ON EXISTENCE OF POSITIVE GENUS SPACE-FILLING TILES THAT DO NOT FORM LINKS OR KNOTS

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ABSTRACT

In this poster, we demonstrate existence of positive-genus space-filling shapes that do not form links and knots. They, therefore, can potentially be assembled and disassembled to construct larger shapes. We observed that these shapes can be obtained by using a 3D graph that is closed under a symmetry operation as Voronoi sites. These 3D graphs can be realized as a curve-network that includes cycles. These 3D graphs should not forms links to allow assembly and disassembly. Another property of these shapes, they can be combined to create higher genus surfaces. The new combined shapes can also fill the space without having any voids between them. From our preliminary structural evaluations, we posit that higher genus tiles may allow for better inter-locking behavior as well. Therefore, many further studies need to be done to characterize and categorize this class of Voronoi sites that could produce different types of higher-genus tiles and the corresponding mechanical behaviors.

1 Introduction and Motivation

Space-filling shapes are defined as unit cells whose replicas together can fill all of space watertight, i.e. without having any voids between them [9]. In other words, a space-filling shape can be used to generate a tessellation of space. While 2D tessellations and space-filling shapes are relatively well-understood, problems related to 3D tessellations and space-filling shapes are interesting and have applications in a wide range of areas from chemistry and biology to engineering and architecture.

The space-filling shapes are usually expected to be "*cellular*" in the sense that a polygon in 2-space (or in general a disc in $\mathcal{R}^2$) is a cell. In other words, 2D regions with holes (e.g. an annulus) cannot be tiled to fill 2D space. This, however, is not true for 3D shapes. Existence of genus-1 tiles that are knotted and linked has been known since 1990's [1, 8, 10]. However, one problem with these tiles, they cannot be assembled or disassemble if they are manufactured as a single pieces. Carlo Séquin developed a clever way to assemble such linked genus-1 tiles by fabricating them two or more separate pieces that can be interlocked [12]. Decomposing single genus-1 tiles into genus-0 pieces helped to assembled and disassemble the whole linked structures. Egon Schulte showed the existence of positive genus tiles that can be assembled and disassembled [11] without decomposing or breaking. He also provided a methodology to construct them. On the other hand, he did not fabricate them to demonstrate the effectiveness of his methodology.

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(a) Five views of a genus-0 space-filling tile. These tiles are cast in aluminum using 3D printed molds.

(b) Two views of a genus-1 space-filling tile obtained by combining two genus-0 tiles.

(c) A genus-2 tile from the combination of three genus-0 tiles .

Figure 1: (Photographs of Fabricated Shapes obtained by our 3D Voronoi Algorithm) Another Example of space-filling arbitrary genus un-linked space-filling tiles. We can obtain any genus by placing genus-0 tiles on top of each other. Each new genus will still be space-filling.

In this work, we have developed a new methodology using Voronoi decomposition of space using un-linked 3D graphs, which are realized as networks of closed curves that are not linked. We demonstrate that using such curve-complexes that are closed under 2D wallpaper symmetry operations it is possible to obtain non-cellular space filling tiles (i.e. positive genus shapes such as a torus) that (1) are connected and (2) can be tiled to tessellate 3D space. Using such curve-complexes, we can obtain higher genus surfaces, an example is shown in Figure 1. There also exists methodologies for designing Voronoi sites that can result in such positive-genus un-linked space-filling shapes [3, 5].

2 Methodology

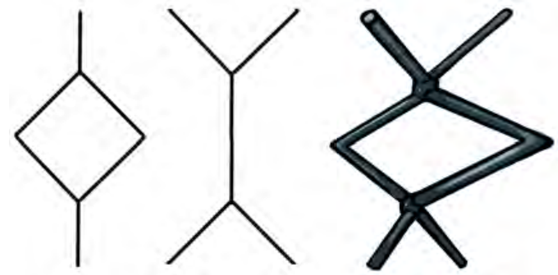
Conventional wisdom suggests that donuts can only form links that cannot be separated without cutting such as genus-1 tiles of Carlo Séquin's [12]. Egon Schulte showed the existence of such structures. We also want to point out that the method by Colin Adams can also lead positive genus tiles that can be separated without cutting if we ignore knotting part [1]. However, neither Schulte and Adams fabricated any example as far as we know.

In this work, we have demonstrated existence of positive genus (homeomorphic to torus, double torus etc.) space-filling shapes that can be assembled and disassembled without cutting. We have also fabricated a few examples. Such space-filling shapes is *not only* theoretically interesting, *but also* opens up possibilities to provide strong interlocking behavior. These particular shapes initially emerged as by-products of our recent work on generalized Abeille tiles[2]. We observed that we can obtain any arbitrary genus space-filling tile by combining certain types of generalized Abeille tiles (See Figure1) . These combinations turns tree-type Voronoi sites that are used to create Abeille tiles into graphs that includes cycles. These cycles produces helps to produce positive genus space-filling shapes.

The existence of this set of positive genus space-filling shapes suggests that there also exist algorithms to compute them systematically. In this work, we briefly present our basic algorithm to design similar shapes. This algorithm to obtain positive genus space-filling shapes, in general, is a specific version of our approach to obtain a variety of space-filling shape types such as Delaunay lofts [14, 13], Generalized Abeille tiles [2] and Woven tiles [7]. Our approach stems from Delaunay's original intent for the use of Delaunay diagrams. He used symmetry operations on points and Voronoi diagrams to produce space-filling polyhedra, which he called Stereohedra [4].

To obtain these new class of shapes, we use higher dimensional Voronoi sites that are closed under 2D (wallpaper) symmetry operations. In Delaunay lofts, these higher dimensional Voronoi sites are curves in the form of $x = f_x(z)$ and $y = f_y(z)$ [14]. In Woven tiles, we use segments of periodic curves as Voronoi sites [6]. In Generalized Abeille tiles, we use 3D trees, such as Y-, X- or T-shaped 1-complexes (i.e. simplexes with maximum dimension 1, e.g. trees and curve networks) [2]. This work resulted from our observation that the unions of the tree-shaped 1-manifolds form general 3D graphs with cycles. Voronoi decomposition of 3D graphs with loops results in positive genus decomposition of 3D space and positive genus space-filling shapes. To guarantee that the resulting structures are positive genus and can be fall-apart without cutting, there is a need for some care in designing the Voronoi sites and symmetry operations.

(a) A curve network of a tree.



(b) A curve network of a graph, which includes cycle.

Figure 2: Comparison of curve networks that are trees and graphs, which include cycles. The one which include cycle can create positive genus space filling tiles when it is used as Voronoi site.

Positive genus space-filling shapes that cannot fall apart without cutting can also be useful in some applications when space-filling chains are needed. However, for space-filling chains, the fabrication can only be done using 3D printing. If our goal is to manufacture each piece individually, symmetry operations must not create linked structures. This problem is related to knot theory and it is trickier than it sounds. To demonstrate the difficulty, assume we have three ellipses. Using these ellipses we can form a linked structure called the Borromean rings. This link will fall apart if we remove any one of the ellipses. In other words, any two ellipses are un-linked and adding the third one makes them linked.

To obtain positive genus structures, we also need to be sure that for each closed curve there must be sufficient empty space in addition to other Voronoi sites that are close to this empty region such that that it can be filled by these Voronoi sites. For another example shown in Figure 1, consider we use two X's top of each other as Voronoi sites. They form a diamond-shaped closed curve in the middle and can potentially result in positive genus tiles. However, we need to orient the other Voronoi sites in such a way that they fill the inside of the diamond shape.

3 Conclusion and Future Work

In conclusion, positive-genus un-linked space-filling shapes provide a significant number of interesting research directions. Particularly in geometry, it will be useful to characterize and categorize the class of Voronoi sites that could produce different types of higher-genus tiles. Since positive genus tiles can provide strong inter-locking behavior, it can motivate studies to evaluate their physical behaviors. We are grateful to reviewers for their detailed review.

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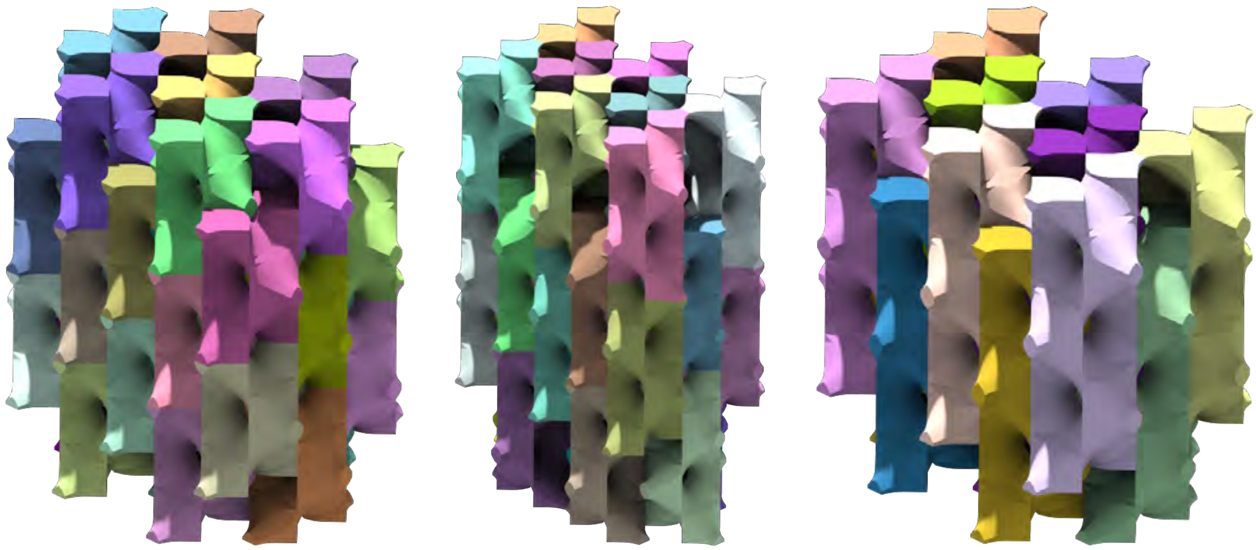

(a) An assembly of genus-1 tiles (b) An assembly of genus-2 tiles (c) An assembly of genus-3 tiles

Figure 3: (Rendering of Virtual Shapes obtained by our 3D Voronoi Algorithm) Examples of assemblies of genus-1, genus2 and genus-3 space-filling tilling. Note that they are obtained by putting genus-0 vaults top of each other. In this way, we can obtain any genus as tall columns.

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