\beta\gamma\delta} - \tfrac{1}{4}\lambda^2\big(F^i_{\alpha\gamma}F^i_{\beta\delta} - F^i_{\alpha\delta}F^i_{\beta\gamma} + 2F^i_{\alpha\beta}F^i_{\gamma\delta}\big),\\
\widetilde{R}_{\alpha\beta\gamma i} &= \tfrac{1}{2}\lambda \mathcal{D}_\gamma F^i_{\alpha\beta},\\
\widetilde{R}_{\alpha i\beta j} &= \tfrac{1}{4}\lambda^2 F^i_{\beta\gamma}F^j_{\alpha\gamma} - \tfrac{1}{4} f_{ijk} F^k_{\alpha\beta},\\
\widetilde{R}_{ijk\ell} &= \frac{1}{4\lambda^2} f_{ijm} f_{k\ell m},
\end{aligned} \tag{63}$$

together with those components related to the above by the Riemann tensor symmetries (48)-(51). Here we are taking the orthonormal basis

$$\begin{aligned}
\tilde{e}^i &= \lambda(\Sigma_i - A^i), \qquad (i = 1, \ldots, \dim(G)),\\
\tilde{e}^\alpha &= e^\alpha, \qquad (\alpha = 1, \ldots, n),
\end{aligned} \tag{64}$$

where e^α is an orthonormal basis for the base space M: thus $ds^2 = e^\alpha e^\alpha$. $R_{\alpha\beta\gamma\delta}$ are the orthonormal components of the Riemann tensor on M, and

$$\begin{aligned}
F^i &= dA^i + \tfrac{1}{2} f_{ijk} A^j \wedge A^k,\\
\mathcal{D}_\gamma F^i_{\alpha\beta} &= D_\gamma F^i_{\alpha\beta} + f_{ijk} A^j_\gamma F^k_{\alpha\beta},\\
D_\gamma F^i_{\alpha\beta} &= E^\mu_\gamma \big(\partial_\mu F^i_{\alpha\beta} + \omega_\mu^{\alpha\gamma} F^i_{\gamma\beta} + \omega_\mu^{\beta\gamma} F^i_{\alpha\gamma}\big).
\end{aligned} \tag{65}$$

(So D_μ is the Lorentz-covariant derivative, and $\mathcal{D}_\mu$ is the Lorentz *and* Yang-Mills covariant derivative.)