Selling Demand Response Using Options

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***Abstract*—Electricity scheduling in US typically operates as a two settlement system, a day-ahead market (DAM) for bulk power scheduling and a real-time market (RTM) for supply-demand balancing. We argue that demand response (DR) can be used as intermediate recourse, as a balance between the DAM and RTM. We show that we can create an intermediate market for DR using call options. We analyze this DR market using options, characterize the equilibrium and study the efficiency properties.**

I. Introduction

We consider a setting where a Load Serving Entity (LSE) supplies energy to a set of consumers and an aggregator manages the load flexibility for these consumers. The LSE can interact directly with the aggregator for getting a given amount of load reduction. In the traditional two settlement market, the LSE buys energy from the DAM, gets random renewable energy close to the delivery time and buys the remaining energy to serve the load from the RTM. However, DR provides an additional resource for the LSE now. It can get load curtailment from the aggregator to reduce the energy purchase cost. *When should the LSE signal the aggregator for getting a required load reduction?*

LSE can signal the aggregator for load reduction at one of the three possible time instants - (i) at a time coinciding with the DAM, (ii) at a time coinciding with the RTM or (iii) at an intermediate time between the DAM or RTM. The LSE will have less incentives to ask for a load curtailment one day in advance because the forecasts about the random renewables and the prices will not be accurate enough to correctly determine the amount of load reduction required. Also, if the LSE has accurate forecast information, then it may be better buying energy from the DAM itself. So, the LSE's preference will be to signal for load curtailment closer to the delivery time. However, the aggregator typically needs some lead time to organize the demand flexibility and to achieve a given load reduction reliably. Also, the cost for load curtailment with very short notice can be considerably high in many scenarios. So, the aggregator may prefer to get the load curtailment signal well before the delivery time. We consider a setting where the LSE requests load curtailment from the aggregator at an intermediate time, between the DAM and the RTM. This enables the LSE to exploit the better forecast available at the intermediate time (as compared with the day-ahead). Also, this gives the aggregator enough lead time for organizing and delivering the load curtailment. *What is an appropriate mechanism for this intermediate time trading of DR between the LSE and the aggregator?*

A spot market with contingent pricing is often considered as the best way to sell demand flexibility from the view point of economic efficiency. This notion has two main difficulties: (a) spot market pricing is typically very volatile and does not offer any guaranteed income to offset the capital costs of organizing the load flexibility and participating in demand response programs (b) an intermediate spot market requires some regulated market infrastructure which typically doesn't exists in today's two settlement systems. Also, it is expensive to establish such an intermediate market infrastructure. We propose a method for trading DR using call options, which overcomes these two difficulties.

In our scheme, the LSE buys a number of call options from the aggregator at a time coinciding with the DAM by paying an option price. A call option gives the LSE the right, but no obligation to get one unit load reduction from the aggregator. The LSE can exercise these options at an intermediate time, i.e., give a notification to the aggregator asking for load reduction, by paying a strike price. So, the LSE can exploit the better forecast information available at the intermediate time to decide the amount of load curtailment. The aggregator also gets some lead time to organize the loads. Also, the payment via the option price acts as a guaranteed income to organize the loads for demand reduction. Options can be viewed as unregulated over-the-counter private transactions between the LSE and aggregator which do not need the infrastructure of an organized market.

A. Our Contributions

We first consider energy scheduling from the perspective of the LSE. We formulate this as a three stage stochastic control problem. We characterize the optimal decisions of the LSE in each of these three stages - the optimal energy purchase from the DAM, the optimal load curtailment decision at the intermediate stage given the available forecast information and the energy purchase from the RTM. Here we indeed consider the LSE and the aggregator as a single entity.

We then consider the interaction of LSE and the aggregator in a spot-market with contingent prices. We show that there exists a competitive equilibrium in this market. Also all equilibria are socially optimal. However, as we mentioned before, there are many practical difficulties for implementing a spot market. Here we show that we can implement an intermediate market using options which overcomes those difficulties. We

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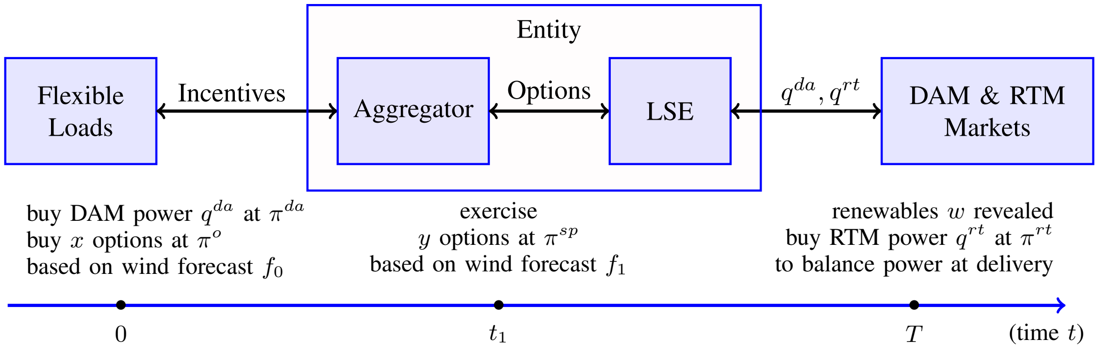

Fig. 1: Players, interactions, and decision time-line.

characterize the equilibrium prices and purchase decisions in such market. We also comment on the efficiency of this mechanism.

B. Related Works

Federal energy regulatory commission has identified the benefits of coordination of demand-side resources to balance the variability of intermittent renewable generation [1]. Many market models, scheduling algorithms and pricing schemes have been proposed for DR. Li *et al.* [2] considered a demand response approach based on utility maximization framework. They showed that time-varying prices can be used as signal to elicit demand response from consumers in such a way that it will achieve social optimality. Roscoe and Ault [3], Samadi *et al.* [4] and Chen *et al.* [5] also studied various real-time pricing based algorithms for DR. Wu *et al.* [6] proposed a game-theoretic model to understand the interactions among EVs and aggregators in a V2G market, where EVs participate in providing frequency regulation service to the grid. Mohsenian-Rad *et al.* [7] and Yang *et al.* [8] also proposed different game theoretic methods for DR.

Gabriel *et al.* [9] considered the problem of setting up electric power contracts from the perspective of a retailer whose objective is to maximize its own profits and minimize settlement risks. Thus, a retailer acts as a mediator between the supplier and the consumer. In [10] Haring *et al.* considered a contract design problem for DR where an aggregator designs and offers these contracts to consumers. [11] also considered a contract design method for load curtailment. [12], [13] and [14] proposed different distributed load control schemes for demand-side management. [15] and [16] considered a parametrized supply function bidding method for demand response. Each consumer submits a parameter which characterizes their supply function and the utility company acts as the market maker who computes the market clearing price.

Varaiya *et al.* [17] proposed the idea of 'risk-limiting dispatch' as a way of integrating renewable energy into the power grid. They formulated this as a multi-stage stochastic control problem where at each stage a utility company can decide the amount of power purchase depending on the the available information. Rajagopal *et al.* [18] extended this idea and characterized threshold based decision policies for power procurement. Huang et.al. [19] also considered a problem of jointly optimizing power procurement and demand response to maximize social welfare. They proposed an algorithm where utility company can announce time varying price to control the users' consumption while ensuring a quality-of-usage guarantee to the power consumer. There are many works on using the tools and techniques from financial engineering for interruptible electricity contracts [20] [21]. Our approach is different from these works.

The paper is organized as follows. We introduce the basic notions and notations in Section II. The problem of energy scheduling with DR from the perspective of a system planner is considered in Section III. In Section IV we consider the intermediate spot market implementation of DR. Option markets for DR is presented in Section V. We conclude paper with a brief description about the future research directions in Section VI.

II. Preliminaries

We consider a setting as shown in Figure 1. The loads get energy for their nominal consumption of l units from the LSE. The loads suffer a disutility of $\phi(y)$ when they curtail their consumption by y units. The LSE has access to zero-marginal cost random renewables w which are revealed at T. It interacts with the market and can purchase q^{da} from the DAM at price π^{da}, and q^{rt} from the RTM at price π^{rt}. It can also extract a curtailment of y units from the loads. The total energy purchase and curtailment should be such that

$$l \leq q^{da} + q^{rt} + w + y. \quad (1)$$

The curtailment decision is made at the intermediate time t_1 based on a forecast f_1 of w and π^{rt}. This intermediate time decision balances the necessary lead time for loads to organize their flexibility against the need for the LSE to have greater visibility about its renewable generation w. The day ahead purchase q^{da} is based on f_0, the information about w and π^{rt} that is available at $t = 0$.

We assume that an aggregator manages a collection of flexible consumers for demand response. It recruits the consumers for DR participation, designs incentives for them and sends load reduction signals to a set of selected consumers when a DR event occurs. In [22] and [23], we have addressed the problem of incentive design, baselining and consumer scheduling for reliably accruing a given net load reduction. In this paper we assume that the aggregator can reliably deliver a net load reduction of y units at time T if it commits for the same at time t_1. For clarity of exposition and simplicity of notation, we assume the LSE interacts only with one aggregator for achieving the load reduction, as shown in Figure 1. This paper addresses the market mechanisms which facilitate this interaction.

Let $p(w|f_1)$ be the conditional probability of the wind given the intermediate forecast state f_1 at time t_1. We assume that $f_1 \in S = [0, 1]$. f_1 can be thought as a sufficient statistics which parametrizes the information available at time t_1. We call this an *information state*. Let

$$p_s(w) = p(w|f_1 = s), \quad P_s(z) = \int_{w=0}^{z} p_s(w)dw.$$

So, $P_s(z)$ is the probability that the wind at time T is less than z given the information state s. Since we are considering only a time window from $t = 0$ to $t = T$, we assume that f_0 is single valued. Let $\alpha(s)$ be probability density function of the information state, i.e.,

$$\alpha(s) = \mathbb{P}\left(f_1 = s|f_0\right).$$

We assume that real-time price π^{rt} is a random variable and denote

$$\overline{\pi}^{rt}_s = \mathbb{E}[\pi^{rt}|f_1 = s].$$

The day-ahead price π^{da} is known at time t_0. We use $\mathbb{E}_s\left[\cdot\right]$ and $\mathbb{E}_w\left[\cdot\right]$ to denote the expectation w.r.t. to the information state and the wind respectively. $\mathbb{E}\left[\cdot\right]$ denotes the joint expectation. We make the following assumptions.

Assumption 1. *(i)* $\mathbb{P}(w \geq z|f_1 = s_1) < \mathbb{P}(w \geq z|f_1 = s_2)$, $\forall z$, *if* $s_2 > s_1$
(ii) $\overline{\pi}^{rt}_{s_2} < \overline{\pi}^{rt}_{s_1}$*, if* $s_2 > s_1$
(iii) π^{rt} *and* w *are conditionally independent given the information state* s*.*

Assumption (i) imposes a stochastic ordering on wind conditioned on the information state $s \in [0, 1]$. Intuitive interpretation is that, higher s indicates (stochastically) more wind. Also, this assumption guarantees that $P_{s_2}(z) < P_{s_1}(z), \forall z$, if $s_2 > s_1$ so that the cdfs $P_{s_1}(\cdot)$ and $P_{s_2}(\cdot)$ don't cross each other. Assumption (ii) similarly imposes an order on the expected value of the real-time price conditioned on the information state. We assume that higher s indicates a lower expected real-time price (because of the higher wind). We note that the above assumptions are not necessary for most of the results we show. However, they considerably simplify the notations and give better insight to the problems.

LSE: The LSE is responsible for satisfying the energy balance specified by equation (1). It can buy q^{da} units of energy at a price π^{da} from the DAM. It can also get a load reduction of y_s units from the aggregator when the information state s is revealed at time t_1 by making a payment $R_s(y_s)$. The purchase form the RTM, q^{rt}, is essentially to balance the energy, i.e., $q^{rt} = (l - q - y_s - w)_+$. The ex-post cost for the LSE given the information state s is,

$$J^{lse}_s = \pi^{da}q^{da} + R_s(y_s) + \pi^{rt}(l - q^{da} - y_s - w)_+ \qquad (2)$$

Aggregator: Aggregator suffers a disutility $\phi(y_s)$ for a load reduction of y_s units, and it gets a payment $R_s(y_s)$ from the LSE. The ex-post cost for the aggregator, given the information state s is,

$$J^{agg}_s = \phi(y_s) - R_s(y_s) \qquad (3)$$

We make the following assumption about the disutility.

Assumption 2. *The disutility function is strictly convex, i.e.,* $\phi''(y) > 0$ *for all* y*.*

Entity: We consider a hypothetical agent, the *entity*, which combines the roles of the LSE and the aggregator. This entity can buy q^{da} units of energy from the DA market, get a load curtailment of y_s units at an intermediate time t_1 when the information state is s, get the wind w at time T and purchase the remaining energy $(l - q^{da} - y_s - w)_+$ from the RTM for the load balance (c.f. equation (1)). The ex-post cost for the entity is then,

$$J^e_s = \pi^{da}q^{da} + \phi(y_s) + \pi^{rt}(l - q^{da} - y_s - w)_+ \qquad (4)$$

So, the function of the entity is equivalent to that of a social planner. It considers only the system cost, not the payment transfer between the agents.

III. Optimal Scheduling for the Entity

In this section we consider the optimal scheduling of energy from the perspective of the entity. In doing so, we separately consider the scheduling problems with and without DR. Later, we will use these solutions as a benchmark for comparing the efficiency of the market mechanisms we propose for DR.

A. Optimal Scheduling without DR

Without DR, the entity can buy energy only from the DA market and the RT market. Let

$$J^e_{ndr}(q) = \pi^{da}q + \mathbb{E}[\pi^{rt}(l - q - w)_+] \qquad (5)$$

$$q^e_{ndr} \in \arg\min_q J^e_{ndr}(q). \qquad (6)$$

$J^e_{ndr}(q)$ is the net expected cost for the entity as a function of the day-ahead purchase q when there is no DR. q^e_{ndr} is the optimal day-ahead purchase.

Proposition 1. $J^e_{ndr}(\cdot)$ *is convex. The minimizer* q^e_{ndr} *is given by the solution of*

$$\pi^{da} - \mathbb{E}_s[\overline{\pi}^{rt}_s P_s(l - q^e_{ndr})] = 0.$$

Remark 1. In order to avoid trivial results, we can assume that $\pi^{da} < \mathbb{E}_s[\overline{\pi}^{rt}_s P_s(l)]$. This will ensure that $q^e_{ndr} > 0$.

B. Optimal Scheduling with DR

Here we consider the energy scheduling with DR from the perspective of the entity. The net expected cost for the entity as a function of the first-stage purchase q, $J^e(q)$, is

$$J^e(q) = \pi^{da}q + \mathbb{E}_s\left[\min_{y_s} J^e_s(y_s)\right], \quad \text{where,} \qquad (7)$$

$$J^e_s(y_s) = \phi(y_s) + \mathbb{E}_w\left[\overline{\pi}^{rt}_s(l - q - y_s - w)_+|f_1 = s\right] \qquad (8)$$

Here $J_s^e(\cdot)$ is the expected second-stage cost conditioned on information state s and the first-stage decision q. Let

$$q^e \in \arg\min_q J^e(q), \;\; J^{*e} = J^e(q^e), \;\; y_s^e \in \arg\min_{y_s} J_s^e(y_s) \tag{9}$$

Proposition 2. *$J^e(\cdot)$ and $J_s^e(\cdot)$ are convex. For any given first stage decision q, the second-stage decision y_s^e is given by solution of*

$$\phi'(y_s^e) - \overline{\pi}_s^{rt} P_s(l - q - y_s^e) = 0 \tag{10}$$

if $\phi'(0) < \overline{\pi}_s^{rt} P_s(l - q)$ and zero otherwise. The first-stage decision q^e is given by solution of

$$\pi^{da} - \mathbb{E}_s[\overline{\pi}_s^{rt} P_s(l - q^e - y_s^e)] = 0. \tag{11}$$

C. Examples

A simple example is worked out in this section for illustrating the scheduling problem. We will later use the same example for comparing with the market based allocations in Section IV and Section V.

1. *Uniform forecast error:* Let $l = 3$. Assume that information set S has three elements, s_L, s_M and s_H indicating low, medium and high wind forecast states respectively. Let $p(w/s_L) = U[0, 3]$, $p(w/s_M) = U[0.25, 3.25]$ and $p(w/s_H) = U[0.5, 3.5]$ where $U[a, b]$ is the uniform distribution of $[a, b]$. Also, let $\alpha(s_L) = \alpha(s_M) = \alpha(s_H) = 1/3$. Take $\pi^{da} = 50, \pi^{rt} = 1000$ and $\phi(y) = 50y + 50y^2$. Figure 2 shows the purchasing decision at each stage. Figure 3 illustrate the way we compute the intermediate load curtailment decisions y_s^e as specified by equation (10). Note that, as intuition suggests, load curtailment is small when the wind forecast is high.

2. *Gaussian forecast error:* Let $l = 5$. Here also, we assume that that information set S has three elements, s_L, s_M and s_H with equal ($= 1/3$) probability. Let $p(w/s_L) = \tilde{\mathcal{N}}(0, 0.5)$, $p(w/s_M) = \tilde{\mathcal{N}}(0.25, 0.5)$ and $p(w/s_H) = \tilde{\mathcal{N}}(0.5, 0.5)$ where $\tilde{\mathcal{N}}(\mu, \sigma)$ is a truncated normal distribution with mean μ and standard deviation σ. We truncated the distribution for negative values and re-normalized the integral to 1. Take $\pi^{da} = 50, \pi^{rt} = 1000$ and $\phi(y) = 50y + 25y^2$. We can compute $q^e = 4.49$, $(y^e_{s_L}, y^e_{s_M}, y^e_{s_H}) = (0.464, 0.222, 0)$.

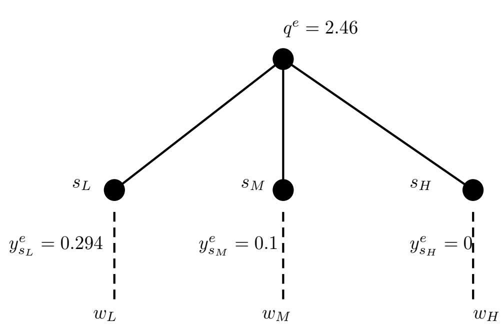

Fig. 2: Purchase Decisions
Uniform Forecast Error

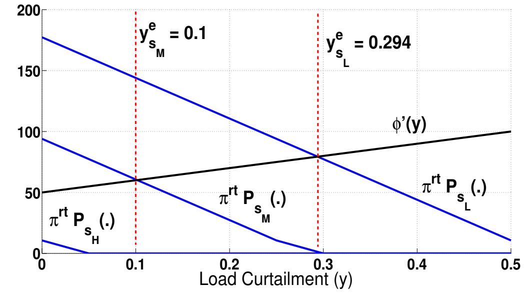

Fig. 3: Load Curtailment Decisions (c.f. equation (10))

IV. Spot Markets with Contingent Prices

In Section III, we considered the optimal scheduling of energy from the perspective of the entity. However, entity is a fictitious system planner and the actual agents in the systems are the LSE and the aggregator. Does there exist a market mechanism that achieves the same scheduling decisions (q^e, y_s^e) as a result of an interaction between these two rational agents? In this section we show that this socially optimal allocation can be achieved in a spot market with contingent prices.

We assume that the agents are rational and price takers. The price taking behavior of agents is a standard assumption in the competitive equilibrium theory in microeconomics. Let π_s^{in} be the price for unit load reduction in the DR market when the information state is s. We call this contingent price because it is contingent on the realized information state s.

The timeline of purchase decisions are as follows. The LSE will buy q units of energy from the DA market at a price π^{da}. Then, depending on the realized information state, it will ask for a curtailment y_s units of energy from the aggregator, paying a price π_s^{in}. Note that this can be treated as purchase from an intermediate market with contingent prices. Finally, the LSE will get the wind energy w and purchase the remaining energy $(l - q - y_s - w)_+$ from the RT market at a price π^{rt}. The net expected cost for the LSE as a function of the first-stage purchase q, $J^{lse}(q)$, is given by

$$J^{lse}(q) = \pi^{da} q + \mathbb{E}_s[\min_{y_s} J_s^{lse}(y_s)], \quad \text{where,} \tag{12}$$

$$J_s^{lse}(y_s) = \pi_s^{in} y_s + \mathbb{E}_w[\overline{\pi}_s^{rt}(l - q - y_s - w)_+]. \tag{13}$$

Here, $J_s^{lse}(\cdot)$ is the expected second stage cost for the LSE given the first stage purchase q. Also, let

$$q^{lse} \in \arg\min J^{lse}(q), \quad y_s^{lse} \in \arg\min_{y_s} J_s^{lse}(y_s) \tag{14}$$

which are the optimal first and second stage purchase decisions for the LSE. Similarly, the net expected cost and the optimal selling decision for the aggregator when the information state is s are given by,

$$J_s^{agg}(y_s) = \phi(y_s) - \pi_s^{in} y_s \quad \text{and} \quad y_s^{agg} \in \min_{y_s} J^{agg}(y_s). \tag{15}$$

Note that the optimal buying/selling decision of agents (LSE/aggregator) are indeed functions of the contingent prices $\{\pi_s^{in}\}$. A market is said to be in equilibrium if the prices are such that the optimal buying and selling decisions of the agents in the markets balance each other. We make this notion more precise below, in the context of the DR market.

Definition 1 (Competitive Equilibrium with Contingent Prices)**.** *The contingent prices $\{\pi^{*in}_s\}$, optimal buying decisions of the LSE $q^{*lse}, \{y^{*lse}_s\}$, optimal selling decisions of the aggregator $\{y^{*agg}_s\}$ constitute a competitive equilibrium with contingent prices, if $(y^{*lse}_s - y^{*agg}_s) = 0, \forall s \in S$.*

Let J^{*lse} be the expected cost for the LSE and let J^{*agg} be the expected cost for the aggregator at equilibrium. Also, let

$$J^{*cp} = J^{*lse} + J^{*agg} \tag{16}$$

which is the system cost at competitive equilibrium with contingent prices. We define the social optimality of an equilibrium by comparing J^{*cp} with J^{*e} (c.f. (9)) which is the optimal system cost from the perspective of the entity.

Definition 2 (Socially Optimal Equilibrium with Contingent Prices)**.** *An equilibrium with contingent prices is said to be socially optimal, if $J^{*cp} = J^{*e}$*

We now give the main result of this section.

Theorem 1. *There exists a competitive equilibrium with contingent prices. All competitive equilibria are socially optimal. At equilibrium, denoting $q^* = q^{*lse}$, $y^*_s = y^{*lse}_s = y^{*agg}_s$,*

$$\phi'(y^*_s) = \pi^{*in}_s = \overline{\pi}^{rt}_s P_s(l - q^* - y^*_s) \tag{17}$$

if $\phi'(0) \le \overline{\pi}^{rt}_s P_s(l - q^*)$ and $y^*_s = 0$ if $\phi'(0) > \overline{\pi}^{rt}_s P_s(l - q^*)$. Also,

$$\pi^{da} - \mathbb{E}_s[\overline{\pi}^{rt}_s P_s(l - q^* - y^*_s)] = 0 \tag{18}$$

A. Examples

The equilibrium allocations are the same as the solution of the entity's problem given in Section III. Equilibrium prices $\{\pi^{*in}_s\}$ can be found easily using the equation (17).

V. Options Markets and Competitive Equilibrium

In the previous section, we showed that the optimal scheduling of energy from the DR market can be implemented using a spot market. More precisely, the competitive equilibrium with contingent prices achieves a socially optimal outcome. However, there are many difficulties in implementing such an intermediate spot market with contingent prices, as described in the introduction.

In this section, we show that an intermediate market for DR can be implemented using options. In particular, we show that there exists a competitive equilibrium for this options market. We also comment about the efficiency of this market equilibrium.

The timeline of purchase decisions are as follows. The LSE will buy q units of energy from the DA market. Also, at the same time, the LSE will buy x units of options from the aggregator at a price π^o. So, the LSE has the right, but not the obligation to get y units of load reduction, $y \le x$, from the aggregator. The LSE can give the notification for a load reduction at any time before the intermediate market closes, by paying a strike price π^{sp} for unit reduction. Clearly, the amount of load reduction the LSE asks for, y_s, will depend on the information state s realized at time t_1. However, note that the strike price π^{sp} doesn't depend on the information state. Given the notification at time t_1, the aggregator will deliver the load reduction at time T. The LSE will observe the wind energy w at time T and purchase the remaining amount of energy $(l - q - y_s - w)_+$ from the RT market.

Since we are considering a competitive market, we assume the agents are rational and price takers. In particular, they make their buying/selling decision based on the market prices π^o and π^{sp}. The net expected cost for the LSE as a function of the first stage decisions q and x, $\tilde{J}^{lse}(q, x)$, is given by,

$$\tilde{J}^{lse}(q, x) = \pi^o x + \pi^{da} q + \mathbb{E}_s[\min_{y_s, y_s \le x} \tilde{J}^{lse}_s(y_s)], \quad \text{where} \tag{19}$$

$$\tilde{J}^{lse}_s(y_s) = \pi^{sp} y_s + \mathbb{E}_w[\overline{\pi}^{rt}_s (l - q - y_s - w)_+]. \tag{20}$$

Here $\tilde{J}^{lse}_s(\cdot)$ is the second stage cost for the LSE given the first stage purchase q and x. Also, let

$$(\tilde{q}^{lse}, x^{lse}) \in \arg\min_{(q,x)} \tilde{J}^{lse}(q, x), \quad \tilde{y}^{lse}_s \in \arg\min_{y_s} \tilde{J}^{lse}_s(y_s) \tag{21}$$

which are the optimal first and second stage purchase decisions for the LSE. Similarly, the net expected cost and the optimal selling decision for the aggregator when the information state is s are given by,

$$\tilde{J}^{agg}(x) = \mathbb{E}_s[\phi(y_s) - \pi^{sp} y_s] - \pi^o x, \quad x^{agg} \in \arg\min_x \tilde{J}^{agg}(x) \tag{22}$$

Unlike the spot market with contingent prices, here we have only two prices π^o and π^{sp}. Also, in the spot market with contingent prices the trading takes place in the intermediate time. In options market, the trading of options takes place in the day ahead market. LSE buys x units of options for a specified option price π^o and strike price π^{sp}. In the intermediate market, only the exercise of the options takes place. In a spot market with contingent prices, the equilibrium notion is that the optimal trading decisions of the LSE and the aggregator should balance each other for every possible contingency of the intermediate market. So, at equilibrium $y^{lse}_s = y^{agg}_s$ for all $s \in S$. In an options market, the equilibrium notion is that, the number of options that the LSE is willing to buy and the number of options that the aggregator is willing to sell should balance each other. So at equilibrium, $\tilde{x}^{lse} = \tilde{x}^{agg}$. Not that the contingent information state s doesn't appear explicitly here. We formally define this equilibrium notion below.

Definition 3 (Competitive Equilibrium for Options Market)**.** *The options price π^{*o} and the strike price π^{*sp}, the optimal buying decision of the LSE x^{*lse} and the optimal selling decision of the aggregator x^{*lse} constitute a competitive equilibrium, if $x^{*lse} = x^{*agg}$.*

Let $\tilde{J}^{*lse}$ be the expected cost for the LSE and let $\tilde{J}^{*agg}$ be the expected cost of the aggregator at equilibrium. Also, let

$$\tilde{J}^{*op} = \tilde{J}^{*lse} + \tilde{J}^{*agg} \tag{23}$$

which is the system cost at the competitive equilibrium of options market. We define the notion of social optimality in this context as below.

Definition 4 (Socially Optimal Equilibrium of Options Market)**.** *An options market equilibrium is said to be socially optimal, if* $J^{*op} = J^{*e}$

We now give the main results of this section.

Theorem 2. *There exists a competitive equilibrium for options market. The equilibrium need not be socially optimal. Also, at equilibrium, denoting* $\tilde{q}^* = \tilde{q}^{lse}, x^* = x^{*lse} = x^{*agg}$ *and* $\tilde{y}^*_s = \tilde{y}^{*lse}_s$,

$$\tilde{y}^*_s = \begin{cases} 0 & \text{if } P_s(l - \tilde{q}^*) < (\pi^{*sp}/\overline{\pi}^{rt}_s) \\ x^* & \text{if } P_s(l - \tilde{q}^* - x^*) > (\pi^{*sp}/\overline{\pi}^{rt}_s) \\ l - \tilde{q}^* - P^{-1}_s(\pi^{*sp}/\overline{\pi}^{rt}_s) & \text{otherwise} \end{cases} \quad (24)$$

$$\pi^{da} - \mathbb{E}_s[\overline{\pi}^{rt}_s P_s(l - \tilde{q}^* - \tilde{y}^*_s)] = 0 \quad (25)$$

$$\pi^{*o} + \pi^{*sp}\mathbb{E}_s[\mathbb{I}\{\tilde{y}^*_s = x^*\}] - \mathbb{E}_s[\overline{\pi}^{rt}_s P_s(l - \tilde{q}^* - \tilde{y}^*_s)\mathbb{I}\{\tilde{y}^*_s = x^*\}] = 0 \quad (26)$$

$$\pi^{*o} + \pi^{*sp}\mathbb{E}_s[\mathbb{I}\{\tilde{y}^*_s = x^*\}] - \phi'(x^*)\mathbb{E}_s[\mathbb{I}\{\tilde{y}^*_s = x^*\}] = 0 \quad (27)$$

A. Examples

1. Uniform forecast error (3 states): We consider the same example as in Section III. Assume that $l = 3$, and $p(w/s_L) = U[0,3]$, $p(w/s_M) = U[0.25, 3.25]$ and $p(w/s_H) = U[0.5, 3.5]$, $\alpha(s_{L,1}) = \alpha(s_{L,2}) = \alpha(s_H) = 1/3$. Take $\pi^{da} = 50, \pi^{rt} = 1000$ and $\phi(y) = 50y + 50y^2$.

Figure 4 shows the prices which supports an equilibrium (π^{*o}, π^{*sp}). Each point on the curve is an (option price, strike price) pair corresponding to an equilibrium. The purchase decisions corresponding at these equilibria will be different.

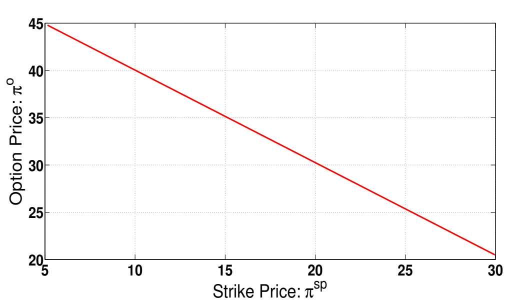

Fig. 4: Equilibrium Prices - Uniform Forecast Error (3 states)

In Figure 5, we shows the load curtailment decisions (c. f. equation (24)) at one particular equilibrium where $\pi^{*o} = 21$ and $\pi^{*sp} = 30$. Also, note that the curtailment decisions are different from the optimal curtailment decisions from the perspective of the entity which is given in the example in Section III. So, this is not a socially optimal equilibrium.

2. Uniform forecast error (9 states): We consider another example with more intermediate states. We use this to compare

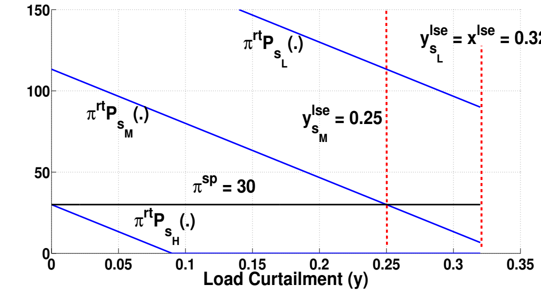

Fig. 5: Load Curtailment Decisions (c.f. equation (24))

social optimal cost J^{*e} and the system cost at competitive equilibrium $\tilde{J}^{*op} = \tilde{J}^{*lse} + \tilde{J}^{*agg}$ (c.f. definition 4).

Let $l = 4$. Let the information set S have 9 elements and s_k denote the state k. Let $p(w/s_1) = U[0,3]$, $p(w/s_2) = U[0.5, 3.5]$, $p(w/s_3) = U[0.75, 3.75]$, $p(w/s_4) = U[1,4]$, $p(w/s_5) = U[1.25, 4.25]$, $p(w/s_6) = U[1.5, 4.5]$, $p(w/s_7) = U[2,5]$, $p(w/s_8) = U[2.5, 5.5]$ and $p(w/s_9) = U[3,6]$ where $U[a,b]$ is the uniform distribution of $[a,b]$. Also, let $\alpha(s_k) = 1/9$. Take $\pi^{da} = 50, \pi^{rt} = 1000$ and $\phi(y) = 50y + 50y^2$.

Figure 6 compares J^{*e} and $\tilde{J}^{*op}$ at various equilibria.

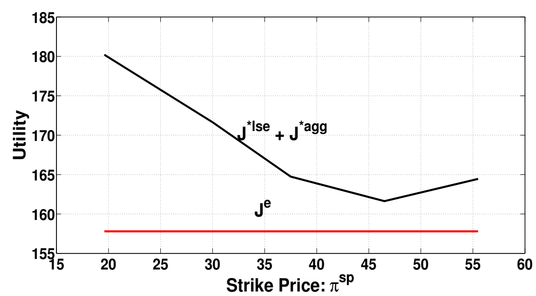

Fig. 6: Efficiency - Uniform Forecast Error (9 states)

Remark 2 (Efficiency). Our simulations suggest that the inefficiency ratio $J^{*e}/\tilde{J}^{*op}$ is low as shown Figure 6. Getting a provable upper bound on this ratio is an ongoing work

VI. Conclusions

We have studied a market model for demand response. In particular, we argued that demand response can be used as an intermediate recourse between the real-time market and the day-ahead market. We show that such an intermediate market can be implemented using options and have characterized the equilibria in such markets.

This paper is only the first step in developing practical methods for using demand response market and in particular for using it as an intermediate recourse. In the future works we will address the problem of options market with multiple stages and the cases where the LSE has a market power. Establishing a theoretical upperbound on the efficiency loss is also a very interesting problem.

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VII. Proofs

A. Proof of proposition 1

Proof. We have

$$J^e_{ndr}(q) = \pi^{da}q + \int_{s=0}^{1}\int_{w=0}^{(l-q)} \overline{\pi}^{rt}_s(l-q-w)p_s(w)dw\ \alpha(s)ds \tag{28}$$

By taking partial derivative we get

$$\frac{dJ^e_{ndr}}{dq} = \pi^{da} - \int_{s=0}^{1} \overline{\pi}^{rt}_s P_s(l-q)\alpha(s)ds$$
$$\frac{d^2J^e_{ndr}}{dq^2} = \int_{s=0}^{1} \overline{\pi}^{rt}_s p_s(l-q)\alpha(s)ds$$

J^e_{ndr} is convex since the second derivative is positive. Hence J^e_{ndr} is strictly convex. Solution is obtained by equating the first derivative to zero and uniqueness follows from strict convexity. □

B. Proof of proposition 2

Proof. We have

$$J^e_s(y_s) = \phi(y_s) + \int_{w=0}^{(l-q-y_s)} \overline{\pi}^{rt}_s(l-q-w-y_s)p_s(w)dw.$$

Taking partial derivatives

$$\frac{\partial J^e_s}{\partial y_s} = \phi'(y_s) - \overline{\pi}^{rt}_s P_s(l-q-y_s)$$
$$\frac{\partial^2 J^e_s}{\partial y_s^2} = \phi''(y) + \overline{\pi}^{rt}_s p_s(l-q-y_s)$$

Second derivative is positive. This implies $J^e_s(y_s)$ is strictly convex. Using first derivative the optimal load curtailment y^e_s is given by,

$$y^e_s = 0 \text{ if } \phi'(0) > \overline{\pi}^{rt}_s P_s(l-q)$$

else the optimal load curtailment y^e_s is given by equating the first order condition to zero i.e.,

$$\frac{\partial J^e_s}{\partial y_s} = \phi'(y_s) - \overline{\pi}^{rt}_s P_s(l-q-y_s)\big|_{y_s=y^e_s} = 0$$

By strict convexity it follows that the optimal load curtailment decision is unique. Also, when $\phi'(0) > \overline{\pi}^{rt}_s P_s(l-q)$, $\frac{\partial y^e_s}{\partial q} = 0$. Otherwise from (10), by taking partial derivative w.r.t. q we get,

$$\phi''(y^e_s)\frac{\partial y^e_s}{\partial q} = -\overline{\pi}^{rt}_s p_s(l-q-y^e_s)\left(1 + \frac{\partial y^e_s}{\partial q}\right)$$

From this it can be deduced that $-1 < \frac{\partial y_s}{\partial q} \leq 0$. Taking partial derivatives of J^e w.r.t q and using the optimality condition for y^e_s gives

$$\frac{dJ^e}{dq} = \pi^{da} - \int_{s=0}^{1} \overline{\pi}^{rt}_s P_s(l-q-y^e_s)\alpha(s)ds$$
$$\frac{d^2J^e}{dq^2} = \int_{s=0}^{1} \overline{\pi}^{rt}_s p_s(l-q-y^e_s)\left(1 + \frac{\partial y^e_s}{\partial q}\right)$$

Second derivative is positive since $\frac{\partial y_s}{\partial q} < -1$. Hence we get the optimality condition (11), by equating the first derivative to zero. □

C. Proof of Theorem 1

Proof. The unique solution of (15) is given by,

$$y_s^{agg}(\pi_s^{in}) = \begin{cases} 0 & \text{if } \phi'(0) > \pi_s^{in} \\ (\phi')^{-1}(\pi_s^{in}) & \text{otherwise} \end{cases}$$

Now, for a given q, y_s^{lse} is unique and is given by,

$$y_s^{lse}(\pi_s^{in}) = \begin{cases} 0 & \text{if } P_s(l-q) < (\pi_s^{in}/\overline{\pi}_s^{rt}) \\ l - q - P_s^{-1}(\pi_s^{in}/\overline{\pi}_s^{rt}) & \text{otherwise} \end{cases}$$

Proof for the above is similar to that of Proposition 2 and hence we skip the details. $y_s^{agg}(\pi_s^{in})$ is a continuous and increasing function of π_s^{in}. $y_s^{lse}(\pi_s^{in}; q)$ is a continuous and decreasing function of π_s^{in}. So, for any given q, there exists a $\pi_s^{in}(q)$ such that $y_s^{agg}(\pi_s^{in}(q)) = y_s^{lse}(\pi_s^{in}(q); q)$. Similar to proposition 2 the minimizer q^{lse} satisfies the first order condition,

$$\pi^{da} - \int_{s=0}^{1} \overline{\pi}_s^{rt} P_s(l - q^{lse} - y_s^{lse})\alpha(s)ds = 0$$

Note that J^{lse} is not strictly convex (which was not the case in 2). Choose $\pi_s^{*in} = \pi_s^{in}(q^e)$ as contingent prices. Then uniqueness of second-stage purchase implies $y_s^{*lse} = y_s^{*agg} = y_s^e$. As a result, $q = q^e$ satisfies the optimality condition and is one of the minimizers. So we can set, $q^{*lse} = q^e$. From Proposition 2 we know that this decision is socially optimal. Hence by convexity equilibria are socially optimal. □

D. Proof of Theorem 2

We first prove a series of lemmas before giving the proof of Theorem 2.

Lemma 1. *The function $\tilde{J}_s^{lse}(\cdot)$ is convex for all $s \in S$. The unique minimizer $\tilde{y}_s^{lse}$ is given by,*

$$\tilde{y}_s^{lse} = \begin{cases} 0 & \textit{if } P_s(l-q) < (\pi^{sp}/\overline{\pi}_s^{rt}) \\ x & \textit{if } P_s(l-q-x) > (\pi^{sp}/\overline{\pi}_s^{rt}) \\ l - q - P_s^{-1}(\pi^{sp}/\overline{\pi}_s^{rt}) & \text{otherwise} \end{cases} \tag{29}$$

Proof. Since

$$\tilde{J}_s^{lse}(y) = \pi^{sp} y_s + \overline{\pi}_s^{rt} \int_{w=0}^{(l-q-y_s)} (l - w - q - y_s) p_s(w) dw,$$

we get,

$$\frac{\partial \tilde{J}_s^{lse}}{\partial y_s} = \pi^{sp} - \overline{\pi}_s^{rt} P_s(l-q-y_s), \quad \frac{\partial^2 \tilde{J}_s^{lse}}{\partial y_s^2} = \overline{\pi}_s^{rt} p_s(l-q-y_s).$$

Since the second derivative is positive, $\tilde{J}_s^{lse}(y)$ is strictly convex. Then using first derivative and the strict convexity property the expression for the unique minimizer $\tilde{y}_s^{lse}$ follows. □

Lemma 2. *The function $\tilde{J}^{lse}(q,x)$ is jointly convex in q and x. The minimizers $(\tilde{q}^{lse}, x^{lse})$ are given by,*

$$\begin{aligned} &\pi^{da} - \mathbb{E}_s[\overline{\pi}_s^{rt} P_s(l - \tilde{q}^{lse} - \tilde{y}_s^{lse})] = 0 \\ &\pi^o + \pi^{sp} \mathbb{E}_s[\mathbb{I}\{y_s = x^{lse}\}] \\ &\quad - \mathbb{E}_s[\overline{\pi}_s^{rt} P_s(l - \tilde{q}^{lse} - \tilde{y}_s^{lse}) \mathbb{I}\{y_s = x^{lse}\}] = 0 \end{aligned} \tag{30}$$

Proof. Let $s_1, s_2 \in S$ be such that $y_s = x$ for $0 \leq s \leq s_1$ and $y_s = 0$ for $s_2 \leq s \leq 1$. Note that s_1 and s_2 depends on q and x from the first stage. Then, $\tilde{J}^{lse}(q,x)$ (c.f. (19)) can be written as,

$$\begin{aligned} &\tilde{J}^{lse}(q,x) = (\pi^o x + \pi^{da} q) \\ &+ \int_{s=0}^{s_1} \pi^{sp} x \alpha(s) ds \\ &+ \int_{s=0}^{s_1} \overline{\pi}_s^{rt} \int_{w=0}^{(l-q-x)} (l - q - x - w) p_s(w) dw \alpha(s) ds \\ &+ \int_{s=s_1}^{s_2} \pi^{sp} y_s \alpha(s) ds \\ &+ \int_{s=s_1}^{s_2} \overline{\pi}_s^{rt} \int_{w=0}^{(l-q-y_s)} (l - q - y_s - w) p_s(w) dw \alpha(s) ds \\ &+ \int_{s=s_2}^{1} \left(\overline{\pi}_s^{rt} \int_{w=0}^{(l-q)} (l - q - w) p_s(w) dw \right) \alpha(s) ds \end{aligned}$$

We give simplified expressions for the partial derivatives of $\tilde{J}^{lse}$ w.r.t q and x below,

$$\begin{aligned} \frac{\partial \tilde{J}^{lse}}{\partial q} &= \pi^{da} - \int_{s=0}^{1} \overline{\pi}_s^{rt} P_s(l - q - \tilde{y}_s^{lse}) \alpha(s) ds \\ \frac{\partial \tilde{J}^{lse}}{\partial x} &= \pi^o + \pi^{sp} \int_{s=0}^{s_1} \alpha(s) ds \\ &\quad - \int_{s=0}^{s_1} \overline{\pi}_s^{rt} P_s(l - q - x) \alpha(s) ds \end{aligned}$$

Once again differentiating the above expressions w.r.t q and x we get,

$$\begin{aligned} \frac{\partial^2 \tilde{J}^{lse}}{\partial x \partial q} &= \int_{s=0}^{s_1} \overline{\pi}_s^{rt} p_s(l - q - x) \alpha(s) ds \\ \frac{\partial^2 \tilde{J}^{lse}}{\partial q^2} &= \int_{s=0}^{s_1} \overline{\pi}_s^{rt} p_s(l - q - x) \alpha(s) ds \\ &\quad + \int_{s=s_2}^{1} \overline{\pi}_s^{rt} p_s(l - q) \alpha(s) ds \\ \frac{\partial^2 \tilde{J}^{lse}}{\partial x^2} &= \int_{s=0}^{s_1} \overline{\pi}_s^{rt} p_s(l - q - x) \alpha(s) ds \end{aligned}$$

It follows that the Hessian will be of the form $\begin{bmatrix} (a+b) & a \\ a & a \end{bmatrix}$, where $a = \int_{s=0}^{s_1} \overline{\pi}_s^{rt} p_s(l - q - x) \alpha(s) ds$, $b = \int_{s=s_2}^{1} \overline{\pi}_s^{rt} p_s(l - q) \alpha(s) ds$.

Now, by Silvester's criterion, this Hessian is positive semi-definite. Hence by convexity the minimizers of LSE cost satisfy (30). □

Lemma 3. *The function $\tilde{J}^{agg}(x)$ is convex in x. The unique minimizer x^{agg} is given by,*

$$(\phi'(x^{agg}) - \pi^{sp}) \mathbb{E}_s[\mathbb{I}\{y_s = x^{agg}\}] - \pi^o = 0 \tag{31}$$

Proof. Taking the derivative of $\tilde{J}^{agg}(x)$, we get

$$\begin{aligned} \frac{\partial \tilde{J}^{agg}}{\partial x} &= -\pi^o - \int_{s=0}^{s_1} (\pi^{sp} - \phi'(x)) \alpha(s) ds \\ \frac{\partial^2 \tilde{J}^{agg}}{\partial x^2} &= \int_{s=0}^{s_1} \phi''(x) \alpha(s) ds. \end{aligned}$$

Since the second derivative is positive, $\tilde{J}^{agg}(x)$ is convex in x. By convexity, the minimizer x^{agg} satisfies (31). □

Next we show the continuity of $\tilde{q}^{lse}(\pi^o, \pi^{sp})$ and $x^{lse}(\pi^o, \pi^{sp})$ in (π^o, π^{sp}) using the implicit function theorem. We omit the details for the proof of continuity of $x^{agg}(\pi^o, \pi_{sp})$.

Lemma 4. *The minimizers of LSE cost* $\tilde{J}^{lse}(q, x)$ *i.e.* $(\tilde{q}^{lse}(\pi^o, \pi^{sp}), \tilde{x}^{lse}(\pi^o, \pi^{sp}))$ *are continuous in its arguments.*

Proof. By Lemma (2), the minimizers $\tilde{q}^{lse}, x^{lse}$ satisfy conditions (30). Define,

$$f(q, x, \pi^o, \pi^{sp}) = \begin{pmatrix} f_1(q, x, \pi^o, \pi^{sp}) \\ f_2(q, x, \pi^o, \pi^{sp}) \end{pmatrix}$$

$$\begin{aligned} \text{Where} \quad f_1(q, x, \pi^o, \pi^{sp}) &= \frac{\partial \tilde{J}^{lse}}{\partial q} \\ &= \pi^{da} - \mathbb{E}_s[\overline{\pi}_s^{rt} P_s(l - q - y_s)] \\ f_2(q, x, \pi^o, \pi^{sp}) &= \frac{\partial \tilde{J}^{lse}}{\partial x} \\ &= \pi^o + \pi^{sp} \mathbb{E}_s[\mathbb{I}\{y_s = x\}] \\ &\quad - \mathbb{E}_s[\overline{\pi}_s^{rt} P_s(l - q - y_s) \mathbb{I}\{y_s = x\}] \end{aligned} \tag{32}$$

Then the minimizers $\tilde{q}^{lse}, x^{lse}$, satisfy $f = 0$. From Lemma (2) the partial derivatives of $\frac{\partial \tilde{J}^{lse}}{\partial q}$ and $\frac{\partial \tilde{J}^{lse}}{\partial x}$ exist. Hence $f(q, x, \pi^o, \pi^{sp})$ is continuously differentiable w.r.t q and x. Also the derivatives of f_1 and f_2 w.r.t π_o and π^{sp} exists and is given by,

$$\frac{\partial f}{\partial \pi^o} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \frac{\partial f}{\partial \pi^{sp}} = \begin{pmatrix} -\mathbb{E}_s[\mathbb{I}\{0 < y_s < x\}] \\ \mathbb{E}_s[\mathbb{I}\{y_s = x\}] \end{pmatrix}$$

This implies that f is continuously differentiable w.r.t q, x, π^o and π^{sp}. From Lemma (2),

$$H = \begin{bmatrix} \frac{\partial f}{\partial q} & \frac{\partial f}{\partial x} \end{bmatrix} > 0$$

at points (q, x, π^o, π^{sp}) where $\pi^o > 0, \pi^{sp} \geq \pi^{da}$ and $f = 0$. Then by implicit function theorem there exists continuous functions $g_1 : (\pi^o, \pi^{sp}) \to q$ and $g_2 : (\pi^o, \pi^{sp}) \to x$ such that the minimizers of LSE cost $\tilde{J}^{lse}(q, x)$ are given by $\tilde{q}^{lse} = g_1(\pi^o, \pi^{sp})$ and $x^{lse} = g_2(\pi^o, \pi^{sp})$. Hence the minimizers $(\tilde{q}^{lse}(\pi_o, \pi_{sp}), x^{lse}(\pi^o, \pi^{sp}))$ are continuous in its arguments. □

Similarly we can show that the minimizer of the aggregator's cost, $x^{agg}(\pi^o, \pi^{sp})$ is a continuous function of its arguments.

We now show the existence of competitive equilibrium for the options market.

Proof. Let

$$z(\pi^o, \pi^{sp}) = x^{lse}(\pi^o, \pi^{sp}) - x^{agg}(\pi^o, \pi^{sp})$$

By Lemma 4, $z(\pi^o, \pi^{sp})$ is a continuous function of π^o and π^{sp}. According to the definition, prices π^{*o}, π^{*sp} supports an equilibrium if $z(\pi^{*o}, \pi^{*sp}) = 0$. So, it is sufficient to show that there exists two pair of prices (π_1^o, π_1^{sp}) and (π_2^o, π_2^{sp}) such that

$$z(\pi_1^o, \pi_1^{sp}) > 0, \; z(\pi_2^o, \pi_2^{sp}) < 0$$

Then the existence of prices (π^{*o}, π^{*sp}) at which $z(\pi^{*o}, \pi^{*sp}) = 0$ follows from the continuity of the function $z(\cdot, \cdot)$.

We only give an intuitive proof sketch. Details are omitted due to page limitation.

If π_1^o and π_1^{sp} are very small, the LSE will prefer to get more load reduction. But the aggregator may not want to offer any load reduction because the reward is not enough to offset the disutility from load curtailment. So, $x^{lse}(\pi_1^o, \pi_1^{sp}) - x^{agg}(\pi_1^o, \pi_1^{sp})$ will be positive.

On the other hand, if π_2^o and π_2^{sp} are very high, the aggregator may offer more load curtailment. But the LSE may prefer very small or no load curtailment at all. It may be better for the LSE to purchase energy from DAM or RTM. So, in this case, $x^{lse}(\pi_2^o, \pi_2^{sp}) - x^{agg}(\pi_2^o, \pi_2^{sp})$ will be negative.

Now, the existence of (π^{*o}, π^{*sp}) follows from the continuity of $z(\cdot, \cdot)$. □