Hamiltonian Cycle Art:
Surface Covering Wire Sculptures and Duotone Surfaces

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Abstract

In this work, we present the concept of ”Hamiltonian Cycle Art” that is based on the fact that any mesh surface can be converted to a single closed 3D curve. These curves are constructed by connecting the centers of every two neighboring triangles in the Hamiltonian triangle strips. We call these curves *surface covering* since they follow the shape of the mesh surface by meandering over it like a river. We show that these curves can be used to create wire sculptures and duotone (two-color painted) surfaces.

To obtain surface covering wire sculptures we have developed two methods to construct corresponding 3D wires from surface covering curves. The first method constructs equal diameter wires. The second method creates wires with varying diameter and can produce wires that densely cover the mesh surface.

For duotone surfaces, we have developed a method to obtain surface covering curves that can divide any given mesh surface into two regions that can be painted two different colors. These curves serve as a boundary that define two visually interlocked regions in the surface. We have implemented this method by mapping appropriate textures to each face of the initial mesh. The resulting textured surfaces look aesthetically pleasing since they closely resemble planar TSP (traveling salesmen problem) art and Truchet-like curves.

Keywords: Shape Modeling, Computer Aided Sculpting, Texture Mapping, Computational Aesthetics, Computational Art

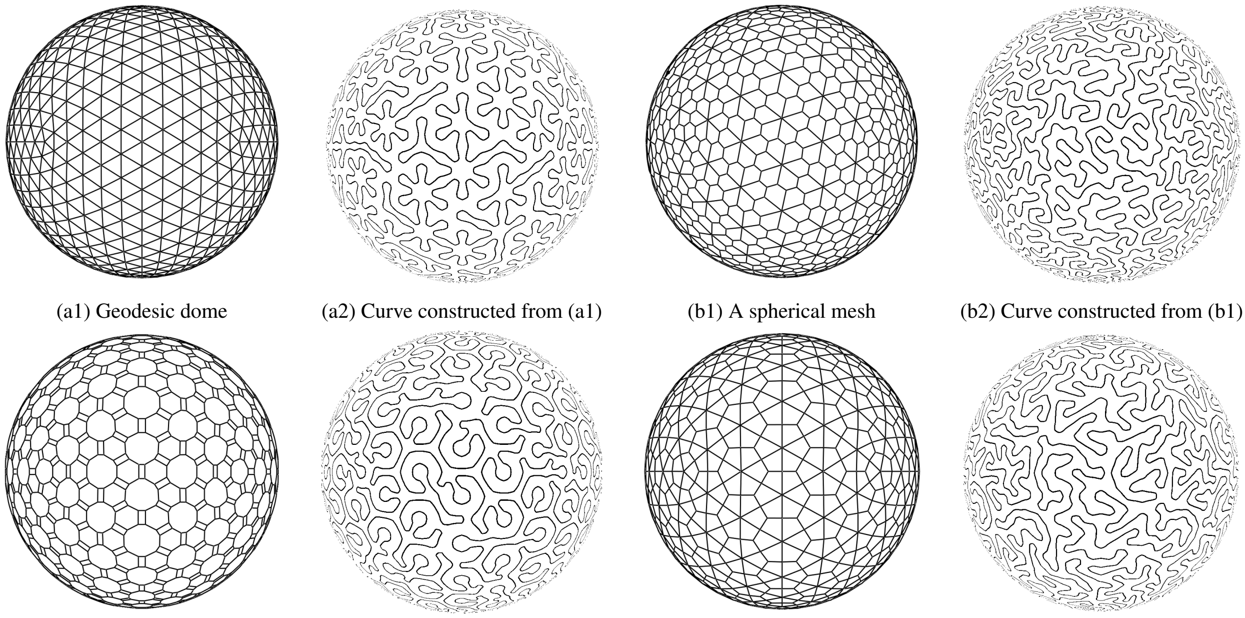
1. Introduction and Motivation

In this work, we introduce a simple approach that provides methods to create a variety of artworks. Our approach is based on converting any given mesh surface into a closed 3D curve that follows the shape of the given surface. Our work is based on Gabriel Taubin's work on constructing Hamiltonian triangle strips on quadrilateral meshes [1, 2, 3].

In graph theory, a Hamiltonian path is a path in an undirected graph that visits each vertex exactly once. A Hamiltonian cycle (or Hamiltonian circuit) is a Hamiltonian path that is a cycle. Note that not every graph has a Hamiltonian cycle. Hamiltonian triangle strips are defined in duals of triangular meshes. Taubin showed that it is always possible to construct a triangular mesh from any given quadrilateral mesh such that the dual of the triangular mesh has an Hamiltonian cycle. Moreover, he presented simple linear time and space constructive algorithms to construct these triangle strips. His algorithms are based on splitting each quadrilateral face along one of its two diagonals and linking the resulting triangles along the original mesh edges. With these algorithms every connected manifold quadrilateral mesh without boundary can be represented as a single Hamiltonian generalized triangle strip cycle.

Using Taubin's algorithms to construct a closed curve in 3D is straightforward. One can simply connect centers of triangles in the triangle strip to obtain a control polygon in 3D. The resulting control polygon can be turned into a smooth curve using a parametric curve such as B-Spline as shown in Figure 1 [2]. These curves can be used for creating artworks. Designers of these curves have a significantly large number of aesthetic possibilities. There are two ways to control aesthetic possibilities for surface covering curves:

- *Designing Mesh Structures:* The shape of any



(a1) Geodesic dome (a2) Curve constructed from (a1) (b1) A spherical mesh (b2) Curve constructed from (b1)

(c1) A spherical mesh (c2) Curve constructed from (c1) (d1) A spherical mesh (d2) Curve constructed from (d1)

Figure 1: Examples of surface covering curves on a sphere: Spherical mesh surfaces are converted into closed 3D curves which follow the shapes of the original spheres. Back-faces in meshes and back-face parts of the curves are not drawn for cleaner images.

given surface can be approximated by a wide variety of meshes. Therefore, designers, by choosing different meshes, can obtain aesthetically different curves. Examples that show the effect of the structure of the underlying mesh on a spherical shape are shown in Figure 1. In this figure, the control meshes are obtained using a variety of subdivision schemes available in TopMod3D such as honeycomb and pentagonal subdivisions [4, 5, 6]. The spherical shapes are obtained by simply moving vertices of the meshes into a unit sphere. For a detailed discussion of how these structures can be obtained see Akleman et al. [7].

- *Controlling Shapes of Curves:* Mathematically speaking, there are many ways to form surface covering curves for any given mesh [1, 8]. This mathematical property provides additional aesthetic possibilities since designers can have additional control over the shapes of the curves. We prefer wavy curves like a meandering river since they resemble space filling curves [9] or TSP (traveling salesmen problem) art [10] embedded on surfaces.

1.1. Surface Covering Wire Sculptures

To convert curves into 3D wires, we sweep a polygon or a line along the curve. This process normally requires rotation minimizing frames to avoid undesirable twists [11]. In our case since the curves are on surfaces it is possible to avoid twists by using surface normals to obtain Frenet frames (see [12] for details). Therefore, it is easy to convert these curves into 3D structures that can be shaded, rendered and 3D printed.

We have developed two methods to construct corresponding 3D ribbons and wires from given curves as extruded lines and polygons along the curves [2]. The first method, called constant-diameter, simply turns the curves into constant thickness ribbons or equal diameter wires. The second method, which we call variable-diameter, creates ribbons with varying thicknesses (or wires with changing diameters) that can densely cover the mesh surface. We have developed a system that converts polygonal meshes to surface filling curves, ribbons and wires. All the images of wire sculptures are direct screen captures from the system.

Figure 2 shows an example obtained by using constant and variable diameter methods. Our variable diameter method guarantees that the sizes are relative to the underlying triangles. Therefore, the actual widths of ribbons are different in different parts of the mesh. Fig 8 shows visual effects of constant vs. variable and ribbon vs. wire for the same spherical mesh.

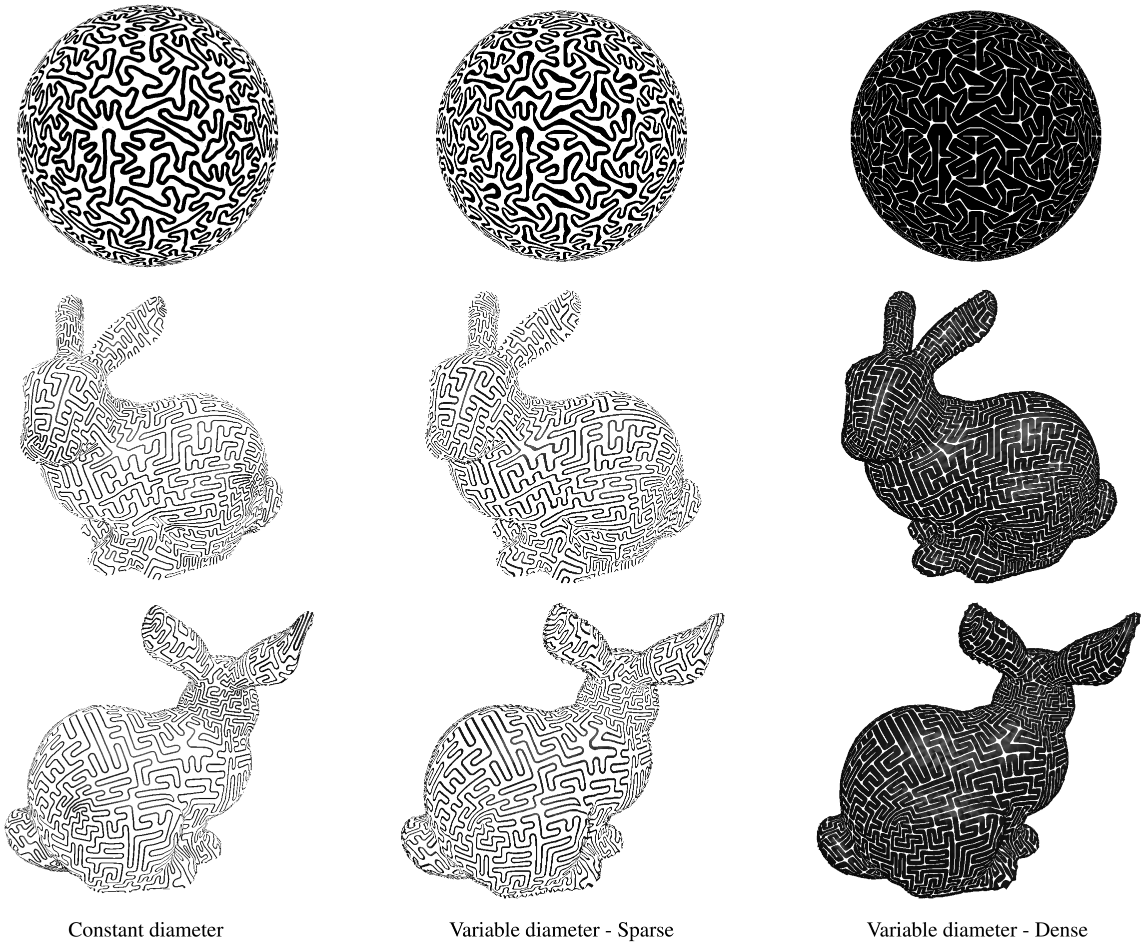


Figure 2: Dense covering of surfaces using ribbons with changing width. The parts of the curves that are occluded by original surfaces are not drawn for cleaner images.

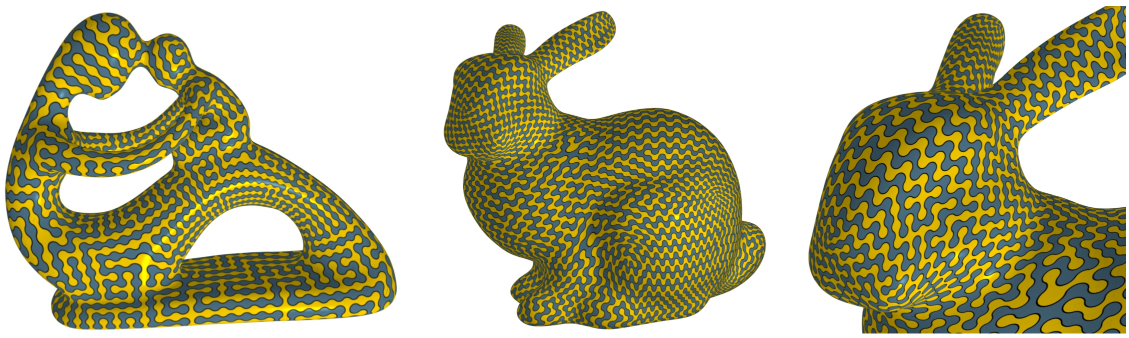


Figure 3: Fertility and Stanford bunny as duotone surfaces.

1.2. Duotone Surfaces

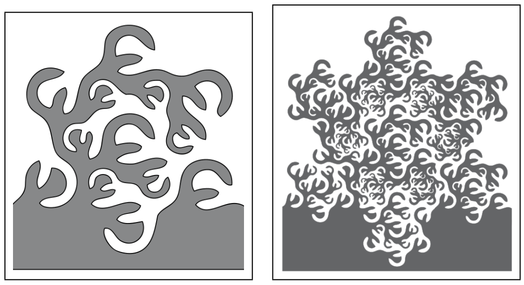
The Jordan Curve Theorem states that any simple closed curve in the plane separates the plane into two regions: the part that lies inside the curve, and the part that lies outside it [13]. Although the theorem seems to be very intuitive, the proof is complicated since closed curves can be complicated sometimes, for example fractal curves. Many artists have used this property to create artworks over plane by creating interesting curves such as Fractal art, Traveling Salesmen Problem(TSP) art and Truchet-like curves. Interestingly, Jordan's theorem is only correct for genus-0 surfaces. Any single curve on a surface with positive genus does not necessarily separate the surface into two regions.

To obtain a duotone surface, we present a simple approach to construct surface covering curves that can separate surfaces into two regions [3] (see Figures 1 and 3). Our method is based on a useful property of vertex insertion schemes such as Catmull-Clark subdivision: If such a vertex insertion scheme is applied to a mesh, the vertices of the resulting quadrilateral mesh are always two colorable. Using this property, we can always classify vertices of meshes that are obtained by a vertex insertion scheme into two groups. We show that it is always possible to create a single curve that covers the whole surface such that all vertices in the first group are on one side of the curve while the other group of vertices are on the other side of the same curve.

We have implemented our approach using Truchet tiles where the boundary curve is not explicitly constructed but appears as the boundary of two regions formed by Truchet tiles. Therefore, our implementation can be considered as an embedding of duotone Truchet tiles over surfaces [14]. We therefore call our textured surfaces duotone surfaces. However, unlike duotone Truchet tiles, our duotone surfaces guarantee only two regions separated by a single curve.

2. Previous Work

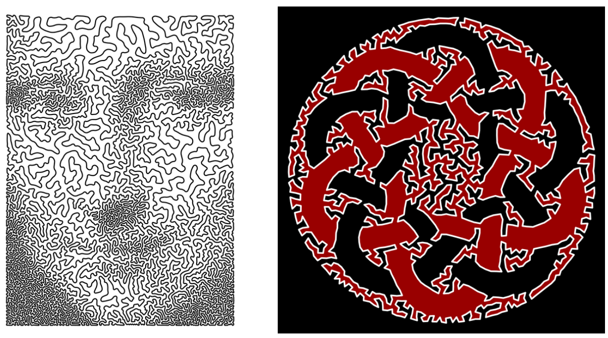
Our surface covering wire sculptures visually resemble space-filling curves, discovered by Giuseppe Peano [15] by his construction of a continuous mapping from the unit interval onto the unit square. Space filling curves became very popular among mathematician-artists after Benoit Mandelbrot's seminal work on Fractal Geometry [9]. In his book, he categorized space filling curves as fractals since they can be constructed using a replacement algorithm starting from a simple curve. Mathematician and artist Douglas McKenna [16], who also created many images in Mandelbrot's Fractal Geometry of Nature, enumerated over 20 million new space-filling recursive designs in the plane.



(a) Two iterations (b) Three iterations

Figure 4: An example of Mandelbrot's duotone space filling curve art. This particular sequence of images was created by Alexis Monnerot-Dumaine under the pseudonym Prokofiev on 24 January 2010.

Duotone coloring of the plane using the Jordan curve theorem is an artistic technique to create planar art using complicated curves. In artistic applications, the most widely used examples of such complicated curves are also space filling fractal curves [9]. Mandelbrot created several examples of duotone art especially using space filling curves. Mandelbrot also discovered a simple way to treat open space filling curves as closed by assuming they were drawn on an infinite plane (See Figure 4). The main aesthetic advantage of space filling curves over other fractal curves for creating duotone art is that they result in indistinguishable inside and outside structures as shown in Figure 4.



(a) Curve TSP art (b) Duotone TSP art

Figure 5: Two types of Travelling Salesman Problem (TSP) art. These images were created by Robert Bosch and used by permission.

Robert Bosch and Adrianne Herman [17] invented another way to create closed curves in the plane based on the Travelling Salesman Problem (TSP) [18]. In TSP, there exists a set of cities and a traveling salesman who resides in one of the cities. The salesman wants to visit each of the other cities exactly once and then return home and would like to visit the cities in an order that will minimize the total length of his tour. Determining an optimal itinerary for the salesman is one of the most well known and well-studied problems in mathematics, computer science, and operations research. Robert Bosch and Adrianne Herman noticed that for interestingly placed city locations, the piecewise curve showing the salesman's itinerary looks artistic (see Figure 5.a). They used points on a grid to create an original artwork. This method was simple but required a large number of dots to produce a decent picture because the dots tended to clump together. Craig S. Kaplan [10] used weighted Voronoi stippling to create positions of the cities. With weighted Voronoi stippling, using substantially fewer dots, it is possible to obtain a more organic appearance. Moreover, by distributing cities with a density that locally approximates the darkness of a source image, and passing the cities to a program that finds a TSP tour, they have produced TSP-art that resembles the source image. An additional advantage of the optimal tours is that they are guaranteed to be closed simple curves. Therefore, Bosch also used TSP curves to create a duotone coloring of the plane [19] (see Figure 5.b).

A method that is closely related to Taubin's work is Truchet tiles, which were originally introduced by Sebastien Truchet as all possible patterns formed by tilings of right triangles oriented at the four corners of a square [20, 21] in a square grid structure. Truchet's triangulations of a grid can be considered a special case of triangulation of quad meshes. In Taubin's scheme any particular triangulation of a quadrilateral mesh corresponds to a set of cyclic triangle strips, which may not be Hamiltonian. Taubin, in addition, showed that by re-tiling the quadrilateral mesh it is always possible to obtain Hamiltonian strips.

An extension of Truchet tiling that is related to this work is introduced by Clifford A. Pickover [22] as a single tile consisting of two circular arcs of radius equal to half the tile edge length centered at opposed corners. The two possible orientations of this tile, and tiling the plane using tiles with random orientations gives visually interesting curves called Truchet curves [23, 24]. Truchet curves are not necessarily single curves, but they guarantee to separate the plane into two regions and therefore they are also used to create duotone planar artworks [14].

In this paper, we show that it is always possible to obtain a single closed curve that covers a surface similar to TSP art and space filling curves. In terms of visual aesthetics, our curves most closely resemble Truchet curves. In fact, if our method is applied to a planar grid, the result will be a single Truchet curve. Although our curves cover space similar to space filling curves, they are not strictly self-similar, i.e. fractals. However, our results exhibit similarities that are visible in our examples. These similarities are just result of the structure of the underlying mesh and initial choices. Unlike TSP art, our curves do not guarantee to provide the shortest route, but they visually resemble random TSP art.

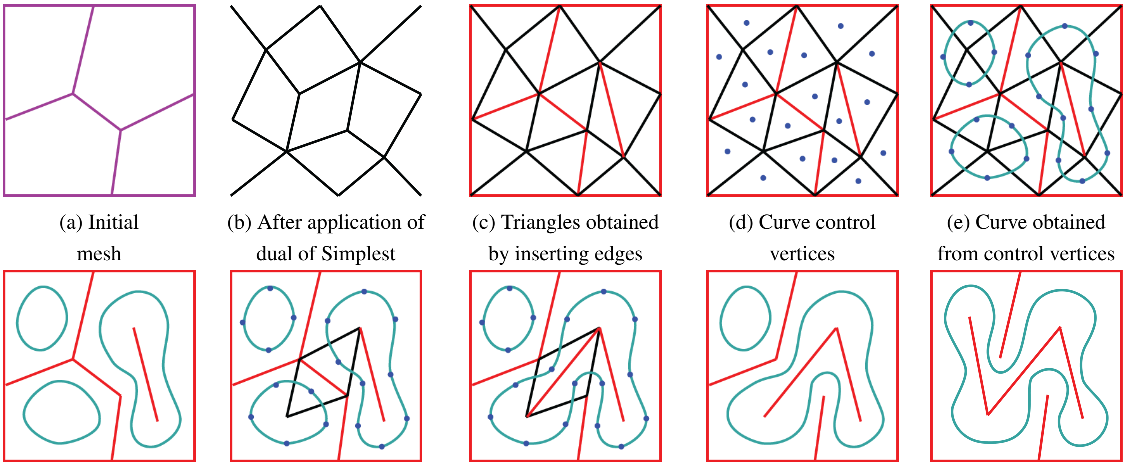
Duotone surfaces can be considered as embedding duotone plane art such as TSP or Truchet art on surfaces. Our approach is based on the construction of a single curve on a surface that can separate the surface into two regions. With this property, resulting surfaces can always be colored by two colors. In terms of visual aesthetics, our results in duotone surfaces more resemble duotone Truchet planar art.

3. Surface Covering Curve Construction

Our approach can be considered a 2-step process: (1) identify an Hamiltonian cycle that connects vertices of the dual of a given mesh, (2) use the 3D positions of vertices (i.e. face-centers of the original mesh) as control vertices of a smooth curve. The resulting curve is guaranteed to be closed and follow the overall shape of the surface. We point out that it is NP-hard to find "an Hamiltonian cycle" for even cubic 3-connected planar graphs [25]. It is also known that existing exponential-time algorithms for Hamiltonian cycles are not sufficient to find single strips for triangular meshes of more than 100 triangles [18]. Fortunately, many researchers have observed that the hardness of finding Hamiltonian cycles can be simplified by minor variations of the problem statement. For example, by adding a few new triangles, it is possible to significantly simplify the Hamiltonian cycle problem without changing the input geometry and visual quality [26].

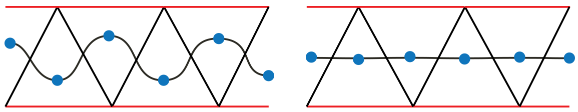
Taubin showed that from a quadrilateral manifold mesh, it is possible to construct a triangular mesh with an associated Hamiltonian cycle in linear time. The construction algorithm simply splits each quadrilateral into triangles and flips edges until the triangles are ordered into a single strip [1].

The curve generation consists of 6 steps:



(a) Initial mesh
(b) After application of dual of Simplest
(c) Triangles obtained by inserting edges
(d) Curve control vertices
(e) Curve obtained from control vertices
(f) Resulting curves
(g) A diagonal edge selected
(h) Diagonal edge flipped New control vertices
(i) Combined curve
(j) Final result

Figure 6: Visual presentation of the Hamiltonian cycle construction and curve generation algorithm. Note that if the initial mesh is quadrilateral, step b can be skipped.



Un-weighted average
Weighted average

Figure 7: The effect of weighted average that favors one vertex of triangle.

- *Initial Mesh:* The initial mesh can be any manifold mesh surface of arbitrary topology. Although, we do not have any restriction, it is better to have only convex faces for aesthetic results. See Figure 6(a) where the original edges are drawn in purple color.

- *Quadrangulation:* Taubin's construction requires a quadrilateral mesh. Therefore, if the initial mesh is not quadrilateral, we need to convert the mesh into a quad mesh. To get the initial quad mesh we apply a quad-conversion subdivision scheme [7]. We prefer to use the dual of Simplest subdivision [7]. This subdivision can be obtained as two operations: (1) Simplest subdivision [27], and (2) dual operation. After this subdivision, each original edge of the initial mesh turns into a quadrilateral (see Figure 6(b) where newly created edges are drawn in black color.)
  Remark: This step can be skipped if the initial mesh is a quad mesh.

- *Initial Triangulation:* We insert edges to turn all quadrilaterals into triangles as shown in Figure 6(c). These newly added edges are shown in red color. Now, every triangle has two black edges and one red edge. Any initial triangulation is acceptable for the algorithm.

- *Control Vertex Position Computation:* For each triangle, we compute a center point as a weighted average of its vertex positions. Let $p_{0,0} = p_{0,1}$ denote the position of the vertex that is at the intersection of two black edges, and $p_{1,0}$ and $p_{1,1}$ denote the positions of the other two vertices (see Figure 9(a)). In other words, we treat the triangle as a quadrilateral of which two consecutive vertices share the same position (See Figure 7.a). Based on this idea, the control vertex position is computed as follows:

$$p_{cv} = \frac{p_{0,0} + p_{0,1} + p_{1,1} + p_{0,1}}{4}$$

Since $p_{0,0} = p_{0,1}$, this computation is a weighted average of vertex positions of triangle as follows:

$$p_{cv} = \frac{2p_{0,0} + p_{1,0} + p_{1,1}}{4}$$

These points, which serve as control vertices of a surface filling curve, are shown in Figure 6(d).
Remark: Using weighted average helps to avoid higher frequency components when the connections are not supposed to create high frequencies, as visually shown in Figure 7. Weighted average moves the control vertex to the middle of the triangle height along the curve direction.

- *Initial Curves:* We construct a control polygon by connecting center points. For this purpose, if two vertices share a black edge, which is created by the dual of Simplest subdivision, we connect these two vertices with an edge. After this operation, each original face is replaced by a closed curve as shown in Figures 6(e) and 6(f).

- *Combining Curves:* We, now, randomly choose a diagonal edge (i.e. red edges in the Figures 6) and flip it if it is between two separate curves (see Figure 6(g)). After the flip, we recalculate triangle centers again and reconstruct the curve. As shown in Figure 6(h) this operation connects the two curves into one. We continue this operation until we obtain one curve as shown in Figure 6(j).
  Remark 1: Twisted edges form a spanning tree for the dual of the initial mesh. In other words, this spanning tree connects all faces of the initial mesh as can be seen in Figure 6(j).
  Remark 2: After the flip operation, each triangle still has two black edges and one red edge.

By the curve generation algorithm, we create a single control polygon that passes through control points in 3D space. To obtain a smooth curve, the control polygon can be approximated or interpolated using a parametric curve such as B-Spline or Catmull-Rom curve [28].

4. Surface Covering Wire Sculptures

To obtain wire sculptures the curves must be converted into wires by sweeping polygons along the curves. It is also possible to convert them into ribbons by sweeping lines along the curves. For aesthetic purposes, the resulting 3D structures must look smooth and must not self-intersect. We have developed two methods for converting curves to smooth wires and ribbons, which we call the constant and variable diameter methods.

The constant diameter method is simply a line or a polygon extruded along the curve. One of our goals is to create dense covering in such a way that the ribbons or the wires cover the surface without leaving large gaps. The constant diameter method provides nice thin and smooth curves but cannot densely cover the surface without self-intersection. The variable diameter method allows us to provide dense covering.

The variable diameter method consists of three steps.

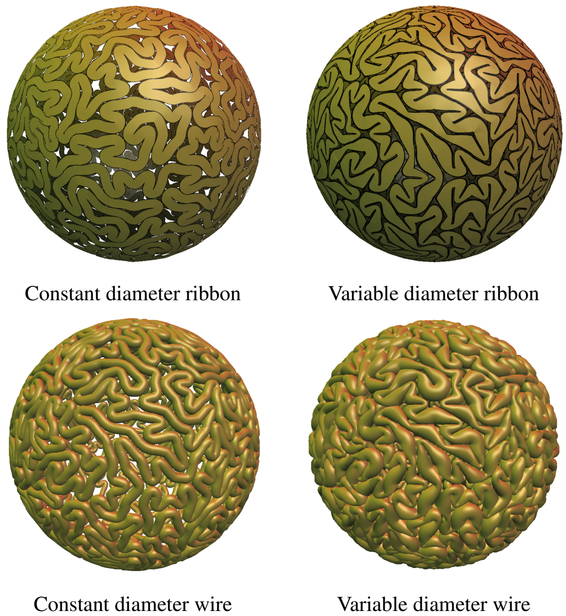
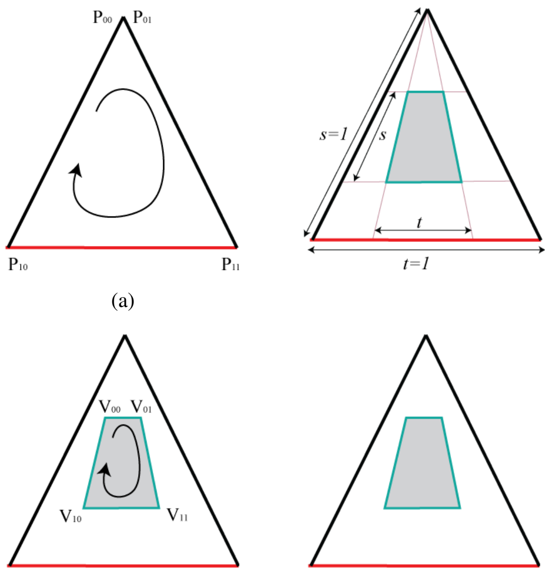


Figure 8: Another example that shows the visual effects of constant vs. variable and ribbon vs. wire for the same mesh. Back-face parts of the ribbons/wires are also shown.

- *Create a trapezoid inside of each triangle using two size parameters:* We again treat the triangle as a quadrilateral of which two consecutive vertices share the same position (see Figure 9(a)). Based on this idea, it is easy to compute the positions of corners of a trapezoid that is drawn inside of this triangle simply using the bilinear equation. Let $v_{0,0}$, $v_{0,1}$, $v_{1,0}$, $v_{1,1}$ denote the positions of four corners of a quadrilateral drawn inside of the triangle (see Figure 9(b)). Then

$$v_{m,n} = \sum_{i=0}^{1}\sum_{j=0}^{1} \frac{(1-(-1)^i s)(1-(-1)^j t)}{4} p_{i+m,j+n}$$

  where t and s are two parameters between 0 and 1; and the summations $i+m$ and $j+n$ are taken modulo 2.
  Remark 1: For $s = t = 0$ the bilinear equation gives weighted average we have already used for computing control vertices.
  Remark 2: Rotation order of vertices for trapezoid is important since it must give the same normal direction as the triangle. The bilinear equation guarantees the consistency of rotation order.
  Remark 3: To create solid wires, the 2D strips shown in Figure 9(b) can be extruded perpendicular to the surface to the desired slab thickness (See Figure 11.a).



(a) (b)

Figure 9: Computation of the trapezoid inside of the triangle.

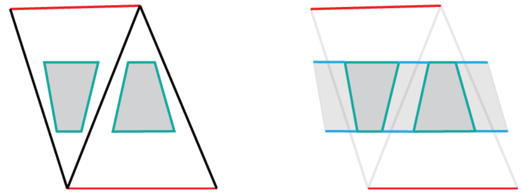

(a) Trapezoids (b) Connections

Figure 10: Connecting the trapezoids with quadrilaterals to obtain a control points for smooth ribbons.

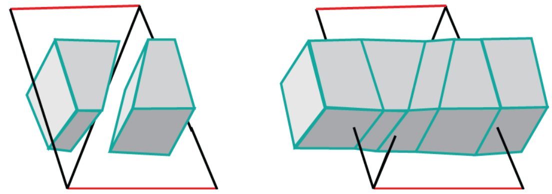

(a) Trapezoidal Prisms (b) Connections

Figure 11: Connecting the extruded trapezoids with prisms to obtain a control points for smooth wires.

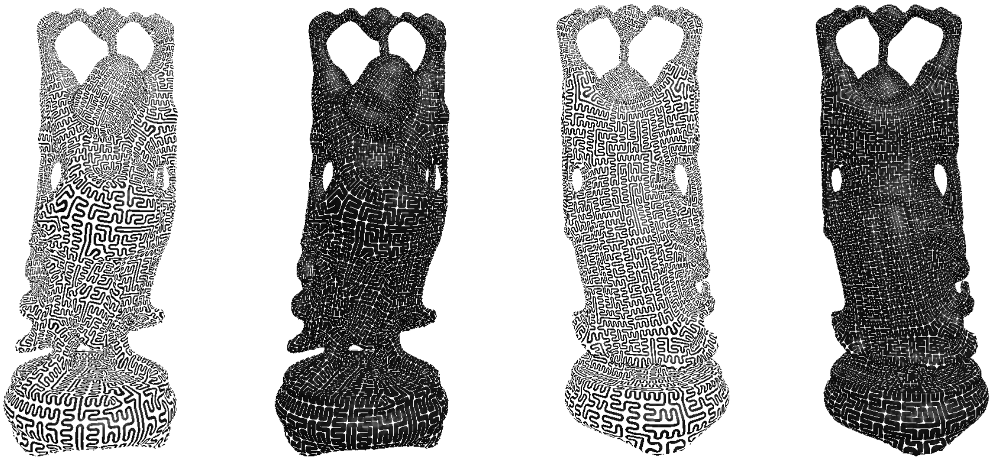
- *Connect trapezoids in two consecutive triangles using a quadrilateral connector:* This operation simply inserts two edges to form connectors as shown in Figure 10(b). In the figure, the newly inserted edges are colored in darker blue. This operation turns the initial triangular strip into a quadrilateral strip, which is used as a control polygon for smooth ribbons.
  Remark 1: The rotation order of vertices for connectors must also be consistent with two neighboring trapezoids. Since we start with a manifold mesh, the original triangles always have a consistent rotation order to start with.
  Remark 2: This operation also guarantees that if a part of the original triangle strip forms a parallelogram, the same part of the resulting quadrilateral strip also forms a parallelogram. In other words, if original data is not wavy, the resulting ribbon is guaranteed not to be wavy.
  Remark 3: To create wires, we connect trapezoidal prisms using hexahedral connectors, which are 3D versions of connectors in the ribbon case. As a result, we obtain a generalized toroidal shape (See Figure 11.b), which is used as a control polygon for smooth wires.

- *Obtain smooth ribbons or wires with a subdivision scheme:* For smoothing the resulting quadrilateral strips we use Catmull-Clark subdivision, which gives B-spline surfaces for regular structures such as quadrilateral strips [29]. As a result, the variable diameter method provides almost the same shapes for thin ribbons. However, even for thicker ribbons it does not self-intersect until it covers the underlying surface with almost no gap.
  Remark: To smooth wires, we simply smooth the generalized toroidal shape, which can again be smoothed using Catmull-Clark subdivision. The result is the same as a B-spline surface since the toroidal shape consists of only quadrilaterals and valence 4 vertices [29].

4.1. Examples and Results

We have developed a system that converts polygonal meshes into surface filling wires and ribbons. We provide s and t parameters to control the size of trapezoids. A user can interactively change the thickness of ribbons and wires by changing the parameters s and t. A very dense covering ribbon is obtained with value $s \approx 1$ and $t \approx 1$. Small values of s and t provide sparse covering. Our variable diameter method guarantees that the sizes are relative to the underlying triangles. Therefore, the actual widths of ribbons are different in different parts of the mesh.

If mesh models are created by a good quadrangulation scheme such as the quadcover method [30],



Sparse | Dense | Sparse | Dense

Figure 13: Buddha model covered with sparse and dense ribbons. These images are obtained using variable diameter method. Back-face parts of the ribbon are not drawn for cleaner images. The original quadrilateral mesh is obtained by wave-based anisotropic quadrangulation.

Thinnest | Thin | Thick | Thickest

Figure 14: Examples of quadrilateral meshes covered with constant diameter wires. Note that if all faces of the original mesh are approximately the same size as here, the constant diameter method can densely cover the surface without significant self intersection. Back-face parts of the wires are shown. The original quadrilateral mesh of the Bunny was obtained by the Quadcover method. The original quadrilateral mesh of the Fertility is obtained by mixed-integer quadrangulation.

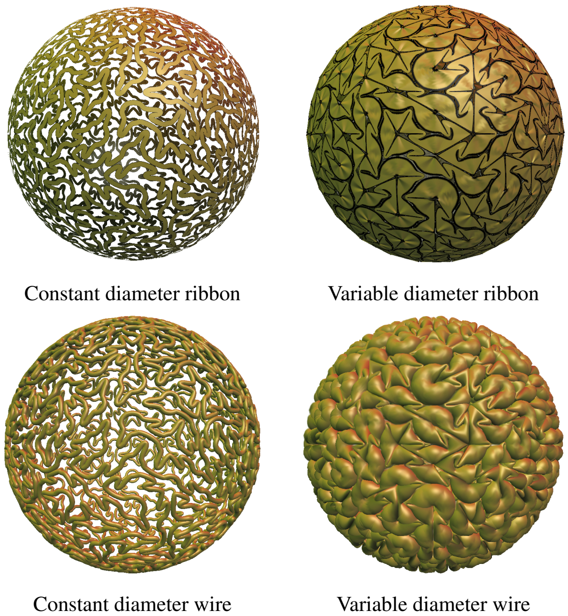


Figure 12: An example that shows the visual effects of constant vs. variable and ribbon vs. wire for the same mesh. Back-face parts of the ribbons/wires are also shown.

mixed-integer quadrangulation [31], and wave-based anisotropic quadrangulation [32], then even the constant diameter method can cover the surface without significant gaps as shown in Figures 13, and 14. This is mainly because such quadrangulation methods create almost-regular quadrilaterals. Moreover, there are only a limited number of non-valence 4 vertices.

5. Duotone Surfaces

Duotone surfaces are based on a special types of surface covering curves that can divide the surface into two regions. We observe that it is possible to obtain such curves if the vertices of the initial quadrilateral mesh are 2-colorable. Such 2-colorable quadrilateral meshes can be obtained by some subdivision schemes such as Catmull-Clark [29] (See Figure 15) and the dual of Simplest [27] subdivisions.

In this work, we use Catmull-Clark subdivision to obtain 2-colorable quadrilateral meshes. Figure 15 illustrates the remeshing scheme of Catmull-Clark subdivision, called vertex insertion. As shown in the figure, the vertex insertion scheme preserves original vertices of the mesh, called vertex-vertices. It also subdivides each edge by inserting a new vertex in the middle of each edge, called edge-vertices, and inserts a vertex in the

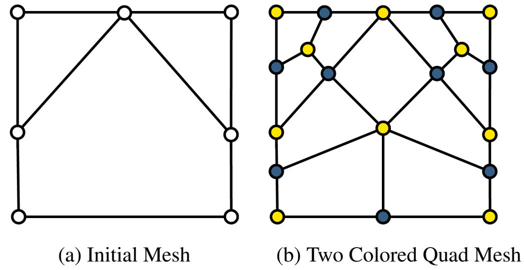


Figure 15: Conversion of a given mesh into a 2-colorable quadrilateral mesh by using Catmull-Clark subdivisiob.

middle of each face, called face-vertices. It also inserts edges between every face-vertex and its edge-vertices. If edge-vertices are labeled with one color and other vertices are labeled with another color, we obtain 2-colored quadrilateral meshes. In the figure, edge-vertices are labeled with dark blue color and rest of the vertices can be labeled with yellow color.

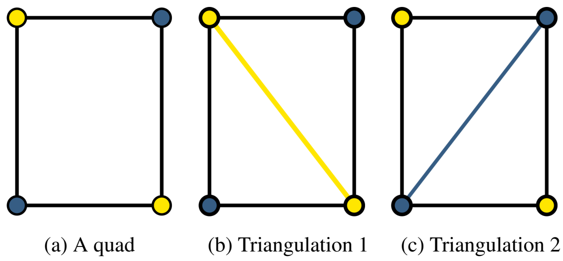


Figure 16: Triangulations of a quad face.

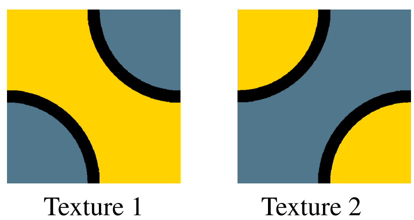


Figure 17: Texture maps that can cover vertices as defined in triangulations 1 and 2. These particular textures are called Truchet tiles which are used to create duotone planar art.

5.1. Texture Map Assignment

The underlying graph of a 2-colorable quadrilateral mesh is bipartite [33]. In other words, the vertices are now divided into two disjoint sets U_0 and U_1 such that every edge connects a vertex in U_0 to one in U_1. Moreover, the diagonal vertices of each quadrilateral of the mesh are in the same set, i.e. they have the same label as shown in Figure 16(a).

Our goal is to cover this mesh with a texture in such a way that vertices in U_0 will be colored yellow and vertices in U_1 will be colored blue. For every quadrilateral, there are two possible ways to assign a texture: there can be a connection either between two yellow vertices or two blue vertices. These two possible cases can be conceptualized as two possible triangulations of a quadrilateral as shown in Figures 16(b) and 16(c). The choice of triangulation of a given quadrilateral uniquely defines how to texture map that particular quadrilateral by using textures such as the ones shown in Figure 17.

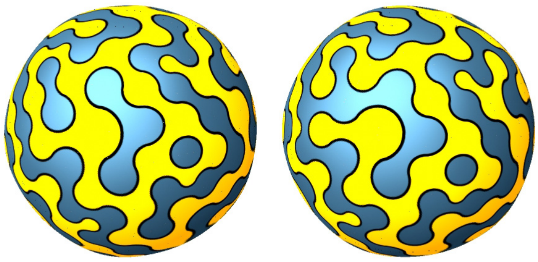


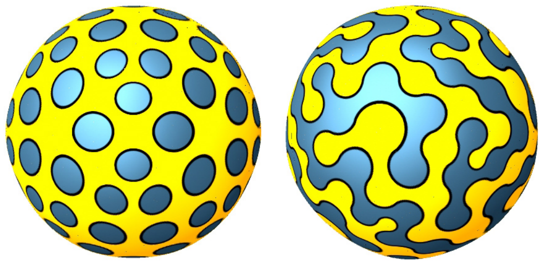
Figure 18: Duotone surfaces with disconnected regions. Our goal is to make all regions to be connected.

If we randomly triangulate all quadrilaterals and apply textures based on triangulations, we most likely obtain a two-colored surface that consists of disconnected regions (see some examples in Figure 18). Such random triangulations correspond to the embedding of duotone Truchet planar art to surfaces. Our goal in duotone surfaces is to connect all disconnected regions in the same color. The next section presents how to obtain such duotone surfaces.

5.2. Combining Disconnected Regions

It is possible to view the triangulated mesh as a graph that consists of three subgraphs: (1) The original bipartite graph; (2) the yellow graph that connects all vertices in U_0, e.g. the graph that consists of only yellow edges; (3) the blue graph that connects all vertices in U_1, e.g. the graph that consists of only blue edges. If both blue and yellow graphs are connected, the corresponding texture map will consist of two completely connected regions as we want. On the other hand, if only one of them is connected, there will be disconnected regions in the other one. For instance, Figure 19(a) illustrates an extreme case in which the yellow graph is connected allowing the yellow region to be connected, but the blue graph consists of isolated vertices which resulted in isolated blue regions.

For the surface of a 2-colorable quadrilateral mesh to have only two regions, we require both blue and yellow graphs to be completely connected. In a sphere, this means that neither of these graphs can have a cycle since a cycle in one graph makes the other one disconnected. Thus, both graphs must be trees covering all yellow and blue vertices respectively. If one of these graphs is a tree, the other one is also a tree[1]. Therefore, it is straightforward to obtain duotone surfaces as shown in Figure 19(b).



(a) (b)

Figure 19: Two duotone surfaces that exhibit completely different behavior. In (a) the yellow graph is completely connected, therefore the yellow region is connected. On the other hand, the blue graph consists of only disconnected vertices, therefore it produces individual circles on the surface. In (b) both yellow and blue graphs are trees resulting in two connected regions.

Lemma 1. *For an embedded bipartite graph on a genus-0 surface, say U_0 and U_1 are the two edge disjoint vertex sets and Y, B are Yellow and Blue graphs respectively. If one of the Y/B graphs is a tree, then the other is also a tree.*

Proof. Assume B is not a tree when given that Y is a tree. Then B has at least one cycle and/or B is a forest. It is impossible that B is a forest when Y is a tree, since in that case we can find a cycle in Y surrounding a tree in B, which contradicts Y being a tree. It is also impossible that B contains cycles. A cycle in B implies that a set of connected edges in $E(Y)$ are isolated, which contradicts our given hypothesis that Y is a tree spanning all vertices of Y. □

For a surface with a positive genus, constructing Hamiltonian triangle strips on quadrilateral meshes is sufficient to construct two connected components, which consist of yellow and blue edges respectively. Taubin [1] presents a simple linear time and space constructive algorithm, where each quadrilateral face is split along one of its two diagonals and the resulting triangles are linked along the original mesh edges. The triangles are flipped until we obtain a Hamiltonian strip. The Hamiltonian strip is actually the representation of the curve that serves as the boundary of blue and yellow regions. The diagonal edges in the resulting triangulation consists of two connected components. This leads directly to the following theorem.

Theorem 1. *For a bipartite graph, say U_0 and U_1 are the two edge disjoint vertex sets and Y, B are Yellow and Blue graphs respectively. Then, we can make Y and B connected respectively.*

5.3. Algorithm

Based on this discussion, for generating duotone surfaces we need an algorithm to make Y and B graphs connected. Our earlier algorithm to obtain surface covering curves is used for generating duotone surfaces with minor modifications as follows:

- *Conversion to 2-Colorable Quad Mesh:* Convert the input mesh to a 2-colorable quadrilateral mesh using a subdivision scheme such as Catmull-Clark [29] and dual of Simplest [27] subdivisions. We convert the input mesh into a 2-colorable quadrilateral mesh using Catmull-Clark subdivision. Let $M = (V, E, Q)$ denote the final quadrilateral mesh, where V, E and Q denote the set of all vertices, edges and quadrilateral respectively.
  Remark: If the mesh is already two-colorable, we skip this step.
- *Vertex Coloring:* Color the vertices of quadrilateral mesh with two colors. As seen in Figure 16, we color the vertices in V to either Blue or Yellow such that no edge exists in E whose end vertices have the same color. Say, $U_0 = \{v \in V \text{and } color = Blue\}$ and $U_1 = \{v \in V \text{and } color = Blue\}$.
- *Initial Triangulation:* Create an initial triangulation by inserting "diagonal" edges between either two yellow vertices or two blue vertices of each quadrilateral as seen in Figure 16(b) and (c) respectively. This will result in a yellow graph and a blue graph. This triangulation defines a set of curves as discussed earlier in the surface filling curves algorithm.
  Remark: We assign a Truchet tile (texture) to each quadrilateral face of M such that the texture placement is consistent with colors of diagonal edges as shown in Figure 17. This texturing results in a two colored surface that consists of disconnected regions as shown in Figure 18
- *Connecting the Disconnected Regions:* Choose a diagonal edge and flip it if it is between two separate curves. After the flip, reconstruct the curve. As discussed in section 3, this operation connects the two curves into one. In this case, the flip operation also changes the color of the diagonal edge. As a result both yellow and blue graphs change.
- *Obtaining only Two Regions:* We continue edge flipping operation until we obtain one single curve. One side of this single curve consists of only yellow vertices and edges; and the other side of this single curve consists of blue vertices and edges only. In each step, we map a Truchet tile (texture) to each quadrilateral face of M consistent with the colors of diagonal edges as discussed earlier..

Note that for a surface with a positive genus, there exist solutions in which both Y and B are connected and neither is a tree. This particular algorithm does not guarantee to make either Y or B a tree. On the other hand, it is always possible to make at least one of them a tree. An algorithm to guarantee that Y is a tree, as suggested by one of the anonymous reviewers, would be the following:
(1) Select the set of all quad diagonals which connect Yellow vertices, which forms a connected graph containing all yellow vertices.
(2) Compute any spanning tree of this graph, which becomes the final Yellow graph.
(3) For any quad not spanned by the tree, add its Blue edge into the Blue graph. This algorithm guarantees to produce a connected Blue graph following an argument similar to Lemma 1 as there are no Yellow cycles. Moreover, the Yellow graph is guaranteed to be a tree.

5.4. Conversion to Subdivision Surface

One final issue is that direct texture mapping of polygonal meshes results in G^1 discontinuities since a polygonal mesh is not G^1 continuous across the edges. We simply turn the polygonal mesh into a subdivision surface. Note that Catmull-Clark subdivision surfaces are already G^2 continuous everywhere except at extraordinary vertices. As our texture maps have the same color around vertices, discontinuous regions around extraordinary vertices will not be visible. On the other hand, the original Truchet textures are only G^1 continuous in edge boundaries, i.e. the two circle boundaries meet in the same point with the same tangent, but the centers of the circles are not the same (see Figure 17). Thus, we obtain only a G^1 continuous texture map although the surface itself is G^2 continuous in edge boundaries. As shown in Figure 21, it can be seen that G^1 continuity is sufficient to obtain good looking results.

5.5. Examples and Results

To obtain duotone surfaces, we have implemented texture mapping as a stand alone software using C++. The initial Catmull-Clark subdivision is done using publicly available software. The resulting mesh is exported as a non-textured .obj file. Our texture mapping software reads this .obj file and assigns appropriate textures and texture coordinates to each quadrilateral of the 2-colorable quadrilateral mesh. Now the textured mesh is exported as an .obj file. We then import this textured mesh into Maya [34] and turn it into a subdivision surface since Maya provides good quality subdivision surfaces [35]. All images in this paper were rendered in Maya as subdivision surfaces using default lighting. Figures 20 and 22 show several examples of duotone surfaces that are obtained by this process. To obtain higher frequency images, we simply obtain denser polygonal meshes using subdivision as shown in Figure 22. We assume that the meshes do not have high aspect-ratio or concave quadrilaterals. Such quadrilaterals might result in visually uninteresting results. Since we could not find references to any methods doing similar work, we could not compare our results against existing standards.

Strict Truchet tiles are not the only ones that can be used for texturing duotone surfaces. It is in fact possible to create a wide variety of aesthetic results using more colorfully designed tiles such as the ones shown in Figure 21.

6. Discussion and Future work

In this paper, we demonstrated that Hamiltonian triangle strips can be used to create artworks. We showed that any given mesh can be converted into a single closed 3D curve. We use these curves to create artworks, namely virtual wire sculptures and two-color textured (duotone) surfaces. We hope that more artists will be interested in experimenting with surface covering curves to develop other types of artworks.

Our wire sculptures and duotone surfaces are mostly related to sculpture and it should be possible to convert

Figure 20: Buddha as a duotone surface.

these virtual sculptures into physical sculptures by using 3D printers. Printing the wire sculpture should be more economical than 3D printing original models since the major cost of the printing is the material. For instance, Carlo Sequin used similar approach to reduce printing costs of dissecting puzzles [36].

One issue is that virtual wire sculptures contain significantly more faces than the faces of the original meshes. If the original mesh consists of only triangles, each original triangle turns into a minimum of 48 quadrilaterals as control shapes for smooth wires. Therefore, even the control shapes of our wire sculptures consist of more than several million polygons. Unfortunately, the internet based 3D printing services currently do not allow uploading models with such large polygon counts. These polygon counts are really a minimum to have nice results. Therefore, unless internet 3D printing services increase their limits it is not possible to print these models using these services.

Another issue is that printed sculptures may not have enough stability to hold their shape. In other words, the weight of the sculpture may not be supported by 3D printed wires. Since we have not printed any wire sculptures it is hard to evaluate stability issues. We are planning to print some of these wire sculptures and observe the stability issues in construction of physical sculptures.

One constraint of the method is that the results depend on the mesh structures. We think this is an advantage for the people who like to manipulate mesh structures. However this can be a problem for others, in particular when they want to get fat wires or large duotone regions with fine-resolution meshes. In such cases, they can always decimate the mesh, but decimation may not necessarily provide the desired meshes. It is, therefore, possible that some people can be frustrated from lack of full control resulting from mesh dependency. As a future work, it would be interesting to compute surface covering curves independent of mesh structures and provide good interfaces to control resulting curves.

One advantage of our method for people who want to create similar artworks is that the Hamiltonian strip is not unique and there exist many possible solutions [8]. Using this property, it should be theoretically possible to control the resulting wires and surface coloring by altering the number of branches. We prefer a high branch count for both yellow and blue trees which result in a more wavy/meandering boundary between two regions [1]. In our current implementation, this is hard-coded and we do not provide an interface for sculptors to control the amount of waviness. On the other hand it is possible to develop a simple interface by controlling the number of branches with a slider. For further applications to be used for novice users, there is also a need for a simple user interface to allow them to design surface curves directly.

Duotone surfaces can also provide sculpting opportunities. For instance, the two regions on the duotone surface can be obtained by cutting the surface into two 2-manifolds with boundaries. To create a sculpture, these two manifolds with boundaries can be turned to solid shapes which can be interlocked together to form the original shape. One future application could be to use this idea to design dissection puzzles [36].

We used to think that it would be possible to produce bad results from bad quality meshes. Based on the suggestion of one reviewer, we tested our hypothesis by creating duotone surfaces from “bad-quality-meshes”. Unexpectedly the resulting duotone surfaces turned out better than we predicted. Of course, our reaction is subjective and some people may not like these results (See Figure 23 for comparison).

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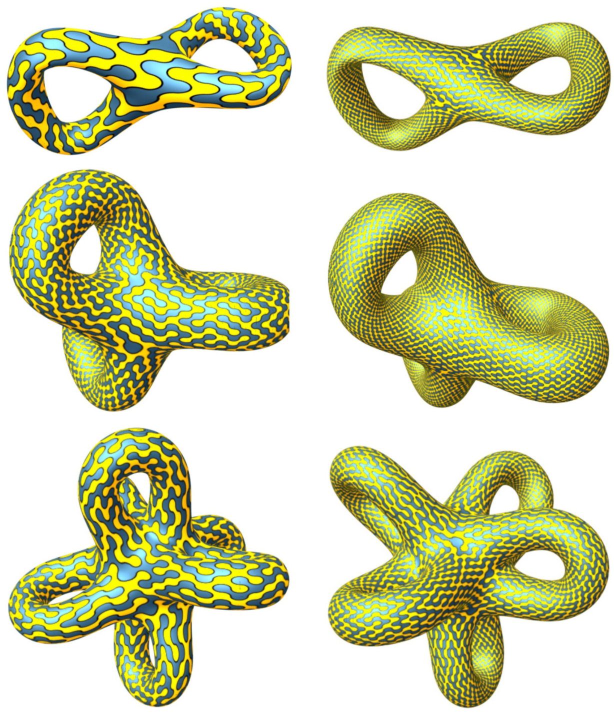

Figure 22: Positive-genus duotone surfaces.

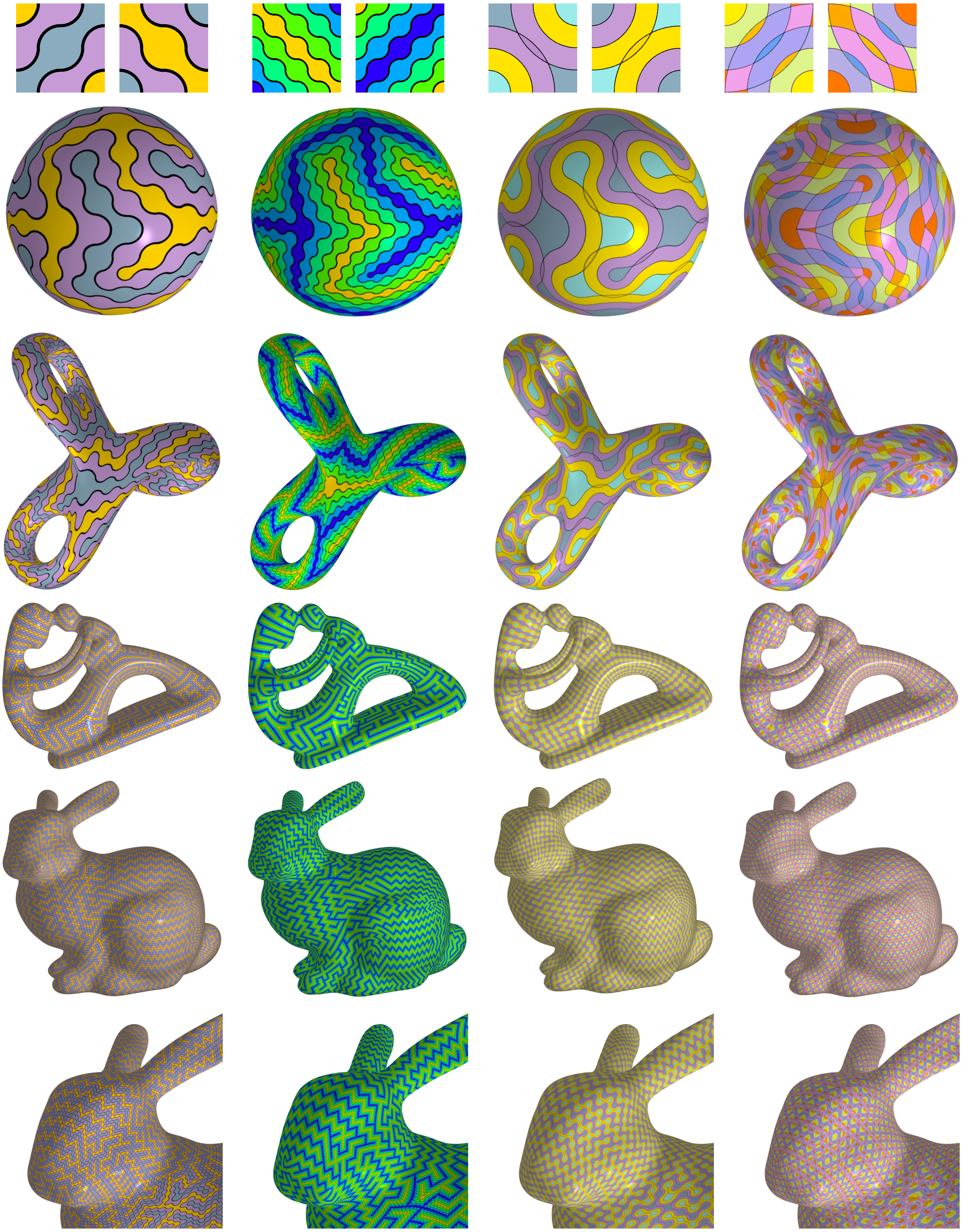

Figure 21: The top row shows four possible tiles that can be used to obtain more colorful versions of duotone surfaces. Duotone surfaces in each column are created using these tiles.

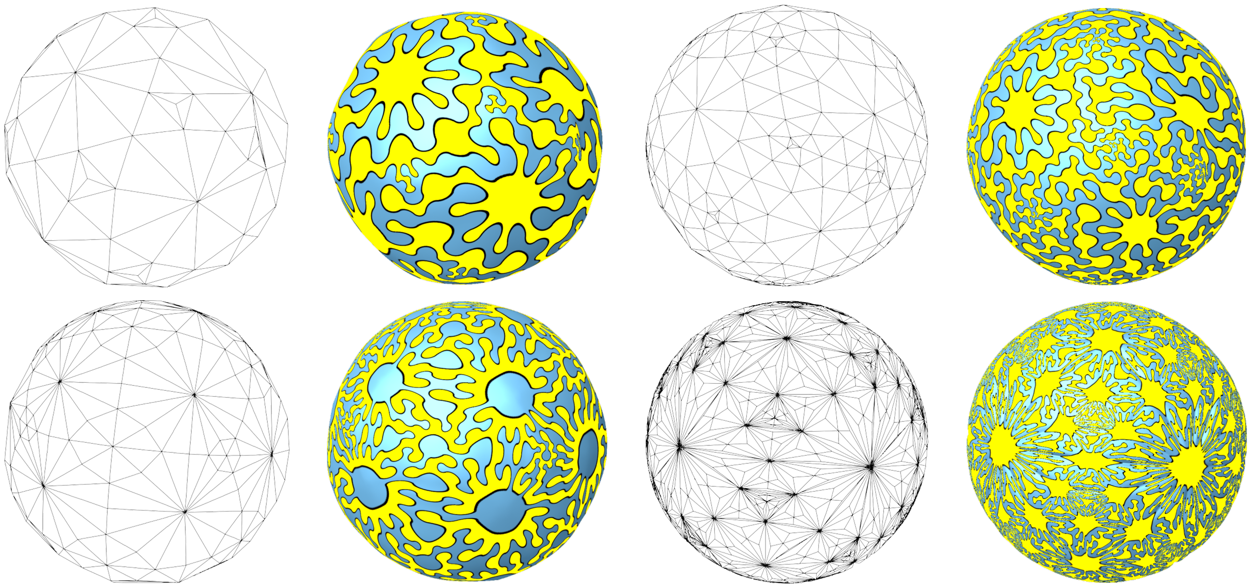

Figure 23: Examples of duotone surfaces that are produced from triangular meshes that includes skinny and small triangles.