3D Gosper Sculptures

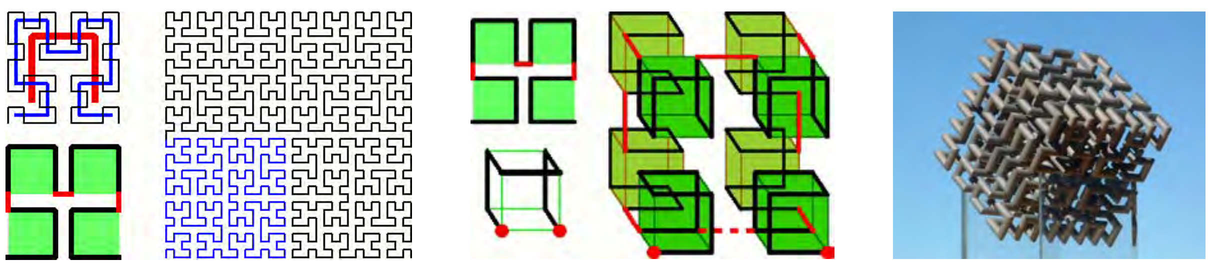
Carlo H. Séquin
CS Division, University of California, Berkeley
sequin@berkeley.edu

Abstract

Twenty years ago, I presented a recursive procedure to fill 3D Euclidean space with a single path inspired by the 2D Hilbert curve. A finite portion of this path then made an attractive constructivist sculpture. Now I am trying to do corresponding constructions inspired by the 2D Gosper curve. The goal is to create a modular polyline that connects nearest neighbor vertices in the face-centered cubic (FCC) lattice, while favoring the characteristic turning angles of 60 and 120 degrees. A complete recursive solution may not exist; thus the focus is on finite-size, modular sculptures. Promising looking designs are being realized as small sculptural maquettes on different 3D printers.

1. Introduction

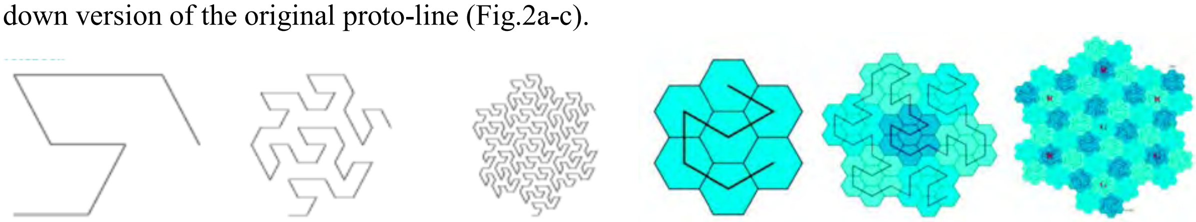
When designing abstract geometrical sculptures, it is often productive to look for symmetries and modularity that permit multiple re-uses of particular shape elements, possibly in different configurations. Suitable elements can often be found in 2D patterns or in plane-filling curves. This is how, in 2001, I came up with "Hilbert Cube 512" (Fig.1c) [2]. This was a third-generation recursive Hilbert curve in 3D space, developed based on a Hamiltonian circuit on the edges of a cube. Each vertex in this path was then replaced with a scaled-down version of an open Hamiltonian path on the cube. These cubelets were oriented in a way to maximize symmetry and to form an overall closed path that roughly follows the initial Hamiltonian circuit (Fig.1b).



(a) (b) (c)

Figure 1: *(a) 2D Hilbert curve; (b) recursive 3D Hilbert curve; (c) "Hilbert Cube 512."*

In this paper I study how another intriguing 2D space-filling curve, the *Gosper* curve (Fig.2) [6][9][10], may lead to a 3D space-filling curve or to finite-size sculptures that preserve the characteristic 60° and 120° turns of this curve. One difficulty is that there is more than one recursive construction to create a Gosper curve. The original presentation of the Gosper curve is based on a prototypical poly line, composed of seven unit-length segments. Each segment in this *proto-line*, is then replaced by a scaled-down version of the original proto-line (Fig.2a-c).



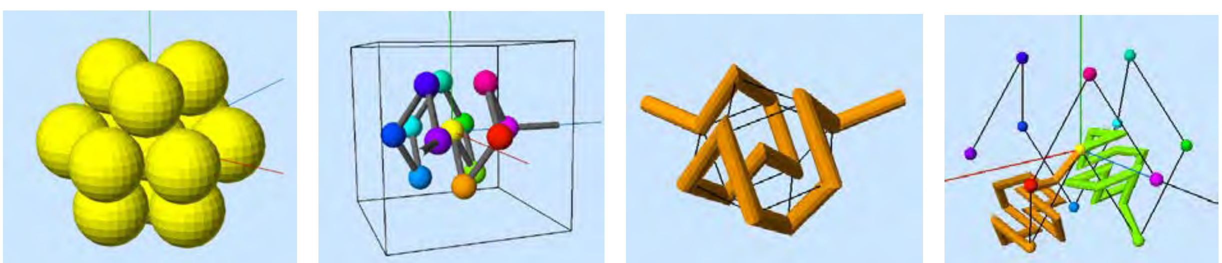
(a) (b) (c) (d) (e) (f)

Figure 2: *Two different Gosper curve constructions: (a-c) based on a 7-segment polyline; (d-f) based on seven hex-tiles [6].*

Another construction is based on a hexagon tiling, in which the proto-line connects the centers of adjacent hexagons [6]. With every recursive generation, a single hexagon is replaced with a clump of seven scaled-down hexagons. This is called the *node-Gosper* curve (Fig.2d-f). Both methods lead to similar-looking recursive polylines with characteristic turning angles of 60° and 120°; they can be drawn by selectively connecting adjacent vertices in a triangular lattice.

2. Basic Considerations and a First "Gosper Sphere"

To take the Gosper curve into 3D, a plausible first approach is to replace the roughly circular, compact clumps of vertices that define the original Gosper proto-line with a Hamiltonian path on a roughly *spherical* clump of equally spaced vertices. This is equivalent to replacing the basic square unit of the 2D Hilbert curve (Fig.1a) with a cubical unit (Fig.1b). For the Hilbert curve, an infinite recursive application of the same basic substitution step can lead to a curve that homogenously fills all of 3D Euclidean space. To construct a recursive 3D Gosper curve, a natural choice for a roughly spherical *proto-clump* might be the densest packing of 13 spheres (Fig.3a), in which the spheres are placed at the corners and at the center of a cuboctahedron. Figure 3b shows a Hamiltonian path on those 13 vertices, where all the bending angles are either 60° or 120°. This then constitutes a possible 3D *proto-line* for a 3D Gosper curve. Figure 3c shows a different, more sculptural view of the same proto-line.



(a) (b) (c) (d)

Figure 3: *(a) 13-atom proto-clump; (b) a Hamiltonian path in this clump; (c) corresponding prototype polyline; (d) two segments of the proto-line have been replaced with reduced proto-line copies.*

Line-Segment Substitution

The question now arises, how to best carry out the recursion step. Following the first approach (Fig.2a-c), every segment of the proto-line is replaced with a scaled down version of the path shown in Figure 3c. But this raises some problems. In the 2D case, there are four ways that a reduced proto-line can be placed onto the line segment that it replaces; in can optionally be mirrored, and it can also be rotated through 180°, which changes the direction in which one could pass through the proto-line. In 3D space, there are infinitely many ways in which the scaled-down copy can be inserted between two given points (Fig.3d). It can be rotated freely around the axis passing through the two endpoints. In addition, one can flip the direction of the inserted proto-line.

On the other hand, there are some constraints. The 13 small proto-lines must not interfere with one another, and, at the vertices where they join, they should create one of the desirable bending angles of 60° or 120°. It also is important to remember that all the vertices of the final 3D Gosper curve should fall on the sites of a regular lattice, and that the curve should densely populate all local lattice sites. For aesthetic reasons, we would like the construction to be as isotropic as possible. When starting with the 13-atom clump, these constraints reduce the choice of lattice to the face-centered cubic (FCC) lattice.

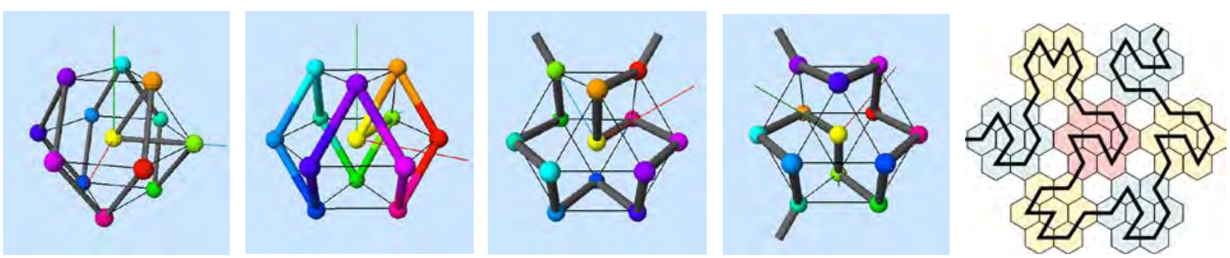
Node Replacement

Because it is difficult to figure out how to place all the proto-line copies so that all the above constraints are met, it seems easier to follow the node-replacement approach (Fig.2d-f). One starts by partitioning a suitable lattice into compact clumps that completely tile the lattice, – like the cube units in the 3D Hilbert curve (Fig.2b). For a 3D Gosper curve, the FCC lattice seems to be the natural choice. Every vertex has 12 nearest neighbours and thus offers 12 directions in which to continue the polyline curve, and the turning angles between subsequent neighbour-connections are either 0°, 60°, 90°, or 120°. Unfortunately, the natural clump of 13 atoms (Fig.3a) does not completely tile the FCC lattice.

In the fall of 2020, we had several Zoom meetings among the Bay-Area Artists & Mathematicians (BAAM), where we discussed ways to create completely-space-filling 3D Gosper curves. No nicely recursive method to create a homogeneous and isotropic path on the FCC lattice has emerged yet. A different proto-clump, and perhaps even a different 3D lattice, may be needed to yield a fully recursive space-filling curve; but there is a strong belief that a pure recursive solution may not exist [8].

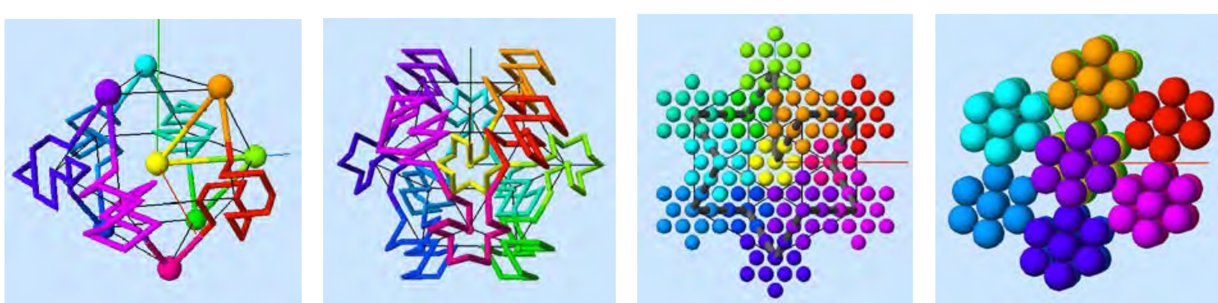
3. Finite Sculptures

In this paper, I thus focus on a different goal: to construct a finite-size, compact sculpture based on the Gosper curve. As in *Hilbert Cube 512*, I like to start with a *closed* Hamiltonian circuit on suitable starting clump (Fig.4a) to obtain a closed-loop Gosper curve, with no open ends sticking out. Such a closed-loop base-circuit may have angles of 60° at eight vertices and angles of 120° at five vertices (Figs.4b). I then place scaled-down copies of this clump at all 13 vertices. These replacement clumps are open-ended Hamiltonian paths, so that they can connect to two of their neighbouring clumps. Thus, two different replacement models are needed: one model with the two connection-stubs 60° apart (Figs.4c), and another one with 120° between connection points (Figs.4d). Both replacement paths can readily be realized with bending angles of 60° and 120°, so the recursive construction process can continue readily.



(a) (b) (c) (d) (e)

Figure 4: *Hamiltonian paths on 13-atom clump: (a) closed base-circuit; (b) highlighted bend-angles. Open paths with: (c) 60° between connectors, (d) 120° between connectors. (e) Hex-tiling with voids.*



(a) (b) (c) (d)

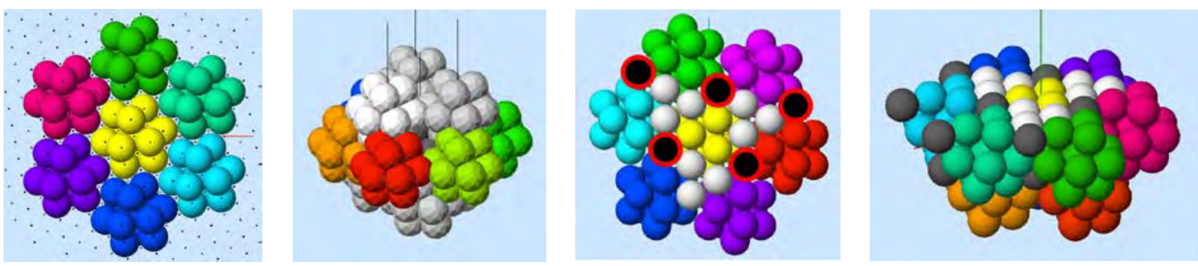
Figure 5: *Building a Gosper sculpture: (a) replacing the five 120° bends with clump Fig.4d; (b) replacing all bends with appropriate clumps; (c) clump connectivity marked in black; (d) view from another angle, revealing the voids between clumps.*

Stub-to-Stub Connections

With the replacement clumps suitably positioned and oriented to line up the stub-to-stub connections (Fig.5a,b), they will assemble into a single-loop Gosper-style path with the desired bending angles, and all the atoms will fall onto vertex sites of a shared FCC lattice (Fig.5c). On the other hand, not all lattice sites are actually visited by this sculptural path. Several sites between adjacent clumps are left unused (Fig.5d). Figure 4e shows an equivalent clumping and tiling in the 2D plane. These stub-to-stub connections render the 13 clumps quite visible and do not result in a dense space-filling path.

4. Looking for Denser Clumping

With the goal of obtaining a denser clumping in the envisioned sculpture, I followed more closely the packing of the hex-tiles in Figures 2d-f. I used the same arrangement to assemble seven spherical clumps so that the "interlocking" pattern of Figure 2e emerges in the equatorial plane of the assembly (Fig.6a). This approach places three atoms of every clump into the layers of spheres immediately above and below the equatorial layer. Additional clumps (white, grey) are now fit above and below the first seven clumps (Fig.6b); they each put an additional three atoms into these two layers. However, for each clump in the main layer, one lattice site in one of these adjacent layers remains unused. Thus, overall, one in 14 lattice sites still remains unoccupied (Fig.6c).

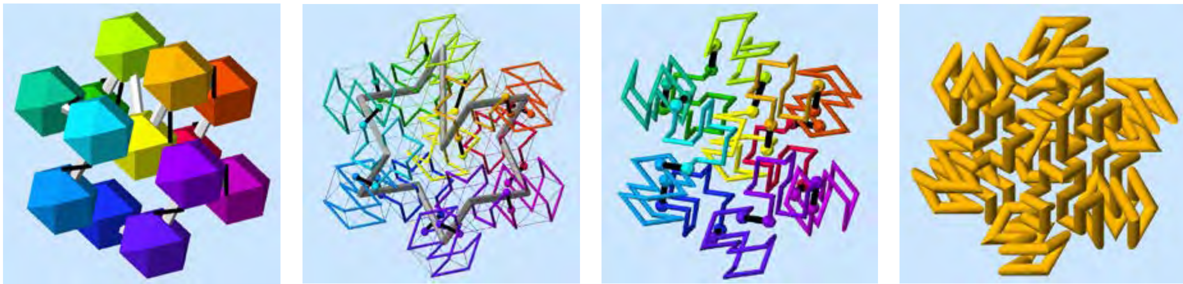


(a) (b) (c) (d)

Figure 6: *(a) Dense clumping of the 13-atom clumps in the equatorial plane (top view). (b) Three layers of 13-atom clumps (side view). (c) Void lattice sites (top view). (d) Voids filled with an extra (dark) atom added to each 13-atom clump.*

In a first approach to make a finite, compact, roughly spherical sculpture, I simply ignored this flaw. These isolated voids in lattice occupancy may not be noticeable by people who are not explicitly looking for them. Stub-to-stub connections cannot be used with this new clump packing. But, two neighboring clumps touch with more than one pair of atoms, and a connection can be established between any pair of touching atoms.

Figure 7a shows such connections as black rods between cuboctahedral clumps. White is the path of the initial base circuit that defines what clumps should be connected. My hope was that I could find a connection pattern, where the two connection points for each clump always are 120° degrees apart. This would then allow me to use just a single type of replacement clump, so that all 13 clumps would have the same internal connectivity and just needed to be rotated to match-up with two of the appropriately placed connector rods. However, after trying quite hard, I have not yet found such a connection pattern, and I am almost certain, that it does not exist. Thus, I made a special replacement clump for the central (yellow) unit, which has its connection points 90° apart. Moreover, because of the chirality of the replacement clump where the connection points are 120° apart, I also needed to make use of the mirrored version. Figures 7b and 7c show a realizable 3D Gosper path. Here one can check what bend-angles result at the ends of the black connector rods. It turns out, the angles are 0°, 60°, 90°, and 120°, and this nicely fits the Gosper paradigm. Figure 7d shows what this composite path would look like as a tubular sculpture.



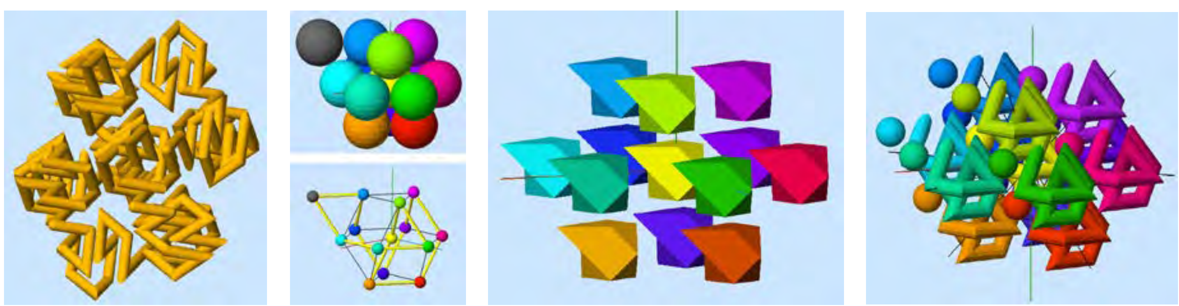
(a) (b) (c) (d)

Figure 7: *"Gosper Ball": (a) Connections between cuboctahedral clumps; (b) connected clumps around the initial base-circuit; (c) resulting 3D Gosper path; (d) tubular sculpture.*

5. Filling the Voids

From most viewing directions, the tubular sculpture depicted in Figure 7d looks reasonably compact and homogeneous. But from a few special directions, the same sculpture shows 13 distinct clumps with some voids between them (Fig.8a). Thus, it is desirable to fill those voids to obtain a more uniform fill-density through the core of the Gosper Ball.

All FCC lattice sites can be filled if we give the 13-sphere clump an additional atom (Figs.8b). Figure 8c shows how these 14-atom clumps might populate densely the FCC lattice sites. However, the extra bump produced by this atomic extension reduces the symmetry of the clumps and the isotropy of the overall structure. It also makes it impossible to rotate individual clumps to line up properly with any pre-defined connection points as exemplified in Figure 7b. This would then require individual designs of the Hamiltonian paths through all 13 clumps in the sculpture in order to obtain the required connections to neighboring clumps.



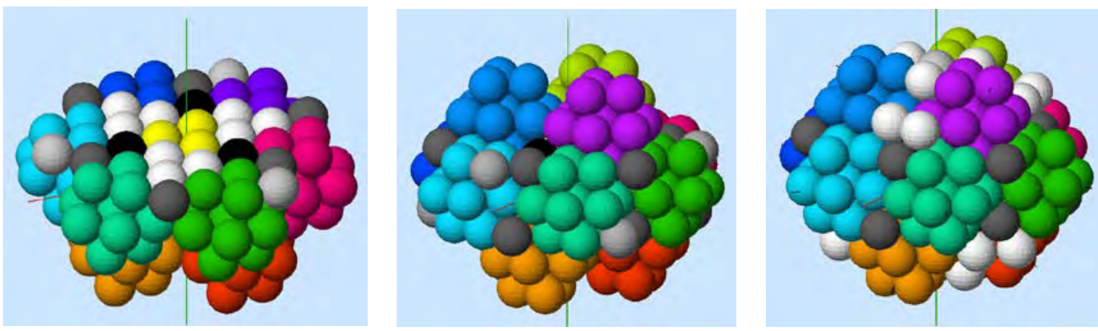
(a) (b) (c) (d)

Figure 8: *(a) Tubular sculpture revealing the voids. (b) Adding a 14th atom to the 13-atom clump. (c) 14-atom clumps on the FCC lattice. (d) The filler atoms in the context of the Gosper path.*

A more flexible approach adds isolated atoms onto all unoccupied FCC lattice sites. This allows one to not only fill the inner voids in the spherical 3D Gosper sculpture, but also to smooth out the rugged surface produced by the assembly of 13 roughly spherical clumps and to make the overall sculpture more spherical. Since there is no solid recursive construction that places the filler atoms automatically with the placement of the 13 clumps, the choice of where to place filler atoms is mostly an issue of aesthetics.

Figure 9a explores some options of which voids one might want to fill. The three black filler atoms touching the three yellow atoms of the central clump are needed to fill these completely internal voids. The six medium gray atoms are part of the surface of the sculpture and are more optional filler places. Finally, the three light grey atoms at the periphery are even more questionable: While filling some concave spaces, they also create additional new concavities next to themselves. Of course, whatever we do in this plane immediately above the equatorial plane, we should also do in the plane below the equatorial plane.

Figure 9b shows three layers of 13-atom clumps. I have maximally filled voids in both atom planes adjacent to the equatorial plane. Thus, these three planes of atoms look rather dense and homogeneous. But now the upper and lower ends of the sculpture, with just three naked clumps each, look rather loose, and the fact that one can still see one of the black filler atoms (Fig.9b) clearly indicates that there are some holes to be filled. Thus, I have added some filler atoms there as well. Figure 9c shows a complete *Gosper Ball* composed of 13 clumps and a total of 38 filler-atoms. With some selective filling-in in all layers, I was able to generate a reasonably smooth surface. – However, one may ask, is this the proper thing to do? Perhaps the raggedness of the Gosper surface is part of the essence of the Gosper curve. Any finite part of a 2D Gosper curve also occupies a "Gosper-island" with a ragged outline. Yet, in 3D, the raggedness of the surface without any filler atoms seems much more pronounced.

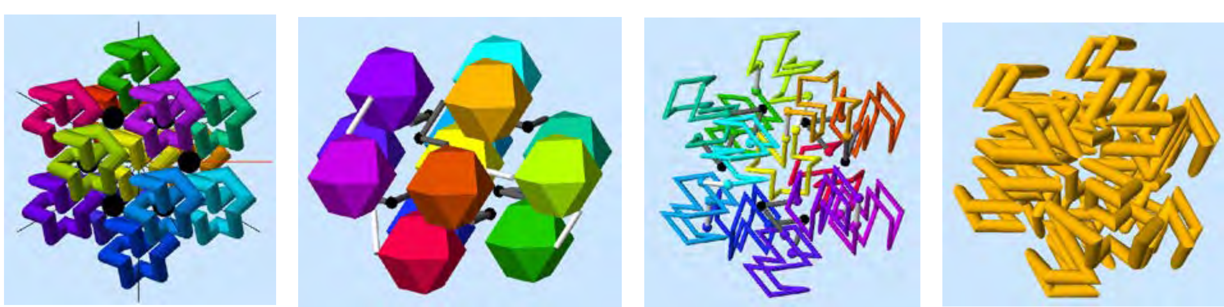


(a) (b) (c)

Figure 9: *(a) Different void filler options. (b) Both near-equatorial planes filled densely. (c) Also filling the voids between the upper and lower three clumps (white atoms).*

Deciding what lattice sites to populate with filler atoms is only half the problem. Next, for all these extra atoms, one must decide how to incorporate them into the final 3D Gosper path. Figure 10a shows the six innermost, required filler atoms in the context of the original 3D Gosper path (Fig.7). Should these atoms be integrated as a local detour into one of the nearby clumps? (Is this always possible without breaking the Gosper characteristics of the local path through that clump?) – Or is it better to use these atoms to transition from one clump to another neighbor? (Does this then require that every clump path must be modified in a different manner to accommodate these new entry- and exit-points?)

Figure 10b shows one possibility to link the six black filler atoms (Fig.10a) to two nearby clumps that need to be connected according to the chosen base-circuit. Six of the required clump-to-clump connections are made by such 2-step links, shown in grey (Fig.10b), while the remaining seven connections are made by direct rod connections, shown in white. Again, I was able to find a solution where the 12 outer clumps have their connection points 120° apart. Yet again, the central (yellow) clump is different; but now it has its connection points only 60° apart. Figure 10c displays the resulting Gosper path, and Figure 10d shows that path rendered as a tubular sculpture.



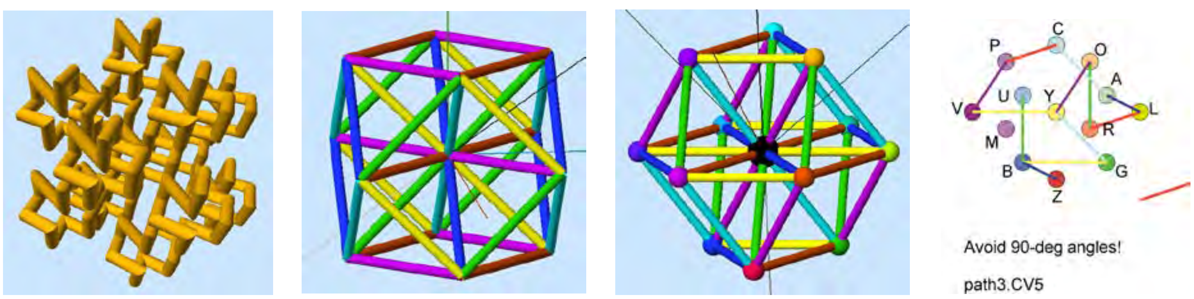
(a) (b) (c) (d)

Figure 10: *(a) The six crucial filler atoms (black) in the core; (b) using them as transition steps; (c) a resulting Gosper path; (d) corresponding tubular sculpture.*

Evaluation

Much tedious handy-work has been spent to make all the connections between the individual clumps. Was it worth the effort? The core of the sculpture definitely looks more compact; but the outer surface of the sculpture looks more "ragged." The 3D sculpture is missing the regular, structured look of the 2D Gosper curve. Also, the connections between clumps have led to some collinear consecutive segments. In some projections of the model, this seems to imply that the tube segments are of varying length.

More experiments with a different base-circuit and with different paths through the 13 clumps and the six filler-atoms need to be explored. A different approach might start with 13 identical clumps with the same closed Hamiltonian circuit on them. All are placed with the same orientation (Fig.13a). Then we break the 13 loops in different ways to make connections to neighboring clumps. But, with all clumps oriented in the same way, it becomes crucial to pick a loop that is as isotropic as possible. On the FCC lattice, tube segments can point in six different directions (Fig.11b,c). Ideally, the loop uses all six directions about equally often. Figure 11d shows a drawing utility that I generated to make it easier to look for such isotropic solutions: Segments in different directions are shown in different colors. However, isotropy is difficult to achieve in a Hamiltonian circuit with only 13 segments. Moreover, it may be even more difficult to obtain a truly isotropic look, because we might also want the pointy 120°-angle-turns to point isotropically in all directions; – there are 24 possible orientations! (In 2D the problem is less pronounced: There are only three segment directions and six pointy angle orientations.)



(a) (b) (c) (d)

Figure 11: *(a) The look of 13 orientation-aligned clumps. (b,c) The six possible segment directions. (c) A drawing utility to find good, isotropic Hamiltonian paths.*

Given all the various difficulties addressed above, perhaps starting with 13 clumps assembled in densest sphere-packing manner may not be the best way to make a "spherical" Gosper sculpture. Before discussing other possible approaches, I want to discuss some additional important, high-level concerns that should be considered before attempting to construct such sculptures.

6. Size, Complexity, Stability …

When choosing the sizes of the base-circuit and of the replacement clumps, overall complexity must be kept in bounds. First, there is an aesthetic concern: At too high a level of complexity, the tube pattern seen on the surface of a sculpture may look more like a texture on the overall (spherical) shape of the sculpture, rather than like a 3D constructivist tubular sculpture.

For a physical sculpture, there is also the issue of stability. My 6-inch model of a third generation *Hilbert Cube 512* (Fig.1c) is quite flexible, even though it is made of metal! If this 512-segment polyline circuit were stretched out, it would form a loop that stretches out for about 600 tube diameters before it turns back to the origin. With the next generation it would be going 8 times farther. Gravity may cause visible bending and warping on such a long loop without any intermediate supports.

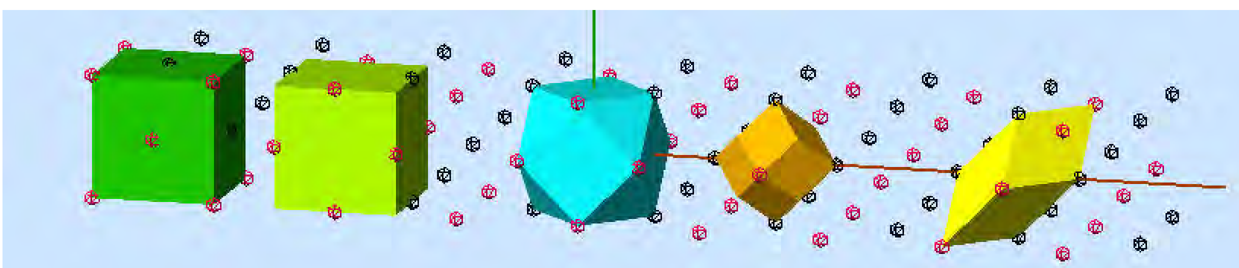
The same limitations apply to a Gosper sculpture. At practical scales, from desk-top models to human-scale sculptures, a single-loop tubular sculpture is probably restricted to a few hundred vertices. This limits the overall design to a maximum of two hierarchical levels. In a 3-level sculpture, the individual clumps at each level would have to consist of less than 10 atoms, and this would not be sufficient to capture the Gosper character within each clump. Thus, the 2-generation designs described above, which are composed of 13 clumps of 13 or 14 atoms, are an appropriate start.

The best way to increase complexity by a limited amount is to add more complexity into the 13 replacement clumps. Starting with a base-circuit that has more than 13 segments seems less important. The base-circuit mostly defines the overall shape of the sculpture. The connectivity between the replacement clumps is not so important, since in a busy sculpture it will be hard to discern how the clumps are connected to form the basic loop. On the other hand, the path through the individual clumps primarily defines the characteristic of the displayed space curve, and it will determine whether the sculpture looks like a 3D Gosper path.

7. Modeling Issues

I ended up doing much of the detailed modeling work twice, with different orientations of the coordinate system. I had started out describing the densest clump of 13 spheres by lining up one of its 3-fold rotational symmetry axes with the z-axis of the modeling system. This led to a pleasing projection of this clump that made it easy to obtain 3-fold rotational symmetry and to place seven spheres tightly packed into the equatorial x-y-plane. However, as the models became more complex, with filler atoms added, it got more difficult to figure out the exact placement of all the features and the mutual offsets between them, which were needed to add exact connecting branches between the various clumps.

Thus, at some point, I started again by constructing first an FCC lattice with all vertices on integer coordinate values. This ended up as a stack of simple uniform square grids parallel to the x-y-plane, alternatingly offset in the x- and y- directions by one unit as the z-height increased (Fig.12). Now all the vertices of the basic cuboctahedron and the placement of all the clumps and filler atoms ended up on integer coordinates, and maintaining precision was no longer a problem.

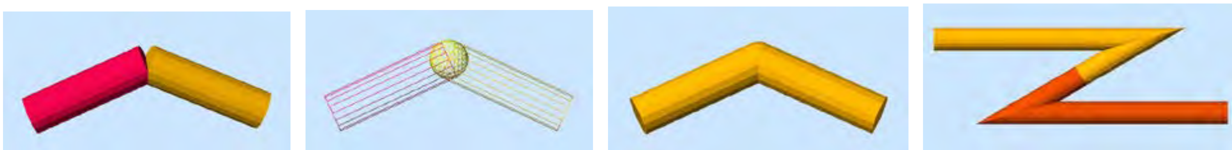


(a) (b) (c) (d) (e)

Figure 12: *The FCC lattice and some of its tiles: (a) cube; (b) cube shifted; (c) cuboctahedron; (d) rhombic dodecahedron; (e) rhomboid.*

Another modeling issue concerns the generation of good "water-tight" boundary representations in the STL files used for the fabrication of 3D-print models. For aesthetic reasons, I found it desirable to model the various Gosper path fragments as sweeps of a circular cross-section, with sharp, mitered corners at all the turns of the sweep path – as generated easily by the Berkeley SLIDE software [4].

It is impractical to model the complete Gosper path as one single sweep. To preserve some modularity, individual clump paths need to be stitched together. One solution is to close off individually all partial sweeps at both ends with *end-caps*. However, when two perpendicularly sealed-off sweeps are joined at an odd angle, the two boundary representations will partly overlap and also leave some wedge-shaped gap (Fig.13a). To correct for the missing material in the gap, a small sphere with the same diameter as the two cylinders is placed at the junction point (Fig.13b); this leads to a suitable geometry for the joint. This approach assumes that the slicing software of the 3D-printer will properly identify multiply-overlapping *internal* areas as simply *internal* [1]. Successful 3D-prints using this approach have been fabricated on Ultimaker printers [7] as well as on Stratasys machines [5].



(a) (b) (c) (d)

Figure 13: *(a) Two cylindrical elements joined at an arbitrary angle yielding overlap and a gap. (b) Adding a sphere to yield a seamless joint. (c) Resulting smooth joint. (d) Colinear joint.*

However, such nicely rounded bends may not be the preferred way to capture the look-and-feel of a Gosper curve. Properly mitered, sharp, spiky joints may be preferable at places where the path makes a 120° turn. In that case, it is better to join separate swept path fragments halfway between two FCC lattice sites, with the two tubular path segments joining in a lined-up collinear manner (Fig.13d). Representing cylinders as prisms with 30 or more sides typically is good enough to produce STL files with "water-tight" boundary representations at these junctions, even without using *end-caps*. If prisms with a much lower number of sides are preferred, I use the *twist*-parameter in the SLIDE sweep construct [4] to align the meshes of the two abutting tube cross-sections.

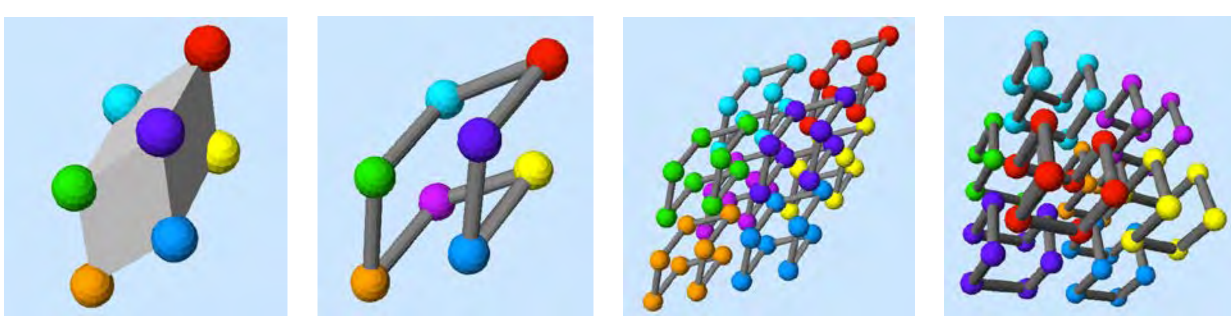
8. Partitioning the FCC Lattice

Much of the problems discussed in the first half of the paper are a result of the fact that the 13-atom clumps do not fit together snuggly in 3D space without any voids in between. On the other hand, the FCC lattice is my top choice, because every site has 12 nearest neighbors. This readily permits to construct Gosper-type paths with unit-length steps and with bending angles of 0°, 60°, 90°, or 120° between subsequent steps. Thus, I will explore next how the FCC lattice can be partitioned into 3D polyhedral tiles that fit together seamlessly.

Once the goal of finding a truly recursive definition of a Gosper path that fills all of 3D space had been pushed aside, I could focus on two separate issues that are important in the construction of a finite 3D sculpture: the definition of a suitable base circuit that defines the overall shape of the sculpture, and the construction of appropriate modular clump tiles.

Rhombic Tiles

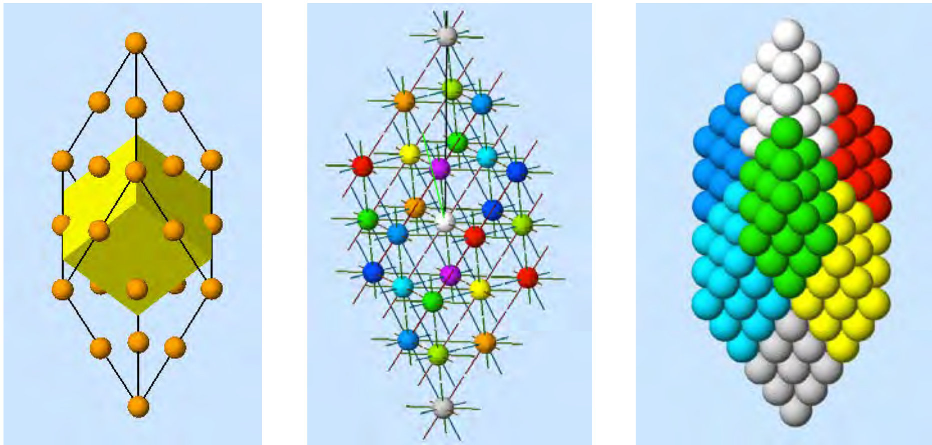
An affine distortion of a cubic lattice, that stretches the unit cube by a factor of 2.0 in the direction of one of its space diagonals (Fig.14a) produces a rhomboid lattice that also places all vertices at the sites of an FCC lattice. This lattice can thus readily be tiled with a rhomboid composed of just 8 atoms. This is just large enough to capture the key angles of 60 and 120 degrees of the Gosper curve. But if this prototype curve (Fig.14b) is used in a recursive manner, the resulting assembly looks more like a stretched 3D Hilbert curve (Figs.14c,d) rather than like a Gosper curve.



(a) (b) (c) (d)

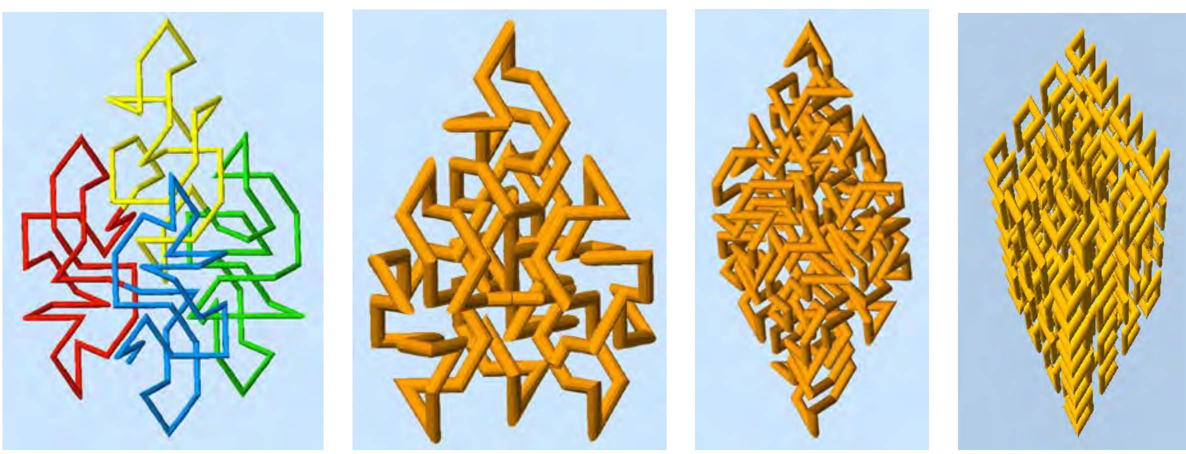
Figure 14: *(a) Stretched cube; (b) proto-path on 8 atoms; (c) an assembly of 8 such clumps; (d) a look at this assembly from the pointed end.*

A larger rhomboid tile composed of 27 lattice sites (Fig.15a,b) offers a richer variation in local path behavior and a better chance of capturing the nature of the Gosper curve. Indeed, an interesting sculpture results when eight 27-atom clumps are assembled (Fig.15c) and properly interconnected (Fig.16).



(a) (b) (c)

Figure 15: *(a) Diagonally stretched cube; (b) atoms placed on FCC lattice. (c) Eight 27-atom clumps.*



(a) (b) (c) (d)

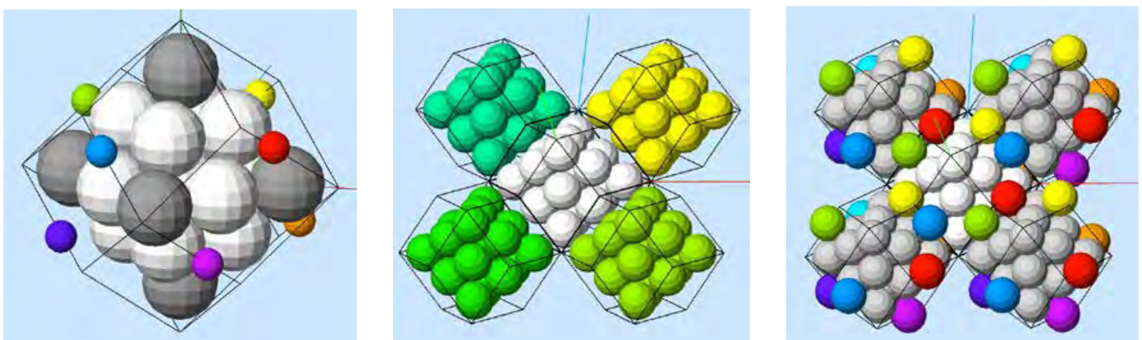
Figure 16: *Rhomboid Gosper sculpture: (a, b) path through the top four 27-atom clumps; (c) "Gosper-Pole_216" using 8 clumps. (d) Diagonally stretched Hilbert Cube.*

Figure 16a shows four suitable Hilbert paths on the 27-atom clumps that capture the spirit of the Gosper curve and that also properly interconnect adjacent clumps. Figure 16b shows this path in the form of a tubular sculpture. Combining two of these assemblies, results in the complete sculpture (Fig.16c), which has a notably different character from an affinely stretched *Hilbert Cube* (Fig.16d).

Rhombic Dodecahedron (RD)

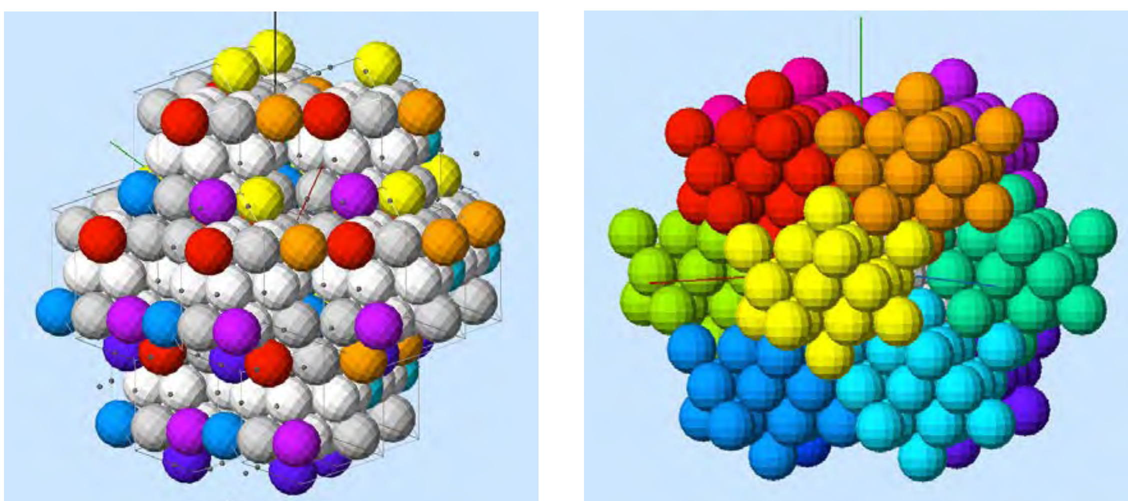
If our goal is to construct a more "spherical" and isotropic 3D Gosper sculpture, then we need to look for a more "spherical" tiling unit. A natural choice then is the *Rhombic Dodecahedron* (RD). To use this cell in the current context, atoms must *not* be placed at the 14 vertices of this polyhedron, but they should be placed *inside* this cell. In addition, where neighboring cells are abutted by using the faces of the RD, empty lattice sites must be filled with atoms that overlap with more than one RD cell.

One way to fill a RD-shaped cell is to add six additional atoms around the 13-atom clump used earlier in this paper, by placing them inside the six valence-4 vertices of the RD (grey atoms in Fig.17a). When trying to fill 3D space with this RD tile containing 19 atoms, not all sites of the FCC lattice are getting covered (Fig.17b). There is a site on every edge of the RD that is left unoccupied. However, an atom placed at such a site extends into the RD cells of the three neighboring tiles that share this edge. Thus, this atom must not be assigned to more than one of those three clumps, or multiple overlapping occupancies would result. In designing the basic tile clump, there are 24 edges on the RD where an atom might be placed; but the tile must only use *eight* of these atoms (colored atoms in Fig.17a). Those must be chosen judiciously, so that in a clump of 13 densely packed tiles no overlaps occur. Figure 17c shows how the interstitial sites are covered in the same 5-clump assembly (shown in Fig.17b). Wherever two rhombic faces connect between adjacent clumps, each clump contributes two shared atoms.



(a) (b) (c)

Figure 17: *The RD clump: (a) 19 atoms inside the RD shell, plus a subset of 8 atoms on the 24 edges; (b, c) an assembly of five RD clumps without and with the colored filler atoms.*

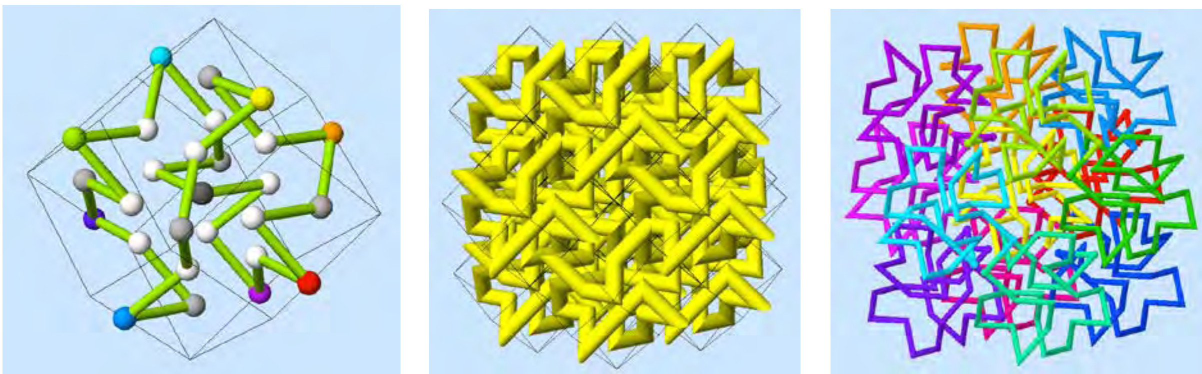


(a) (b)

Figure 18: *Thirteen 27-atom clumps: (a) "filler" atoms highlighted; (b) uniformly colored clumps.*

Figure 18 shows an assembly of 13 clumps, with the shared atom locations shown in the colors used in Figure 17 (Fig.18a), and with all 13 clumps in different colors (Fig.18b). The shared "filler" atoms fill the core of the sculpture densely. On the other hand, they produce a rather irregular surface structure. To obtain a smoother surface, some of these atoms might be selectively removed. However, this would destroy the nice modularity of the underlying RD tiling and would require different connectivity patterns in most clumps. It seems more appropriate to accept this irregular, corrugated surface; – after all, every finite portion of the 2D Gosper curve also occupies an island with a serrated fractal outline (Fig.2f).

Figure 19a shows a hand-crafted Gosper-like path connecting the 27 atoms in this RD cell. All bending angles are either 60° or 120°. In Figure 19b I have packed 13 such RD cells into a densest sphere-packing configuration to obtain an idea what a 3D Gosper sculpture with 351 path segments would look like. Even though the 13-atom base circuit has only two bend-angles of 60° and 120°, respectively (Fig.4a), I still needed to design several different open-ended clump paths, since the filler atoms make these clumps anisotropic and they no longer can be rotated freely into all the required orientations to make the proper connections to two neighbor clumps. The result is shown in Figure19c.

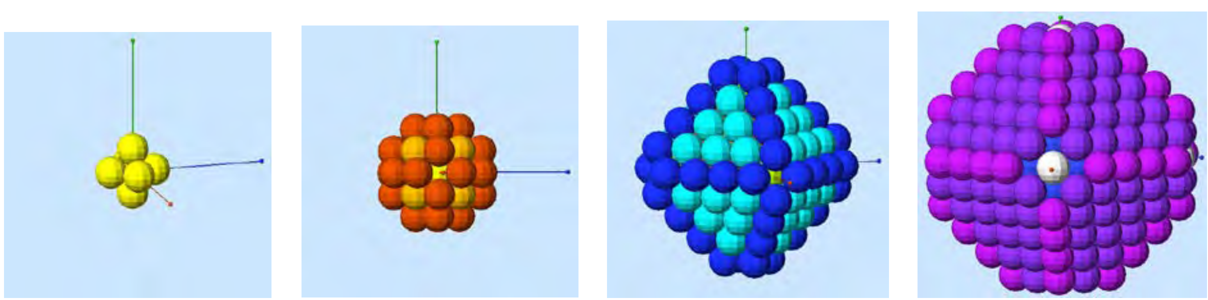


(a) (b) (c)

Figure 19: *Gosper path in RD-27: (a) Connecting the 27 atoms. (b) 13 clumps tightly packed together as a preview of a 351-node Gosper-Lattice. (c) The 13 clumps properly interconnected.*

9. Onion-Shell Approach

The difficult trade-off between the roundness of the clumps and the number of interstitial voids between them prompted me to investigate also a different approach. Here I am trying to form a nicely rounded overall shape as a sequence of nested shells of atoms. I start with an octahedral core composed of six atoms (Fig.20a). This core is surrounded with a second shell with 8 + 6*4 = 32 atoms (Fig.20b). The next shell comprises 6 (lime) atoms on the coordinate axes, 12*4 = 48 (blue) atoms on the edges of an octahedral shape, and 8*6 = 48 (cyan) atoms to fill in the faces, for a total of 102 atoms (Fig.20c). The forth shell has 6 + 48 + 120 = 174 atoms (Fig.20d).

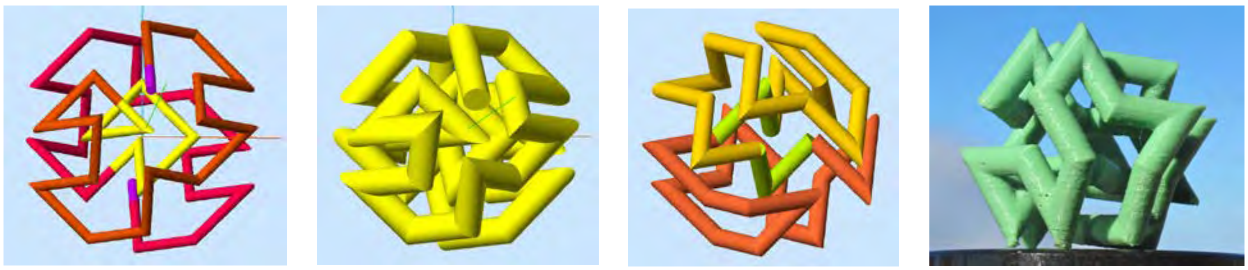


(a) (b) (c) (d)

Figure 20: *Onion shells: (a) central octahedral core, 6 atoms;. (b) 2nd shell, 32 atoms; (c) third shell, 102 atoms; (d) forth shell, 174 atoms.*

The main idea is to draw a bent version of a 2D Gosper pattern in each shell. Fortunately, the outermost shell, which is most important for conveying the Gosper-look, is also the one with the least curvature. Each shell connects with a minimum of two points to the shell below; these points correspond to the beginning and the end of the Gosper curve in the upper (outer) shell.

Figure 21a shows this approach for the first two shells. The Hamiltonian path on the octahedron is shown in yellow. At both ends, it transitions into the second shell and continues for another 15 segments to cover half the atomic sites in that shell, exhibiting two 7-segment Gosper patterns (Fig.2d) on two of the octahedral faces visible in Figure 20b. It ends in the two purple stubs, through which it may connect to shell #3. Figures 21b shows this combined path as a tubular sculpture; one of the connection stubs to the third shell can be seen clearly.



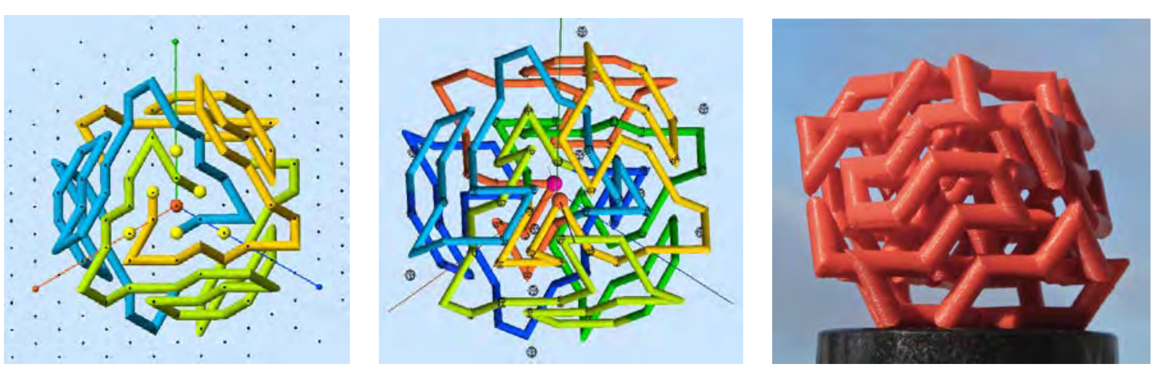
(a) (b) (c) (d)

Figure 21: *Onion shell paths: (a) Octahedral path (yellow), surrounded by two 15-segment paths in the 2^{nd} shell, ending in the purple stubs that will connect to shell #3. (b) The combined path shown as a tubular sculpture. (c) A different path forming an overall closed circuit. (d) The closed path presented as a tubular "level-2 Onion" sculpture.*

The plan is to continue with the same spherically layered approach for a couple more shells. However, I am concerned about the stability issues discussed earlier in the paper. If a large outer shell is supported through only two points on the next lower shell, it may easily bend, shift, or sag so that it touches the inner shell in some manner. Thus, it seems advisable to partition that shell into two or three parts that are separately supported by the lower shell. This may also yield an opportunity to give this sculpture some overall symmetry and thus some modularity in design.

Figure 21c shows this approach for shell #2. The shell is partitioned into two identical (orange and red) paths that have their end-points 90° apart from one another. Two 2-segment (lime) paths on the octahedron vertices make two connections between the half-shell paths, thereby forming a single closed circuit (Fig.21c). Figure 21d shows the same path as a tubular sculpture. I think, this "*Gosper-Onion_38"* has some merits as a stand-alone mini-sculpture, and it looks like a promising start for onion sculptures with more shells.

Figure 22 shows my approach to designing a 3-level *Gosper-Onion*. I have partitioned the outermost shell into three identical (orange, blue, and green) parts, each covering roughly two octants of a sphere. Each path then connects at both ends to fragments of the second shell that each cover one octant of a sphere. The three composite paths then get joined in the "level-1 shell" by three edges of the central octahedron. So far, the structure has nice D_3-symmetry.



(a) (b) (c)

Figure 22: *A 3-level Gosper Onion: (a) Three path fragments arranged with D_3-symmetry. (b) Adding detours into the prototypical path fragment to visit all FCC lattice sites. (c) The complete "Gosper-Onion_140" as a sculptural maquette.*

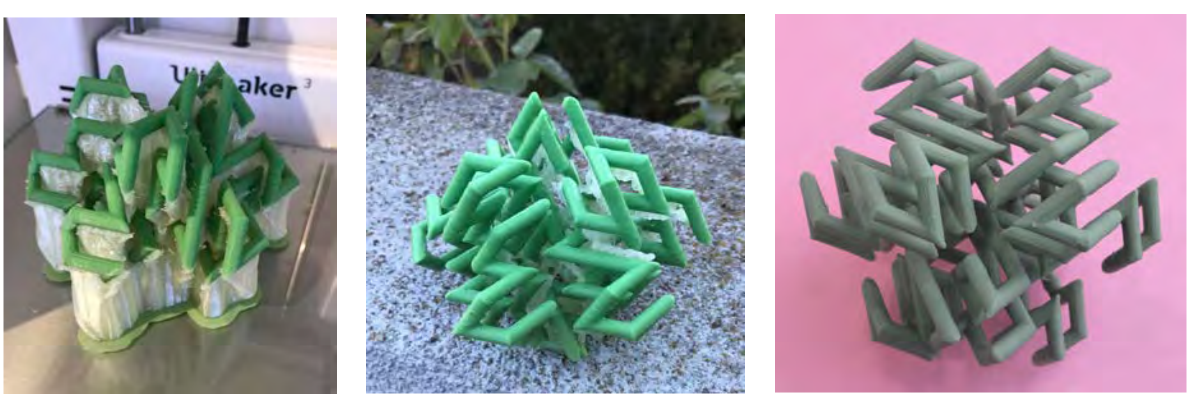
Unfortunately, not all FCC sites have been covered yet. In particular, the two opposing octants through which the D_3-axis passes have sites that still need to be included in the overall path. A site in shell #2 is shown in red, and six sites in shell #3 are shown in yellow in Figure 22a. The orange branch in shell #3 makes a detour to include two of these yellow atoms, and so do the other five copies of the orange half-path in Figure 22b. In addition, the orange branch also makes an extra step to include the vacant red atom on shell #2. This move is not included by the blue and green path copies; it thus breaks the D_3-symmetry. The complete level-3 Gosper-Onion path is shown in Figure 22c as a 3D print.

10. Fabrication of 3D-Print Models

The fabrication of good 3D print models of the above designs turned out more difficult than anticipated. The default setting in the CURA slicing program for the Ultimaker printers [7] is not suitable to provide good enough support for the various tubular elements slanted at different angles. In particular, any downward pointing, spiky, mitered joints need a much smaller filament overhang angle (about 30°) than the default of 60° (Fig.23a). This support-extension then results in a rather solid block of support, which is difficult to remove from the inner parts of any Gosper sculpture (Fig.23b), particularly, if the default *gridded* support structure is used (Fig.23c). After several initial trial runs on the Ultimaker printers readily available to me, it became clear that any fabrication technique that requires manual removal of the support structure is not suitable for Gosper models with more than about 50 path segments. The inner portions of such sculptures are difficult to reach, and the overall single-loop path is too fragile to allow pulling out inner support elements by brute force.

(a) (b) (c)

Figure 23: *Fabrication difficulties: (a,c) Onion shell path (Fig.21b); (b) Gosper-Pole_216 (Fig.16c).*



(a) (b) (c)

Figure 24: *Results on Ultimaker with soluble support: (a) fresh off the printer; (b) with most of the support material washed away; (c) after final manual cleanup. (Photos a,b by N. Weaver).*

After some experimentation with the advanced parameter settings in the CURA slicing software, good results have recently been obtained on an Ultimaker printer using PLA for the model and a water-soluble support material (PVA). A model corresponding to Figure 10d is shown as it came off the printer (Fig.24a), after most of the support has been washed away (Fig.24b), and after manual cleanup of the last remnants (Fig.24c).

I have also started to explore the use of an FDM printer from Stratasys [5] that has a soluble support material. Models that are more complicated will be sent off to the Shapeways 3D printing service [3] for nylon prints using Selective Laser Sintering.

11. Discussion

How do I judge whether I have composed a successful 3D Gosper Sculpture? I think that the following criteria can help answer this question: The result should be a closed polyline with segments of uniform length on a suitable 3D lattice, with the FCC lattice being the preferred choice. The polyline should exhibit the characteristic "claw" shapes seen in Figure 2, and it should favor bending angles of 60° and 120°. The local geometry should have some modularity and repeat the same local pattern several times, while completely filling all lattice sites in an appropriately shaped 3D island domain on the FCC lattice.

The last characteristic seems to be the most difficult one to achieve, because it is hard to find a good clump of atoms that tessellates the FCC lattice without any interstitial voids. Cubic clumps can readily tile the FCC lattice, but clumps with just $2^3 = 8$ atoms (at the corner of a small cube) are not enough to define a suitable 3D Gosper claw; and clumps with $4^3 = 64$ atoms emphasize the cubic shape too strongly to pass for a Gosper curve. A clump of $3^3 = 27$ atoms may be acceptable, particularly, if it is affinely deformed into a rhomboid shape (Fig.16) and if we accept an overall elongated shape for the resulting "Gosper-Pole" sculpture.

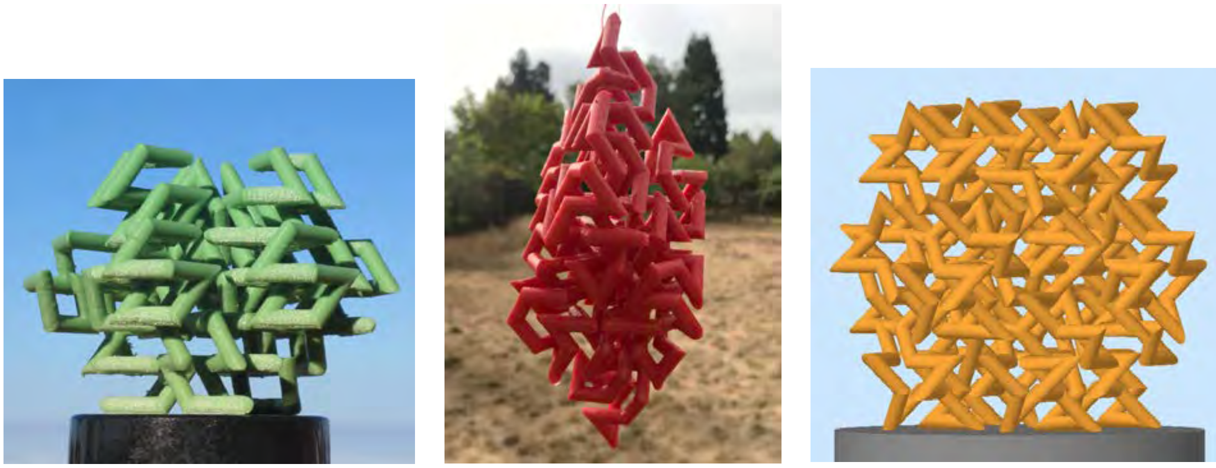
If we want to obtain a more "spherical" sculpture, as implied by the hexagonal patterns in Figure 2, we face a challenging trade-off. It is easier to find small clumps that seamlessly partition the FCC lattice, but they may not offer a rich enough set of path options to represent a convincing Gosper characteristic. As we look for larger and rounder clumps, they will leave more interstitial voids, which must be included in the overall polyline, – which appears difficult to do in an algorithmic manner. The "Onion-Shell" approach does not offer a completely satisfactory solution. While CAD models look quite promising on the computer screen, already the level-3 model has too many consecutive coplanar path segments in its outermost shell. The path patterns in that shell may look like parts of a 2D Gosper curve, but, overall, this would at best yield a "2.5D" Gosper sculpture.

Based on the above criteria, I offer three design approaches as my currently preferred and most promising realizations of 3D Gosper Sculptures. They are in the process of being fabricated as small maquettes on a variety of different 3D printers: "Gosper-Ball_175" (Fig.25a) is based on Figure 10d, "Gosper-Pole_216" (Fig.25b) is based on Figure 16c, and "Gosper-Lattice_351" (Fig.25c) is based on Figure 19c; the last model offers a particularly rich and varied set of projections from different viewpoints. The "Gosper-Onion" models look nice, but they don't really display true 3D Gosper curves.

12. Conclusions

For the Hilbert curve, a well-chosen recursion step can lead to a complete, uniform filling of all of Euclidean 3D space, as well as to finite-size parts thereof that can lead to attractive tubular sculptures with the topology of the unknot. For the Gosper curve, an infinite recursion that fills all of 3D Euclidean space has not yet been found, and it may not exist [8]. On the other hand, this work shows several possible approaches to make compact sculptures that reflect the character of the various Gosper curves, and which could be constructed physically at a human-size scale by using a few hundred tubular segments.

Because of the lack of an elegant recursive formulation of the overall path, I ended up doing a lot of detailed handiwork to draw 3D Gosper-like path fragments and to make the appropriate connections between different clumps in different orientations. An open challenge remains to find a better algorithmic approach to define the overall Gosper-style polyline that makes all the needed connections between the local clumps and possible interstitial lattice sites.



(a) (b) (c)

Figure 25: *Promising styles: (a) Gosper-Ball_175; (b) Gosper-Pole_216; (c) Gosper-Lattice_351. [Model and photo (b) by Nicholas Weaver]*

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