The Geometry of Locational Marginal Prices$^{\pi}$

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***Abstract*—We show that the set of feasible loads under economic dispatch can be decomposed into a union of polytopes, each of which is uniquely determined by a set of congested lines and marginal generators. From this decomposition we derive closed form expressions for the locational marginal prices (LMPs) within each set. We illustrate these results using simple examples, and demonstrate their value in exploiting load flexibility for congestion relief.**

***Index Terms*—Locational marginal prices, economic dispatch, sensitivities, electricity markets.**

NOMENCLATURE

m, n	Number of lines, buses
p	Bus real power net injection vector
g	Bus generation vector
l	Bus load vector
$J(g)$	Production cost functions
H, H_c	Shift factor matrices
C, C_c, C_u	Line capacity vectors
G	Generation upper limit vector
λ	LMP vector
γ	Power balance shadow price
μ, μ_c	Congestion shadow prices
ν_l, ν_u	Generation limit shadow prices
Λ_m, Λ_n	Marginal and nonmarginal generation selection matrices
x	Load dummy variable
$\mathcal{S}(l)$	Set of loads
$\mathcal{C}(l)$	Set of congested lines
$\mathcal{M}(l)$	Set of marginal generators
$\mathcal{G}_l, \mathcal{P}_l$	Generation set, net injection set

I. INTRODUCTION

The electricity industry has experienced a significant shift over the last three decades, moving from vertically integrated monopolies to deregulated open markets. Wholesale power markets in the US are designed based on a Federal Energy Regulatory Commission (FERC) White Paper [1]. Key features of this market design proposal are a two-settlement system, consisting of a day-ahead market for bulk power transactions, and a real-time market for supply and demand power balancing. Another key feature is the congestion pricing mechanism based on the concept of locational marginal pricing (LMP) [2]. The LMP at a bus is the minimal cost to supply an additional unit of power to a load at that bus without violating network security constraints. A review of the evolution of electricity markets is given in [3].

The LMPs are computed as shadow prices of real power balance constraints of an optimal power flow (OPF) problem that maximizes the system social welfare [4]. The LMP at each bus can be decomposed as the sum of three terms, the marginal cost of energy, the marginal cost of network losses, and the marginal cost of network congestion [5]. In a lossless network, and in the absence of congestion, the price of electricity at every bus is identical and is the marginal cost of the marginal generator.

In the two-settlement market structure, the day-ahead market is a forward market in which the generators and loads can submit their offers and bids for each hour of the subsequent day. The ISO first calculates a generation commitment schedule by solving a security-constrained unit commitment (SCUC) problem. Then the clearing price for each hour of this day is computed by solving a security-constrained economic dispatch (SCED) problem for each hour, given the commitment schedule. The real-time market is designed to provide power balance at a much lower time scale, typically 5 to 15 minutes. Consequently, there are two LMPs in the wholesale market, day-ahead LMPs and real-time LMPs. The real-time LMP at each bus is computed by solving an incremental optimal power flow problem that accounts for feasible and security-constrained deviations with respect to the previously computed day-ahead power flow. If the market is well designed, the majority of power is transacted in the day-ahead market and the real-time market is used to resolve disparity between actual and forecasted demand, and account for any network outages or contingencies.

Much research has been devoted to locational marginal pricing since it was proposed in [6], [4], and developed later in [2], [7], as a means to deal with congestion management in electricity markets. For a detailed analysis of the different aspects of the design and use of LMPs in electricity markets, the reader is directed to [8], [9], [5], [10]. In most markets the DC-OPF model is used to run SCED and compute LMPs [11], [12], [13]. This is due to the fact that the LMPs are straightforward to obtain as the Lagrange multipliers of a linear program. Economic dispatch using the AC-OPF model is nonlinear and nonconvex, making it more difficult to solve.

The geometry of the feasible injection region of power networks has previously been studied in [14], using an AC power flow model with various operational constraints. The geometry of LMPs is studied in [15], where the state-space of a power system represented by a DC power flow model is

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partitioned into polyhedral price regions in which the LMPs are constant. This work is extended in [16] for probabilistic forecasting of LMP using a multiparametric programming formulation, however, closed-form expressions are not obtained for the LMP vector. In [17], a method for estimating how the binding constraints change with varying load is proposed. An approach for computing sensitivities of the LMP vector with respect to load and other parameters of the problem is reported in [18], however this can only handle small load variations where the binding constraints do not change. In [19], the DC optimal power flow model is modified to consider losses, and sensitivity expressions for the LMP vector with respect to load variations are obtained. In [20] a continuous locational marginal pricing method is proposed to avoid step changes when a change in the set of marginal generators or congested lines occurs.

The main contributions of this paper are as follows. Using the DC power flow model, the feasible load set of the economic dispatch problem is expressed as a union of polytopes, each of which is characterized by a set of congested lines and a set of marginal generators.[1] Each polytope can be obtained as the Minkowski difference of a marginal generation set and a congested injection set. We consider both linear and quadratic generator cost functions, and in both cases obtain explicit expressions for the LMP vector in terms of the load vector. Finally, the results are applied to exploit load flexibility and derive economic benefit for the loads.

The paper is organized as follows. In Section II we describe the model and the economic dispatch problem. Section III details the theoretical results on congestion sets, closed form expressions for LMPs, and illustrative examples using a simple two-bus network. Section IV describes the application of the results for load flexibility, and Section V concludes the paper.

II. Background and set-up

A. Network modelling

We consider a power network modelled by a connected undirected graph $\mathcal{G}(\mathcal{N},\mathcal{E})$, where $\mathcal{N} := \{1,\dots,n\}$ is the set of n nodes (buses), and $\mathcal{E} := \{(i,j) : i,j \in \mathcal{N}\}$ is the set of m edges (transmission lines), and $(i,j) \in \mathcal{E}$ means that there is a line connecting buses i and j. Throughout this paper we will assume the standard linear DC power flow model, where lines are lossless, voltage magnitudes are all equal, and voltage angle differences are small. The model is common but we use notation from [21]. Power flows on the network are governed by a shift-factor matrix $H \in \mathbb{R}^{2m\times n}$ which linearly maps the vector of nodal net injections to the vector of directional line flows. We denote $C \in \mathbb{R}_+^{2m}$ as the vector of line capacities. Given that nodal injections must sum to zero across the network, we can write the set of feasible power injections as the polytope $\mathcal{P} \subset \mathbb{R}^n$.

$$\mathcal{P} := \left\{p \in \mathbb{R}^n \mid Hp \le C,\ \mathbf{1}^\top p = 0\right\} \tag{1}$$

$\mathcal{P}$ is compact, since $\mathrm{rank}(H) = n-1$ and $\mathbf{1}^\top$ is linearly independent from the rows of H.

[1] A polytope is defined as a convex, compact, polyhedral set.

B. Generation and load modelling

We assume aggregated generation g_i at node i, with a generation vector $g \in \mathbb{R}^{n_B}$ for the whole network. This generation has variable and fixed costs of production, but faces no startup, shutdown or no-load costs, nor ramping constraints. The cost of generation is measured by an increasing convex function $J(g)$ taking the general form[2]

$$J(g) = J_1^\top g + \frac{1}{2} g^\top J_2 g \tag{2}$$

where $J_1 \in \mathbb{R}^n$ and $J_2 \in \mathbb{R}^{n\times n}$ are the linear and quadratic costs of generation respectively. It should be noted that J_2 is a diagonal matrix. We assume the generation capacity is constrained as

$$0 \le g \le G \tag{3}$$

where $G \in \mathbb{R}^n$ is the vector of upper generation limits.

We assume an aggregated load l_i at each bus i, with a load vector $l \in \mathbb{R}^n$ for the whole network. This load is considered inelastic. It should be noted that $p = g - l$.

C. The economic dispatch problem

At the begining of each operating day, the ISO receives supply offers from the generators and demand bids from the loads for each hour of the subsequent day. Each supply offer reports the marginal cost function over a feasible production interval. Using this information, and assuming a lossless DC power flow model, the economic dispatch problem is formulated as

$$\begin{aligned}
\mathbf{ED}:\ \min_{g}\ & J(g) && (4)\\
\text{s.t. } & \mathbf{1}^\top(l-g) = 0 && (5)\\
& H(g-l) - C \le \mathbf{0} && (6)\\
& -g \le \mathbf{0} && (7)\\
& g - G \le \mathbf{0} && (8)
\end{aligned}$$

Associating Lagrange multipliers to each constraint, the Lagrangian for $\mathbf{ED}$ is given by

$$\begin{aligned}
\Phi(g,\gamma,\mu,\nu_l,\nu_u) =& J(g) + \gamma(\mathbf{1}^\top(l-g)) + \nu_l^\top(-g)\\
&+ \nu_u^\top(g-G) + \mu^\top(H(g-l)-C)
\end{aligned} \tag{9}$$

and the Karush-Kuhn-Tucker (KKT) conditions are

$$\begin{aligned}
J'(g) - \gamma + H^\top\mu - \nu_l + \nu_u = \mathbf{0} && (10)\\
\mathbf{1}^\top(l-g) = 0 && (11)\\
\mu^\top(H(g-l)-C) = 0 && (12)\\
H(g-l) - C \le \mathbf{0} && (13)\\
\nu_l^\top g = 0 && (14)\\
\nu_u^\top(g-G) = 0 && (15)\\
-g \le \mathbf{0} && (16)\\
g - G \le \mathbf{0} && (17)\\
-\mu, -\nu_L, -\nu_U \le \mathbf{0} && (18)
\end{aligned}$$

[2] A constant fixed term J_0 could be added to take account of fixed costs, however this term has no influence on the LMPs and is not explicitly considered here.

where $J'(g)$ denotes the gradient of the cost function $J(g)$. The KKT conditions are necessary and sufficient optimality conditions since $\mathbf{ED}$ is a convex quadratic optimization program with linear constraints.

III. Main results

In this section we will demonstrate that the feasible set of loads can be partitioned into a set of convex polytopes, where within each polytope the set of congested lines and marginal generators does not change. We characterize each polytope as the Minkowski sum of a net injection set and a generation set. We will also obtain closed form expressions for the LMP vector within each polytope.

A. Preliminaries

It will be convenient to define as $\lambda \in \mathbb{R}^n$ a particular combination of the Lagrange multipliers

$$\lambda = \gamma \mathbf{1} - H^\top \mu \tag{19}$$

Definition 1: The LMP vector is defined as the vector of sensitivities of the optimal production cost with respect to the loads while satisfying all the physical constraints.

In the absence of losses, the LMP vector equals λ, since the complementary slackness conditions guarantee that the optimal production cost equals the Lagrangian function Φ.

$$\frac{\partial \Phi}{\partial l} = \gamma \mathbf{1} - H^\top \mu = \lambda \tag{20}$$

Definition 2: A line (i,j) is congested for the load vector l if it reaches its capacity limit. The set of all lines that are congested for the load vector l is denoted by $\mathcal{C}(l)$.

The Lagrange multiplier vector μ has $2m$ elements, each of them corresponding to a network line (i,j). The elements corresponding to uncongested lines are zero while the elements corresponding to congested lines are nonnegative. If we assume that we know the congested lines for a certain network operating condition, then we can partition (if needed, after a permutation of rows) the matrices H and C as follows $H = \begin{bmatrix} H_c^\top & H_u^\top \end{bmatrix}^\top$, $C = \begin{bmatrix} C_c^\top & C_u^\top \end{bmatrix}^\top$, where the subscript c denotes congested lines, and u uncongested lines. The vector of the corresponding Lagrange multipliers is $\mu = \begin{bmatrix} \mu_c^\top & \mu_u^\top \end{bmatrix}^\top$ with $\mu_u = \mathbf{0}$ and $\mu_c \geq \mathbf{0}$.

Remark 1: Definition 1 establishes that the LMP vector λ can be decomposed into an energy term and a congestion term $\lambda = \lambda^e + \lambda^c$. The energy term $\lambda^e = \gamma\mathbf{1}$ has the same value at every network bus. The congestion term $\lambda^c = -H^\top \mu$ has a different contribution for each network bus. Moreover, since by row permutation μ can be expressed as $\mu = [\mu_c^\top \ \mathbf{0}^\top]^\top$, $\lambda^c = -H_c^\top \mu_c$ where H_c is the relevant submatrix of H. In the absence of congestion, we see the well known fact that the LMPs are the same at each bus in the network. Since in this case $\mu = \mathbf{0}$ and $\lambda = \gamma \mathbf{1}$.

For each load $l \in \mathbb{R}^n$, the generation levels at each bus are obtained by solving the $\mathbf{ED}$ problem. From the KKT conditions a generator can be in one of the following three states:

1) At capacity: $g_i = G_i$, $\nu_{l,i} = 0$, $\nu_{u,i} \geq 0$.
2) Unused: $g_i = 0$, $\nu_{l,i} \geq 0$, $\nu_{u,i} = 0$.
3) Marginal: $0 < g_i < G_i$, $\nu_{l,i} = \nu_{u,i} = 0$.

A marginal generator is so called, since its marginal cost determines the LMP at its node. That is $J'(g_i) = \lambda_i$, which follows from (10). The set of all marginal generators for the load vector l is denoted by $\mathcal{M}(l)$.

For any feasible load vector l, the generation vector can be decomposed into marginal and nonmarginal generation $g = \Lambda_m g_m + \Lambda_n g_n$, where g_m and g_n denote the marginal and nonmarginal generation vectors respectively. Accordingly, the LMP vector λ can be decomposed as $\lambda = \Lambda_m \lambda_m + \Lambda_n \lambda_n$. Note that $\Lambda_m^\top [\Lambda_m \ \Lambda_n] = [I \ \ \mathbf{0}]$. The nonmarginal generators are fixed either to their lower or upper limit and can be considered parameters of the problem.

Definition 3: The set of loads for which the congested lines of the network and the marginal generation for that load vector l do not change is given by

$$\mathcal{S}(l) = \{x \in \mathbb{R}^n \,|\, \mathcal{C}(x) = \mathcal{C}(l), \ \mathcal{M}(x) = \mathcal{M}(l)\} \tag{21}$$

Theorem 1: The load vectors $x \in \mathcal{S}(l)$ satisfy the following conditions

$$\Lambda_m^\top (J'(g) - \lambda) = \mathbf{0} \tag{22}$$
$$\gamma \mathbf{1} - H_c^\top \mu_c - \lambda = \mathbf{0} \tag{23}$$
$$\mathbf{1}^\top (x - g) = 0 \tag{24}$$
$$H_c(g - x) - C_c = \mathbf{0} \tag{25}$$
$$H_u(g - x) - C_u \leq \mathbf{0} \tag{26}$$
$$\Lambda_n^\top g - g_n = \mathbf{0} \tag{27}$$
$$-\Lambda_m^\top g \leq \mathbf{0} \tag{28}$$
$$\Lambda_m^\top (g - G) \leq \mathbf{0} \tag{29}$$

where g_n is the fixed vector of nonmarginal generation.

Proof: Conditions (22)–(24) follow from the KKT conditions for the load l where the set of congested lines $\mathcal{C}(l)$ and the set of marginal generators $\mathcal{M}(l)$ are known in advance. Conditions (25)–(29) ensure that no uncongested line becomes congested and no nonmarginal generator becomes marginal. Thus, these conditions are valid not only for l but for any load $x \in \mathcal{S}(l)$.

□

B. General results

Let us begin by studying the set of loads $\mathcal{S}(l)$. We can first conclude that $\mathcal{S}(l)$ is convex, compact, and polyhedral, thus a polytope. This is due to the fact that we can prove it is the Minkowski sum of two other polytopes, a generation set $\mathcal{G}_l$, and a set of net injections $\mathcal{P}_l$. This is achieved by reintroducing the vector of net real power injections $p = g - l$ and splitting the KKT conditions. While proving the compactness and convexity of $\mathcal{S}(l)$, this formulation also provides a more intuitive sense for the construction of $\mathcal{S}(l)$.

Definition 4: The generation set $\mathcal{G}_l$ is the set of feasible generation vectors for the set of marginal generation $\mathcal{M}(l)$ and producing congestion in the lines given by the congestion set $\mathcal{C}(l)$.

The set $\mathcal{G}_l$ is given by

$$\begin{aligned}\mathcal{G}_l := \{ & g \in \mathbb{R}^n \mid \exists\, \gamma \in \mathbb{R},\ \mu_c \in \mathbb{R}^{|\mathcal{C}(l)|}, \\ & \text{such that } 0 \le \Lambda_m^\top g \le \Lambda_m^\top G, \Lambda_n^\top g = g_n, \\ & \Lambda_m^\top (J'(g) - (\gamma \mathbf{1} - H_c^\top \mu_c)) = \mathbf{0}\}\end{aligned} \tag{30}$$

where $|\mathcal{C}(l)|$ denotes the cardinality of the congestion set, *i.e.* the number of congested lines. This set is the intersection of a compact convex set, $\{g \in \mathbb{R}^n \,|\, 0 \le \Lambda_m^\top g \le \Lambda_m^\top G, \Lambda_n^\top g = g_n\}$, and a convex set $\{g \in \mathbb{R}^n \,|\, \Lambda_m^\top (J'(g) - (\gamma \mathbf{1} - H_c^\top \mu_c)) = \mathbf{0}\}$, thus it is compact and convex. For both linear and quadratic cost functions it can be shown that this set is also a polytope, although for higher order cost functions this is not necessarily true.

Definition 5: The net injection set $\mathcal{P}_l$ for the load set $\mathcal{S}(l)$ is the set of feasible net injections congesting the lines given by the congestion set $\mathcal{C}(l)$.

The set $\mathcal{P}_l$ is given by

$$\mathcal{P}_l := \left\{ p \in \mathbb{R}^n \mid H_c p = C_c,\ H_u p \le C_u,\ \mathbf{1}^\top p = 0 \right\} \tag{31}$$

where the partition $H = [H_c^\top \ H_u^\top]^\top$ is given by the set of congested lines $\mathcal{C}(l)$. Similarly to the full feasible set of net injections, $\mathcal{P}$, it is easy to show that this set is a compact convex polytope.

Theorem 2: The set $\mathcal{S}(l)$ is the Minkowski sum of $\mathcal{G}_l$ and $-\mathcal{P}_l$[3], both compact convex polytopes, thus $\mathcal{S}(l)$ itself is a compact convex polytope.

Proof. From the definition of the net injection vector p, any load vector $x \in \mathcal{S}(l)$ can be expressed as the difference of two vectors g and p, $x = g - p$, such that $g \in \mathcal{G}_l$ and $p \in \mathcal{P}_l$. So we may write

$$\mathcal{S}(l) = \{x \,|\, x = g - p,\ g \in \mathcal{G}_l,\ p \in \mathcal{P}_l\} \tag{32}$$

□

We are effectively assigning equations (22)–(29), to their relevant physical variable, and then coupling these constraints through $p = g - l$ to obtain the load set.

A caveat to these results is that the sets $\mathcal{S}(l)$ are not disjoint, i.e. their boundaries intersect, such that in the case of limiting congestion or generation capacity, a load could be considered to be in multiple sets at the same time. This is a degenerate case however, and would be incredibly rare in practical application. Another caveat is that one must know ahead of time which generators are marginal and which lines will be congested for a given load to be able to create these sets. At present no guaranteed method exists to predict ahead of time what the qualitative state of the power system will be for any given load, however work in [17] attempts to predict these changes with load variation. In practice one may be able to observe typical patterns of qualitative behavior in the power system, and compute these congestion sets, and the LMP formulae that will follow, offline ahead of time.

Remark 2: The set of loads for which no congestion occurs, i.e. all l for which $\mathcal{S}(l) = \{x \in \mathbb{R}^n \,|\, \mathcal{C}(x) = \emptyset,\ \mathcal{M}(x) = \mathcal{M}(l)\}$, can be written as the Minkowski difference of an economic generation set and an uncongested feasible injection set. The economic generation set is the set of generation that would be selected based on merit ordering, and the uncongested feasible injection set is the set of injections such that no congestion occurs in the network. This is true since an uncongested network can be treated as a single bus and generators can be selected based on merit ordering. This is proved in [22].

We finish this section by studying a general result for the LMP vector in the set $\mathcal{S}(l)$. We will make use of equations (22) and (23). The following theorem exploits the linear structure of the DC power flow model and provides a closed form expression for the LMP vector.

Theorem 3: Let Λ_m the matrix that establishes the marginal generation for a given load l. Define $Q = [\mathbf{1} \ \ -H^\top]$, then $\Lambda_m^\top \lambda$ belongs to the linear space spanned by the columns of $\Lambda_m^\top Q$ and the LMP vector satisfies $\lambda = Q(\Lambda_m^\top Q)^\dagger \lambda_m$ where $(.)^\dagger$ denotes the pseudoinverse.

Proof: The LMP vector for a load vector l is unique and satisfies $\lambda = Qv$ where $Q = [\mathbf{1} \ -H^\top]$ and $v = [\gamma \ \mu^\top]^\top$, from (23). v satisfies the linear equation $\Lambda_m^\top Q v = \Lambda_m^\top \lambda$ and therefore $\Lambda_m^\top \lambda$ belongs to the linear space spanned by the columns of $\Lambda_m^\top Q$. Premultiplying both sides by $(\Lambda_m^\top Q)(\Lambda_m^\top Q)^\dagger$ and using the properties of the pseudoinverse, we obtain

$$\begin{aligned}\Lambda_m^\top Q v &= (\Lambda_m^\top Q)(\Lambda_m^\top Q)^\dagger \Lambda_m^\top Q v \\ &= (\Lambda_m^\top Q)(\Lambda_m^\top Q)^\dagger \Lambda_m^\top \lambda\end{aligned}$$

Thus, $v = (\Lambda_m^\top Q)^\dagger \Lambda_m^\top \lambda$ and the LMP vector satisfies $\lambda = Q(\Lambda_m^\top Q)^\dagger \lambda_m$.

□

Remark 3: The expression $\lambda = Q(\Lambda_m^\top Q)^\dagger \lambda_m$ is valid whenever the set of marginal generation established by the matrix Λ_m does not change. The result can be further simplified, knowing that elements of μ corresponding to uncongested lines are zero. Thus, after a row permutation $\mu = [\mu_c^\top \ \mathbf{0}^\top]^\top$, $H = [H_c \ H_u]$, and then defining $Q_c = [\mathbf{1} \ \ -H_c^\top]$ and $v_c = [\gamma \ \mu_c^\top]^\top$, we have that $Qv = Q_c v_c$ and the LMP vector can be expressed in terms of Q_c as $\lambda = Q_c(\Lambda_m^\top Q_c)^\dagger \lambda_m$. In addition, if Q_c has full column rank and its rank equals the number of marginal generators, then the matrix $\Lambda_m^\top Q_c$ is invertible.

Since the cost of the marginal generation is known and $\lambda_m = \Lambda_m^\top J'(g)$ from (22), then Theorem 3 can be used to obtain a closed-form expression for the LMP vector. We will obtain these expressions for the case of both linear and quadratic generation costs.

C. Linear case

Let $\mathcal{S}(l)$ be the set of loads defined in Definition 3. The generation cost function is $J(g) = J_1^\top g$, and its gradient is $J'(g) = J_1$. From Theorem 3 the LMP vector is given as

$$\lambda = Q_c(\Lambda_m^\top Q_c)^\dagger \Lambda_m^\top J_1. \tag{33}$$

Theorem 4: The LMP vector λ is constant in the set $\mathcal{S}(l)$.

Proof: In the interior of the set $\mathcal{S}(l)$, the LMP vector is given by equation (33) and is constant. □

[3] *i.e.* the Minkowski difference of $\mathcal{G}_l$ and $\mathcal{P}_l$

D. Quadratic case

In this case the generation cost function is $J(g) = J_1^\top g + \frac{1}{2}g^\top J_2 g$ and its gradient is $J'(g) = J_1 + J_2 g$. From condition (22), $\Lambda_m^\top \lambda = \Lambda_m^\top J'(g)$, thus $\Lambda_m^\top \lambda = \Lambda_m^\top (J_1 + J_2 g)$. The generation vector is $g = \Lambda_m g_m + \Lambda_n g_n$ where g_n is the nonmarginal generation vector that is constant in $\mathcal{S}(l)$. Since J_2 is a diagonal matrix, then $\Lambda_m^\top J_2 \Lambda_n = \mathbf{0}$ and the marginal generation vector is equal to

$$g_m = (\Lambda_m^\top J_2 \Lambda_m)^{-1} \Lambda_m^\top (\lambda - J_1) \tag{34}$$

Recalling that $Q_c = [\mathbf{1} \;\; -H_c^\top]$ we note that

$$\begin{bmatrix} 0 \\ C_c \end{bmatrix} = -Q_c^\top p \tag{35}$$

Substituting in Equations (24) and (34), and recalling that $\lambda = Q_c v_c$, for $x \in \mathcal{S}(l)$ we see that

$$\begin{aligned} \begin{bmatrix} 0 \\ C_c \end{bmatrix} &= -Q_c^\top (g - x) \\ &= -Q_c^\top (\Lambda_m g_m + \Lambda_n g_n - x) \\ &= -Q_c^\top (R(Q_c v_c - J_1) + \Lambda_n g_n - x) \end{aligned}$$

where $R = \Lambda_m (\Lambda_m^\top J_2 \Lambda_m)^{-1} \Lambda_m^\top$. Then,

$$Q_c^\top R Q_c v_c = Q_c^\top (x + R J_1 - \Lambda_n g_n) - \begin{bmatrix} 0 \\ C_c \end{bmatrix} \tag{36}$$

and using the properties of the psudoinverse, we finally obtain

$$\lambda = Q_c (Q_c^\top R Q_c)^\dagger \left(Q_c^\top (x + R J_1 - \Lambda_n g_n) - \begin{bmatrix} 0 \\ C_c \end{bmatrix} \right) \tag{37}$$

Thus we state

Theorem 5: The LMP vector λ is a linear function of the load vector $x \in \mathcal{S}(l)$.

After computing the LMP vector in equation (37) for a given load, the assignment of power to each generator can be obtained using equation (34).

Throughout the feasible load set, the LMP is a piecewise linear function of the load, since the gain term will change from set to set.

E. An illustrative example: The two-bus case

We illustrate the results using a simple two-bus example shown in Fig. 1 with line susceptance b. The loads are l_1, l_2, the generation is g_1, g_2, the line capacity is C and the generation limits are G_1 and G_2. We will also assume that $C < G_i$. The shift factor matrix H is given by

$$H = \begin{bmatrix} -1 + \frac{1}{b} & -1 \\ 1 - \frac{1}{b} & 1 \end{bmatrix}$$

1) The linear case: The marginal costs of generation are $J_1'(g) = \pi_1$ and $J_2'(g) = \pi_2$ with $\pi_1 < \pi_2$.

The feasible load set is depicted in Fig 2 and is the union of three sets $\mathcal{S}_1$, $\mathcal{S}_2$ and $\mathcal{S}_3$ that are given as:

$$\mathcal{S}_i = \{ l \in \mathbb{R}^2 \,|\, l \geq \mathbf{0}, l \in \mathcal{G}_i - \mathcal{P}_i \}, \;\; i = 1, 2, 3 \tag{38}$$

where

$$\mathcal{P}_1 = \mathcal{P}_2 = \{ p \,|\, p_1 + p_2 = 0, p_1 \leq C, p_2 \leq C \} \tag{39}$$
$$\mathcal{P}_3 = \{ p \,|\, p_1 + p_2 = 0, p_1 = C, p_2 \leq C \} \tag{40}$$
$$\mathcal{G}_1 = \{ g \,|\, 0 \leq g_1 \leq G_1, g_2 = 0, \lambda = \pi_1 \mathbf{1} \} \tag{41}$$
$$\mathcal{G}_2 = \{ g \,|\, 0 \leq g_2 \leq G_2, g_1 = G_1, \lambda = \pi_2 \mathbf{1} \} \tag{42}$$
$$\mathcal{G}_3 = \{ g \,|\, 0 \leq g_i \leq G_i, \lambda_i = \pi_i, i = 1, 2 \} \tag{43}$$

The congestion free sets are shown in blue and the congested set is shown in red.

In the sets $\mathcal{S}_1$, $\mathcal{S}_2$, there is no congestion and the marginal generators are the generator in bus 1 with marginal cost π_1 in $\mathcal{S}_1$ and the generator in bus 2 with marginal cost π_2 in $\mathcal{S}_2$, respectively. In the set $\mathcal{S}_3$, both generators are marginal and the line conecting bus 1 and bus 2 is congested. The LMP vectors satisfy equation (33) for each load set, $\mathcal{S}_i$.

$$\lambda = Q_1 v_1, \;\; Q_1 = \mathbf{1}, v_1 = \pi_1, \;\; l \in \mathcal{S}_1 \tag{44}$$
$$\lambda = Q_2 v_2, \;\; Q_2 = \mathbf{1}, v_2 = \pi_2, \;\; l \in \mathcal{S}_2 \tag{45}$$
$$\begin{aligned} &\lambda = Q_3 v_3, \;\; Q_3 = [\mathbf{1} \;\; -H_3^\top], \\ &\quad H_3 = \begin{bmatrix} -1 + \frac{1}{b} & -1 \end{bmatrix}, v_3 = \frac{1}{2} \begin{bmatrix} \pi_1 + \pi_2 \\ 2(\pi_2 - \pi_1) \end{bmatrix}, \;\; l \in \mathcal{S}_3 \end{aligned} \tag{46}$$

2) The quadratic case: The marginal costs of generation are $J_1'(g) = \pi_1 g$ and $J_2'(g) = \pi_2 g$ with $\pi_1 < \pi_2$.

The feasible load set is depicted if Fig 3 and is the union of three sets $\mathcal{S}_1$, $\mathcal{S}_2$ and $\mathcal{S}_3$ that are given as:

$$\mathcal{S}_i = \{ l \in \mathbb{R}^2 \,|\, l \geq 0, l \in \mathcal{G}_i - \mathcal{P}_i \}, \;\; i = 1, 2, 3 \tag{47}$$

where

$$\mathcal{P}_1 = \mathcal{P}_2 = \{ p \,|\, p_1 + p_2 = 0, p_1 \leq C, p_2 \leq C \} \tag{48}$$
$$\mathcal{P}_3 = \{ p \,|\, p_1 + p_2 = 0, p_1 = C, p_2 \leq C \} \tag{49}$$
$$\mathcal{G}_1 = \{ g \,|\, 0 \leq g_i \leq G_i, \gamma = \pi_i g_i, i = 1, 2, \lambda = \gamma \mathbf{1} \} \tag{50}$$
$$\mathcal{G}_2 = \{ g \,|\, 0 \leq g_2 \leq G_2, g_1 = G_1, \lambda = g_2 \pi_2 \mathbf{1} \} \tag{51}$$
$$\mathcal{G}_3 = \{ g \,|\, 0 \leq g_i \leq G_i, \lambda_i = \pi_i g_i, i = 1, 2 \} \tag{52}$$

As in the linear case, the congestion free sets are shown in blue, and the congested set is shown in red.

The generation sets $\mathcal{G}_1$ and $\mathcal{G}_3$ have the same marginal generators, however in $\mathcal{G}_1$ both LMPs are equal, whereas in $\mathcal{G}_3$ the LMPs are different because the line is congested. In sets $\mathcal{S}_1$, $\mathcal{S}_2$, there is no congestion and consequently the LMP is equal at both buses. In the set $\mathcal{S}_1$ since both generators are marginal, then $\pi_1 g_1 = \pi_2 g_2$. In the set $\mathcal{S}_2$, since there is no congestion and the second generator is marginal, then the LMPs at both buses are equal to the marginal cost of this generator. In the set $\mathcal{S}_3$, both generators are marginal, however the line is congested and the LMPs are different at each bus. The LMP vectors satisfy equation (37) for each load set.

$$\lambda = Q_1 v_1, \;\; Q_1 = \mathbf{1}, v_1 = \frac{\pi_1 \pi_2}{\pi_1 + \pi_2} \mathbf{1}^\top l, \;\; l \in \mathcal{S}_1 \tag{53}$$
$$\lambda = Q_2 v_2, \;\; Q_2 = \mathbf{1}, v_2 = \pi_2 (\mathbf{1}^\top l - G_1), \;\; l \in \mathcal{S}_2 \tag{54}$$
$$\begin{aligned} &\lambda = Q_3 v_3, \;\; Q_3 = [\mathbf{1} \;\; -H_3^\top], H_3 = \begin{bmatrix} -1 + \frac{1}{b} & -1 \end{bmatrix}, \\ &\quad v_3 = \frac{1}{2} \begin{bmatrix} \pi_1 (l_1 + C) + \pi_2 (l_2 - C) \\ 2(\pi_2 (l_2 - C) - \pi_1 (l_1 - C)) \end{bmatrix}, \;\; l \in \mathcal{S}_3 \end{aligned} \tag{55}$$

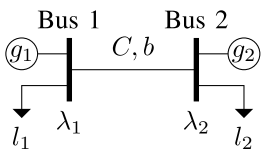

Fig. 1. Two bus network

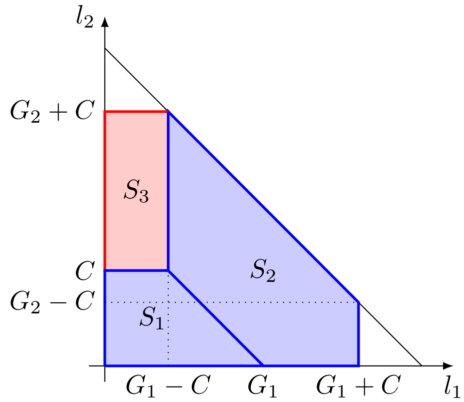

Fig. 2. Load set for the linear case

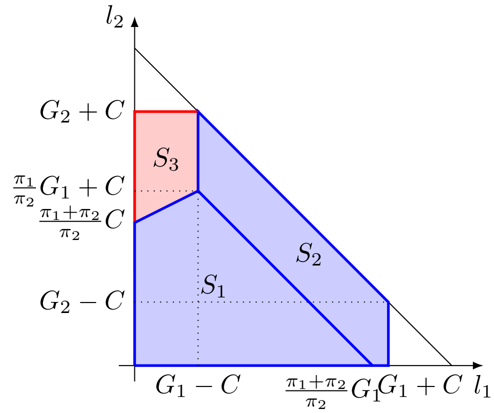

Fig. 3. Load set for the quadratic case

The LMP vector for the set $\mathcal{S}_3$, after computing $\lambda = Q_3 v_3$ is given by

$$\lambda = \begin{bmatrix} \pi_1(l_1 + C) \\ \pi_2(l_2 - C) \end{bmatrix} \tag{56}$$

IV. Applications

While the formulation presented so far is relatively abstract, the closed form expressions for LMPs, and the geometric concept of congestion sets can have many applications. For example the LMP expression could be used in bi-level optimization programs, of which many examples arise in the power systems literature. Here we present an application of congestion sets to illustrate the potential of load flexibility to eliminate congestion in transmission networks and provide economic benefits for market participants.

A. Load Flexibiity for Congestion Free Dispatch

Load flexibility is considered essential for the development of the smart grid. We take a broader definition of load flexibility to encompass both temporal demand response and energy storage. Promising applications include frequency regulation, peak load reduction, reserve capacity, volt/var control, and crucially facilitating increased penetration of renewables and distributed generation on the grid [23], [24], [25].

Another such application of load flexibility is congestion mitigation. Due to out of merit order generation dispatch, congestion causes prices to vary and typically increase across a network, usually resulting in wealth transfers away from end consumers. For instance, from 2008-2013, between 2 and 6% of PJM's total annual billing was attributable to congestion, representing an average annual cost of approximately \$1bn [26].[4] It may be desirable for the loads to avoid congestion in the network so as to reduce their costs.

If we examine the congestion free sets in the two-bus example, shown in blue in Figures 2, 3, we see that whilst the individual sets S_1, S_2, are convex, their union is non-convex. This suggests that if we could use flexibility to move the load vector, we might be able to eliminate congestion for some loads. We illustrate this idea using a simple example, building on the two-bus case with linear generation costs.

We will define three loads, shown in Fig. 4. The red triangular area in Fig. 4 is the set $S_3 \cap \text{conv}(S_1 \cup S_2)$, where $\text{conv}(S)$ denotes the convex hull of S.

$$\begin{aligned} l^{(1)} &= \left(\frac{X - \alpha(G_1 - C)}{(1-\alpha)}, C \right) \in \text{bd}(S_1) \\ l^{(2)} &= \left(G_1 - C, \frac{Y - (1-\alpha)C}{\alpha} \right) \in \text{bd}(S_2) \\ l^{(3)} &= (X, Y) \in S_3 \cap \text{conv}(S_1 \cup S_2) \\ &= (1-\alpha) l^{(1)} + \alpha l^{(2)} \end{aligned}$$

where $\alpha \in (0, 1)$, and $\text{bd}(S)$ denotes the boundary of the set S. $l^{(3)}$ is an arbitrary point in the congested load set $S_3 \cap \text{conv}(S_1 \cup S_2)$, with coordinates (X, Y). $l^{(3)}$ also lies on the line segment connecting $l^{(1)}$ and $l^{(2)}$. This suggests that if $l^{(3)}$ is constant over some time period, it can be met congestion free using flexibility. In other words suppose we need to serve a constant load $l^{(3)}$ over some time period T. Since $l^{(3)}$ cannot be served without congesting the network, suppose the loads admit flexibility and we dispatch $l^{(1)}$ over a time period $(1-\alpha)T$ and $l^{(2)}$ over a time period αT. This results in a congestion free dispatch.

Scenario	$(1-\alpha)T$	αT	Congestion free
Non-flexible	$l^{(3)}$	$l^{(3)}$	No
Flexible	$l^{(1)}$	$l^{(2)}$	Yes

To demonstrate the economic effects of a flexible and congestion free dispatch, we consider each class of market participants as a collective and calculate the system cost (SC),[5] the generation revenue (GR), the load payment (LP), and the merchandising surplus (MS) resulting from each scenario. The MS is the surplus collected by the system operator after paying all generators and receiving all payments from loads, $MS = LP - GR$. It should be noted that the MS is zero in the absence of congestion. We denote the non-flexible scenario (n), and the flexible scenario (f), and we

[4] Specifically this figure refers to the congestion component of LMPs

[5] This is the value of the objective function solved by the system operator *i.e.* the fuel cost of the generators.

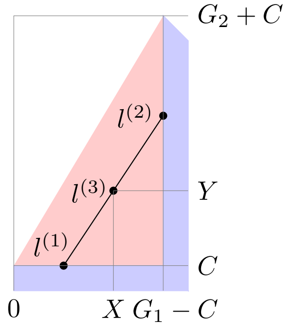

Fig. 4. $l^{(3)} = (1-\alpha)l^{(1)} + \alpha l^{(2)}$

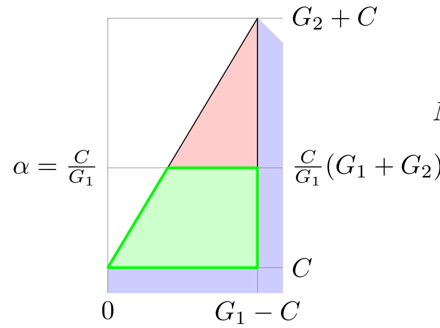

Fig. 5. Set of congested loads which make savings under flexibility

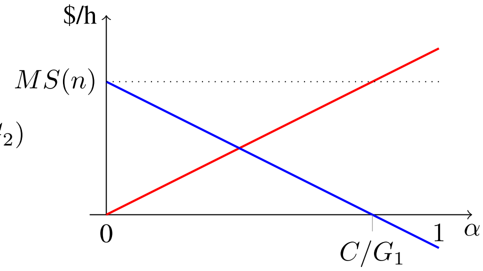

Fig. 6. Load savings in blue, generator profits in red

assume without loss of generality that $T = 1$. The results are shown in Table I.

TABLE I
DATA FOR FLEXIBLE AND NON-FLEXIBLE SCENARIOS

	Non-flexible Scenario
$SC(n)$	$\pi_1 X + \pi_2 Y - (\pi_2 - \pi_1)C$
$GR(n)$	$\pi_1 X + \pi_2 Y - (\pi_2 - \pi_1)C$
$LP(n)$	$\pi_1 X + \pi_2 Y$
$MS(n)$	$(\pi_2 - \pi_1)C$

	Flexible Scenario
$SC(f)$	$\pi_1 X + \pi_2 Y - (\pi_2 - \pi_1)C$
$GR(f)$	$\pi_1 X + \pi_2 Y + (\pi_2 - \pi_1)(\alpha G_1 - C)$
$LP(f)$	$\pi_1 X + \pi_2 Y + (\pi_2 - \pi_1)(\alpha G_1 - C)$
$MS(f)$	0

Examining the differences between the two scenarios, we see that

$$SC(f) - SC(n) = 0$$

$$GR(f) - GR(n) = (\pi_2 - \pi_1)\alpha G_1$$

$$LP(f) - LP(n) = (\pi_2 - \pi_1)(\alpha G_1 - C)$$

We can now draw the following conclusions:

1) The loads receive the same total energy in both cases, since $l^{(3)} = (1-\alpha)l^{(1)} + \alpha l^{(2)}$.
2) The system cost is identical in both cases.
3) Under flexibility, the generator collective receives a nonnegative profit $(\pi_2 - \pi_1)\alpha G_1$.
4) Under flexibility, the load collective makes savings if $LP(f) < LP(n)$. This only occurs if $\alpha < C/G_1$.

Referring to Fig. 4, it is clear that the choice of α is non-unique. Varying α will change the relative location of $l^{(1)}$ and $l^{(2)}$, however the line segment between them will always pass through $l^{(3)}$.[6] It is also clear that for any arbitrary load $l^{(3)} \in S_3 \cap \mathrm{conv}(S_1 \cup S_2)$ there exists a minimum and maximum allowable α, such that $l^{(1)} \in \mathrm{bd}(S_1)$ and $l^{(2)} \in \mathrm{bd}(S_2)$. This, combined with the condition that $\alpha < C/G_1$ for the load collective to make savings, defines a set, shown in green in Fig. 5. Any load in this set can pick a valid α such that $l^{(1)} \in \mathrm{bd}(S_1)$, $l^{(2)} \in \mathrm{bd}(S_2)$, $\alpha < C/G_1$, and the load collective makes savings under flexibility.

[6]One can imagine the points $l^{(1)}$ and $l^{(2)}$ sliding along the boundaries of sets S_1 and S_2 respectively.

These economic transfers to loads and generators represent a redistribution of the original merchandising surplus.

$$MS(n) = LS(f) + GP(f) \tag{57}$$

where LS denotes load savings, and GP denotes generator profits, under flexibility. This redistribution of the merchandising surplus is shown in Fig. 6, with varying α. This also clearly illustrates the transition point at which the load collective begins losing money under flexibility. The generator collective in contrast, only ever makes a profit under flexibility.

There are some caveats to these results. Technically the boundary of each congestion set is a degenerate case where the load can be thought of as being in multiple congestion sets. As such to ensure a truly congestion-free dispatch, the loads $l^{(1)}$, $l^{(2)}$, would need to be in the interior of their respective congestion sets, which it can be shown leads to an increased system cost. Another assumption here is that the loads are allowed to change their behaviour while the rest of the system remains constant. In reality there would be some dynamic gaming effects, for example generators adjusting their offer prices, or perhaps even other loads trying to cause congestion. Additionally for this scheme to make sense, the merchandising surplus must represent a wealth transfer away from the loads. In some system operator areas, a portion of the merchandising surplus is returned to loads through an *auction revenue right* (ARR). This will vary depending on the system, but the overall goal of any load using flexibility should be to reduce their overall cost. Knowledge of the price structure of the system can only help with this endeavor.

B. General Network Results

Two theorems regarding congestion free dispatch in general networks will be stated here but not proved. A full treatment can be found in [22]. We denote the union of congestion sets where no congestion occurs as the *congestion free set*, L_F. In the two bus example $L_F = S_1 \cup S_2$. We first consider the case of a constant nominal load n over some time period T.

Theorem 6: If $n \in \mathrm{conv}(L_F)$, then by allowing flexibility at each node, n can be met congestion free.

In other words, any constant nominal load in the convex hull of the congestion free set can be met congestion free using flexibility, effectively enlarging the congestion free set to its convex hull. This can be seen intuitively from the above example where any constant nominal load $n \in \text{conv}(S_1 \cup S_2)$ can be met congestion free.

The second theorem concerns a nominal load profile over a series of equal time intervals. For example, a load profile over a day with a different load in each hour. We denote the average load of the profile as $\overline{n}$.

Theorem 7: If $\overline{n} \in \text{conv}(L_F)$, then by allowing flexibility at each node, the full load profile can be met congestion free.

This is a strong condition and can be relaxed to the necessary and sufficient condition that the average load of some subset of the load profile, containing any load that is normally congested, lies in the convex hull of the congestion free set.

V. Conclusions

We have seen that the feasible load space of the economic dispatch problem can be decomposed into convex polytopic *congestion sets*, each uniquely defined by a set of congested lines and marginal generators. Moreover we have shown that each congestion set $\mathcal{S}(l)$ can be written as the Minkowski difference of of a generation set $\mathcal{G}_l$ and a net injection set $\mathcal{P}_l$. From this analysis we derive closed form expressions for the LMP vector λ in each congestion set.

We have illustrated these constructions with a simple two bus example for both linear and quadratic generation costs. We then demonstrated the benefit of load flexibility for congestion mitigation, using the notion of congestion sets. Finally some general network results for congestion free dispatch were presented.

Future work will focus on exploiting the closed form expressions for the LMPs in bilevel optimization programs. In particular to show the value that can be derived from load flexibility, not only from congestion mitigation, but from displacing marginal generators over short time intervals. Additionally, more work must be done on determining *allowable* combinations of marginal generation and congested lines for a given load.

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