Growing Shapes: Unfolding simple quadrangulations of the sphere one quad at a time

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Abstract

The object of this research is making sculpture that grows, that is, not merely changing in shape, but developing, one small step at a time, from small to large, maintaining definite, reproducible shape all the while. Recent advances in graph theory have shown that the simple quadrangulations of the sphere can be generated inductively by a set of map operations more restricted in their context of application than sets previously known. I show that this new pair of operations can be realized by local unfolding of quads that have been hinged along both diagonals. Current development of these ideas is demonstrated in a small hinged-plate model that grows from 2 quads to 6 quads.

Introduction

I have been interested for some time in making sculptures that change shape[6][1]. Figure 1 shows my earlier work, *Crumple,* which expanded and contracted by unfolding and re-folding in synchrony with the inflation and deflation of an inner membrane. Such shape changing is a far cry from biological growth: the proportional increase in volume is small, and the sequence of shapes is uncontrolled and indefinite. My aim in this research is to get closer, both mathematically and physically, to my goal of sculptures with controlled growth via unfolding.

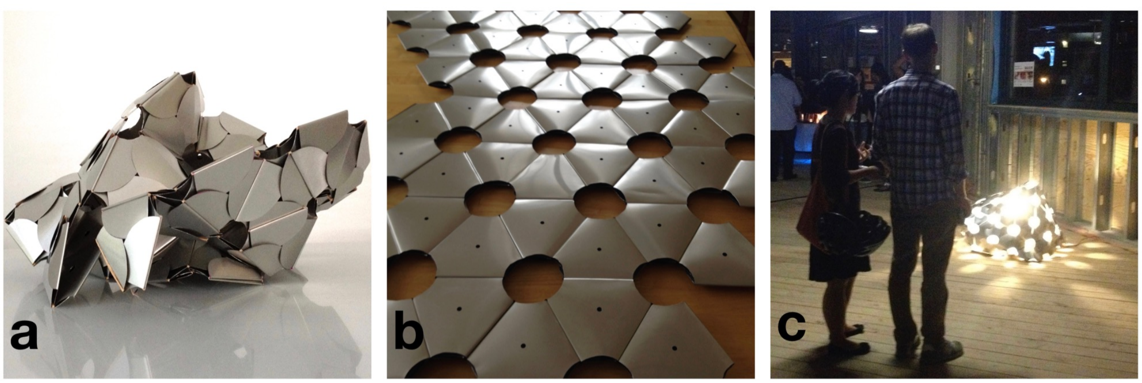

Figure 1: *Crumple, 2013. A kinetic unfolding/folding sculpture: a) maquette, b) fabrication, c) exhibition.*

Background

Complete Folding of Triangulations

Physicists are interested in the folding of surfaces as it relates to simplified models of quantum gravity. In 2005 Di Francesco and Guitter [4] showed that any 3-colorable[2] triangulation can be "folded on a single equilateral triangle by sending each node of a given color onto one of the three vertices of this triangle."

[1] Reference 6 demonstrates how a completely foldable basket can be woven, but does not demonstrate a shape-by-shape sequence of unfolding that can simulate growth.

[2] A graph is 3-colorable if each vertex can be assigned one of three colors such that no edge connects vertices of the same color.

This *complete folding* of a triangulated surface onto one of its triangles is a *phantom folding* differentiated from the familiar *physical folding* of paper. For instance, the mathematical surface is allowed to pass through itself, both when coinciding with itself in the completely folded state, and possibly also in performing the folding moves needed to reach that state. If the triangles of the triangulation are not already of the same size and shape, a topological transformation is permitted to make this the case. Also, it is necessary to slightly relax isometry during folding (for example, elastic hinges might connect the triangles) to facilitate escape from geometric obstructions that may arise in transitioning between partially-folded states.

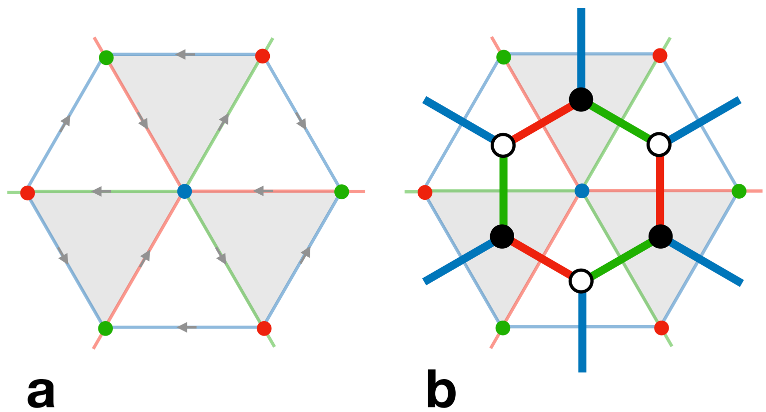

Figure 2: *In Figure 2a, the four equivalent conditions given by Di Francesco and Guitter for a planar triangulation to be completely foldable are: its vertices have a proper 3-coloring (red, green, blue); or its faces have a proper 2-coloring (white, gray); or its edges can be oriented such that each triangle has a well-defined orientation (arrows); or it is Eulerian (every vertex has even degree.) In Figure 2b, the dual is therefore planar, bipartite (2-vertex colorable,) and cubic (all vertices are degree 3,) and can be equipped with a proper 3-coloring of its edges derived from the systematic (but improper) edge coloring in the triangulation (pale red, green and blue in 2a.)*

Complete Folding of Triangulations on the Sphere

We limit ourselves to closed surfaces of spherical topology. Restricting to spherical (a.k.a., planar) topology, Di Francesco and Guitter found four equivalent characterizations of completely foldable planar triangulations (see Figure 2a):

1. The triangulation is vertex 3-colorable;
2. its faces are 2-colorable;
3. its edges may be oriented so that the boundary of each triangle receives a well-defined (clockwise or counterclockwise) orientation;
4. the triangulation is Eulerian (meaning there is an even number of edges—equivalently of triangles—around each vertex.)

The last characterization yields the simplest statement: an Eulerian planar triangulation can be completely folded.

Physical Reality of the Completely Folded State on the Sphere

Modeling physical (self-avoiding) folding is notoriously difficult; we do not yet know very much about general cases. A first possible obstruction to the physical reality of complete folding is that there might not be any arrangement of physical triangles in a stack such that elastic hinges joining triangles adjacent in the

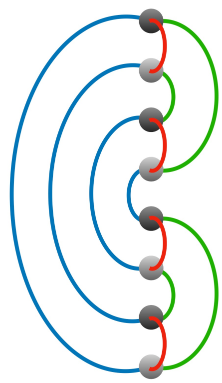

Figure 3: *Example of an edge-colored BCP (bipartite cubic planar) graph with its vertices arranged in a dispersable order: edges of like color do not cross each other. A theorem due to Overbay [7] states that all such dispersable orders for regular graphs alternate in vertex color.*

surface would be uncrossed. A recent advance in graph theory has, for the spherical case, dispensed with this possible obstruction. The dual of an Eulerian planar triangulation (see Figure 2b) is a cubic (a.k.a., 3-regular) planar graph; it is also bipartite, meaning its vertices can be properly 2-colored (since the faces of the primal were 2-colorable.) The edges of the primal triangulation can take on an *improper*[3] edge-coloring by taking the color of the opposite vertex; the edges of the dual, on the other hand, pick up a *proper* coloring by taking the color of the primal edge that they cross. So we now have an alternate mathematical model of our Eulerian triangulation, namely a bipartite cubic planar (*BCP*) graph endowed with a proper 3-coloring of its edges. In seeking a physical stacking order of the triangles, we are seeking a linear order for the vertices in the BCP graph that allows each color class of edges to be drawn without crossings (Figure 3.) Such a linear order of vertices is known as a *dispersable order* in the theory of book embeddings. Three recent papers present proofs that all BCP graphs indeed have a dispersable order [5, 1, 8].

Differences from Origami

Complete folding of a triangulation has some aspects in common with origami, but it is worth underscoring the differences. Origami begins with a unitary, topological disk that is geometrically flat, while the completely foldable triangulations we are considering begin as a topological sphere in whatever geometric configuration was convenient for connecting the individual triangles (that may well have been the completely folded configuration.) No part of the triangulation, other than the individual facets, is guaranteed to be flat. Folded paper enforces a strict constancy of distances measured in the folded plane of the origami paper (isometry,) but elastic joins between triangles are a necessity for transitioning between geometric configurations (or even for achieving the completely folded configuration, if the triangles have thickness.) Finally, unfolding is the problem rather than folding: the completely folded state is a mathematical given, we wish to unfold it into something bigger.

Foldable Quadrangulations

Quadrangulations

We will only deal with quadrangulations of the sphere that are *simple*, that is, they have no parallel edges; equivalently they have no cycles—facial or non-facial—of length 2 or less. All quadrangulations on the sphere are bipartite (2-vertex colorable), so they can have no odd cycles. Thus the smallest cycle we encounter in

[3]an edge coloring is proper if no two edges incident to the same vertex have the same color.

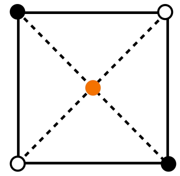

Figure 4: *Quadrangulations on the sphere are vertex 2-colorable, so central triangulations of their faces are vertex 3-colorable, and therefore completely foldable.*

a simple quadrangulation on the sphere is of length 4. Being bipartite, a quadrangulation on the sphere has a proper vertex 2-coloring. We assume one of the two possible bicolorings has already been fixed: thus we are always dealing with a *bicolored* quadrangulation. A new vertex added to the center of each face can bear a third color, and thus, after triangulating each quad face using this central vertex (Figure 4), we obtain a planar triangulation equipped with a proper vertex 3-coloring. Therefore, when folding along both diagonals is permitted, any simple quadrangulation of the sphere can be completely folded onto a single triangle.

Inductive Generation of Simple Quadrangulations of the Sphere

In 2005, Brinkmann et al. [3] showed that two context-restricted operations dubbed P_0 and P_1, can generate all the simple quadrangulations of the sphere, starting from the square. We will show that these two operations can be realized by local unfolding of triangulated quad faces, and therefore local unfolding is one possible way to grow a shape whose surface is defined by a quadrangulation of the sphere. We will explore this possibility with a small physical model.

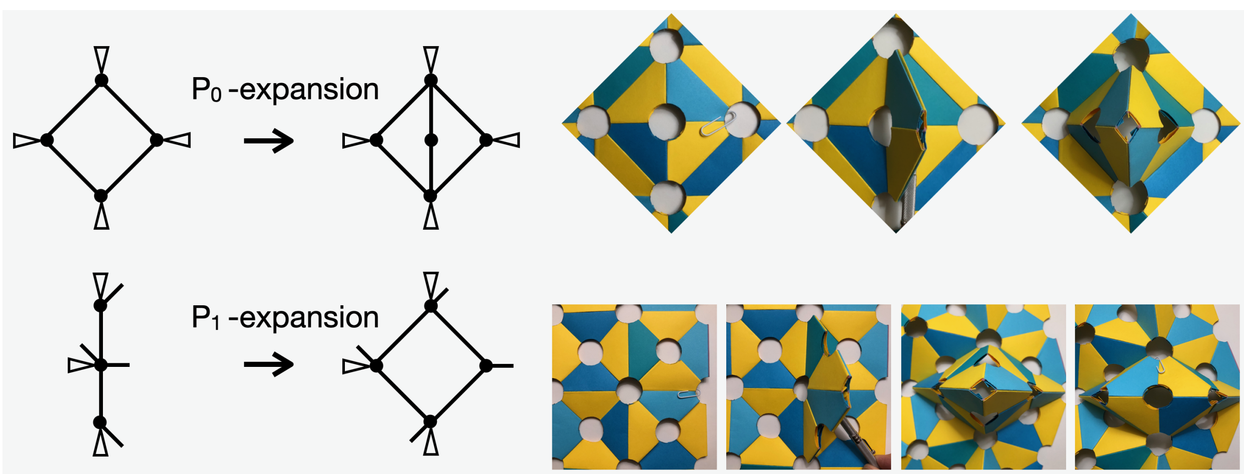

Figure 5: *Quads hinged along both diagonals can unfold in ways that realize the P_0-expansion (top) and the P_1-expansion (bottom.) In the initial position the undeployed quads lie in a piggy-back configuration (i.e., flat against the already deployed surface, with their hinges in alignment with the hinges underneath.) The realization of the P_1-expansion actually unfolds two quads and then folds one away. The narrow triangles in the schematic diagrams represent any number (≥ 0) of additional edges incident at that position.*

P_0 and P_1 are shown in Figure 5 as schematic diagrams along with photographs of the unfoldings that realize them. In the growth, or inductive, direction these operations are called P_0*-expansions* or P_1*-expansions;* in the inverse, or reductive, direction they are called P_0*-reductions* or P_1*-reductions.* The reduction operations are of interest because, when we wish to grow a particular quadrangulation starting from, say, the square embedded on the sphere, the simplest way to discover a growth sequence is to find a sequence of reduction operations that reduces the quadrangulation down to a square. In that way, we discover a growth sequence from the square that is simply the corresponding expansion operations applied in the reverse order.

The P_0-reduction attacks any degree-2 vertex in the quadrangulation, eliminating one of its two neighboring quads. In accordance with the proof provided in [3], we must use P_0-reductions iteratively until no degree-2 vertices are in the quadrangulation, before any resort is made to the P_1-reduction. The P_1-reduction attacks any degree-3 vertex in the quadrangulation, eliminating one of its three neighboring quads—except in some cases where the choice is reduced to two because eliminating one of quads would create a parallel edge. That quad cannot be eliminated (yet) because doing so would not preserve the simple-ness of the quadrangulation.

The unfolding that realizes the P_0-expansion (top of Figure 5) is very simple and trouble-free. The unfolding that realizes the P_1-expansion (bottom of Figure 5) unfolds two quads and then eliminates one with a P_0-reduction. This work-around introduces complications in deploying this unfolding, extra thickness, and possible mechanical interferences with neighboring P_1 folds.

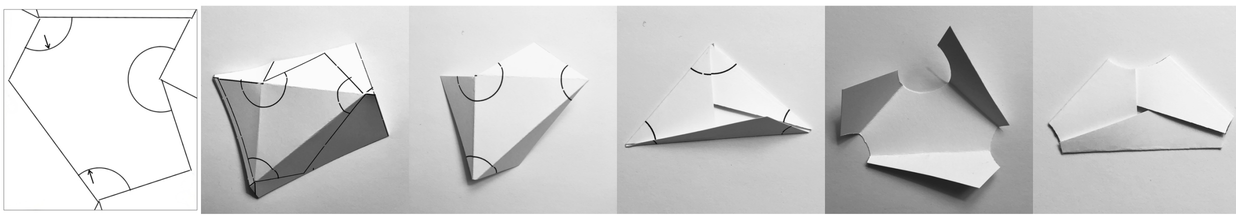

Figure 6: *Paper models of hinged triangles were made (L to R) by printing the pattern, creasing over a sharp edge at the six tick-marks around the perimeter, trimming just inside the straight lines, then putting the piece in the folded position before trimming just inside the circular arcs. After looping elastic bands on the three wings (bands go on the dark triangles only, and stay there permanently) fold the wings down in order of decreasing height, and finish by tucking the shortest wing under the tallest.*

Figure 7: *A growth sequence for a quadrangulation can be worked out with a physical modeling system. Build the quadrangulation, then label the faces by inserting numbered 'black' diagonals (i.e., diagonals connecting the black vertices of the bicolored quad.) Reduce the quadrangulation observing the rules for P_0- and P_1-reductions. At each step in the reduction two quad edges and a numbered black diagonal are eliminated. Record at each step the number of the diagonal removed and whether the face reduction contracted the 'black' or the 'white' diagonal of the face. This all the information needed to connect the hinged triangles and sequence their deployment.*

Making the Models

In 1961, Architect Frederick Bassetti [2] invented a clever way to make hinges using elastic bands and folded cardboard. The models in the photographs were made from 65-lb (176 g/m2) cardstock. The template in Figure 6 was printed at a scale that made the dimension indicated by the arrows equal to 54 mm. At that scale, the little elastic bands made for the Rainbow Loom® work well for both hinge widths. Instructions for folding, trimming, looping on the elastic bands, and assembly are found in the caption of Figure 6. I found a growth sequence empirically for this model (Figure 7) using the Flexeez® construction toy. The growth sequence in action is seen in Figure 8.

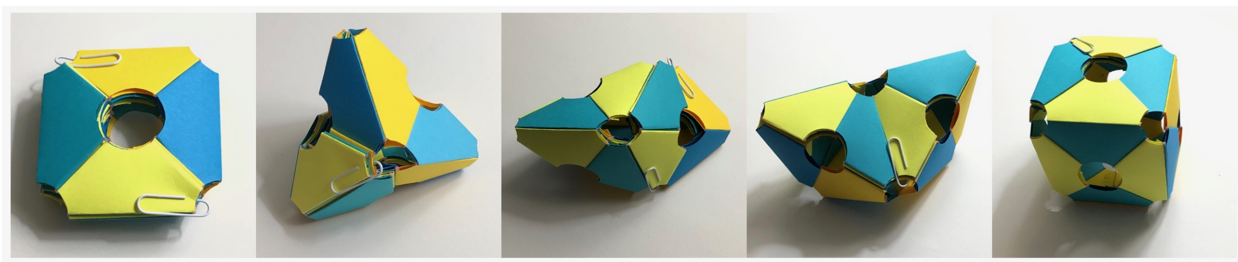

Figure 8: *A growth sequence for the cube. Each growth stage is a simple quadrangulation of the sphere.*

Summary

I have shown that the (P_0, P_1) inductive generation of simple quadrangulations on the sphere can be realized by the unfolding of hinged triangular plates. A number of theoretical and practical issues remain to solved: possible mechanical interferences in the P_1-unfolding, and whether an optimal growth sequence can avoid such interferences; and practical issues of sequencing and actuating the unfolding in a sculpture that works without assistance.

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