Regular Meshes

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Abstract

This paper presents our preliminary results on regular meshes in which all faces have the same size and all vertices have the same valence. A regular mesh is denoted by (n,m,g) where n is the number of the sides of faces, m is the valence of vertices and g is the genus of the mesh. For $g=0$, regular meshes include regular platonic solids, all two sided polygons. For $g=1$ regular meshes include regular tilings of infinite plane. Our work shows that there exist infinitely many regular meshes for $g>1$. Moreover, we have constructive proofs that describe how to create high genus regular meshes that consist of triangles and quadrilaterals $(3,m,g)$ and $(4,m,g)$.

1 Introduction

Polygonal meshes are most commonly used representations in computer graphics applications. The polygonal meshes can represent complicated surfaces by subdividing them into simpler surfaces, which are called *faces* [Mantyla 1988; Baumgart 1972; Akleman and Chen 1999]. This subdivision idea even exists in real life. We often create complicated surfaces by combining simpler patches. Figure 1.A shows one such example, a ball that is created by hand-sewn 12 pentagonal patches. As seen in this real example, the patches can be curved and do not have to be planar.

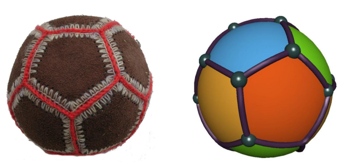

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Figure 1: Real ball and its abstract representation with faces, edges and vertices.

Formally, a *mesh* on a 2-manifold S specifies an embedding $\rho(G)$ of a graph G on the 2-manifold S. Each connected component of $S-\rho(G)$ plus the bounding edges in G makes a *face* of the mesh. Following the conventions adopted by many other researchers [Hoffmann 1989; Mantyla 1988], we allow a mesh to have multiple edges (i.e., more than one edge connecting the same pair of vertices) and self-loops (i.e., edges with both ends at the same vertex). The mesh is *cellular* if the interior of each face is homeomorphic to an open disk. Non-cellular meshes have also been studied in the literature but the corresponding modeling algorithms are more complicated [Hoffmann 1989; Mantyla 1988]. In manifold mesh modeling, we only consider cellular meshes.

In computer graphics applications, most commonly used manifold meshes are piecewise linear: the faces are planar polygons with straight edges. Triangular meshes are particularly popular since triangles with straight edges are always planar. Such planar faces are very useful for computer graphics: planar faces are easy to render. It is also easy to detect collisions with planar faces.

Planarity is a geometric condition. In topological mesh modeling, we ignore any such geometric condition and assume that faces and edges can have any shape. With this assumption, although we lose the advantages coming from planarity/linearity of faces and edges, we gain a lot. Without geometric constraints, it is not only possible to develop simpler and faster algorithms for interactive modeling; it is also possible to represent a wide variety of shapes. Figure 2 shows that manifold meshes can represent one or two dimensional entities such as points, lines, curves and polygons.

1.1 Topologically Regular Meshes

We will define topological regularity as all faces and vertices have the same combinatorial property; i.e having the same size and same valence, respectively.

The valence of a vertex is defined as the number of edge-ends that emanates from that vertex. For instance, the vertex of a point-sphere in Figure 2 does not have any edge emanating from it and its valent is 0. On the other hand, the vertex of the self-loop in Figure 2 has valence 2, since two edge-ends emanates (although both ends belong to the same edge) from the vertex.

The face size is counted as the number of edge sides (also called half-edges [Mantyla 1988]) belonging to a face. In other words, two sides of a same edge can belong to the same face and that edge will be counted twice. For instance, the only face of a line-manifold in Figure 2 is a two-gon in which the edge counted twice since both sides of the edge belong to the same face.

We will represent a regular mesh with a triple (n,m,g) where n is the face size, m is the valence of vertices and g is the genus of the surface. All manifolds in Figure 2 are regular meshes. For instance, point-sphere in Figure 2 is the simplest regular mesh with $(0,0,0)$. Similarly line-manifold in Figure 2 is $(2,1,0)$, self-loop, which is the dual of line-manifold is $(1,2,0)$ and triangle-manifold is $(3,2,0)$.

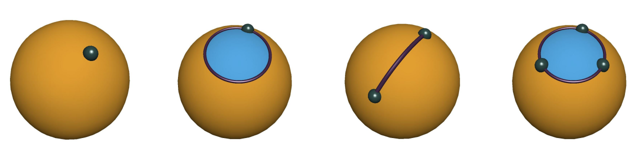

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Figure 2: Four examples of unusual Manifolds: A. Point-Sphere $(0,0,0)$, B. Self-Loop $(1,2,0)$, C. Line-Manifold $(2,1,0)$ and D. Triangle-Manifold $(3,2,0)$.

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1.2 Motivation for Studying Regular Meshes

Our motivation to study regular meshes comes from understanding the power of various mesh modeling approaches. One of our recent work [Srinivasan and Akleman 2004] suggested that subdivision schemes [Zorin and P. Schröder 2000] are more powerful than various fractal schemes such as Iterated Function Systems (IFS) [Barnsley 1988; Mandelbrot 1980]. Another one of our recent results [Akleman et al. 2004] further showed the limitations of subdivision schemes. This paper implicitly shows that vast majority of regular meshes cannot be created by subdivision schemes (There are, of course, some exceptions such as $(4,4,1)$, $(6,3,1)$ and $(3,6,1)$). Moreover, it is possible to create dodecahedron $(3,5,0)$ from tetrahedron $(3,3,0)$ by using pentagonal conversion algorithm [Akleman et al. 2004]. Simplest subdivision [Peters and Reif 1997] can create octahedron $(3,5,0)$ from tetrahedron $(3,3,0)$.

Another motivation for studying regular meshes is texture mapping. The regular meshes in the form of $(3,m,g)$ and $(4,m,g)$ provide nice triangular and quadrilateral subdivisions of high genus surfaces. These quadrilateral or triangular patches can seamlessly be covered by aperiodic tiles [Stam 1997; Neyret and Cani 1999; Cohen et al. 2003]. $(3,m,g)$ and $(4,m,g)$ can also provide a framework to describe control meshes for patch modeling [Takahashi et al. 1997].We also think that regular meshes will eventually be useable for topological simplification of meshes. Regular meshes can also be useful for morphing high genus surfaces from one to another. We note that there has been extensive literature in mathematical research on the related topics [**?**; Cromwell 1997; Grunbaum and Shephard 1987]. For example, regular meshes on surfaces of genus 1 and 2 has been investigated by Brahana [Brahana 1926] and regular polyhedra for infinite genus is discovered by Coxeter $(6,6,\infty)$ $(6,4,\infty)$ and $(4,6,\infty)$ [Coxeter 1937; Gott 1967; Bulatov 2005; Green 2005]

2 Euler Equation

In topological mesh modeling, our only concern is mesh structure; how faces, edges and vertices are related with each other. Euler equation is the fundamental equation that gives the relationship between the number of faces, f, the number of edges, e, and the number of vertices v. Using Euler-Poincare equation, without using geometric properties, we can identify some essential properties of manifold meshes. Euler-Poincare equation is given as follows:

$$f - e + v = 2 - 2g \tag{1}$$

where g is the total number of handles in the surface, called genus.

Using Euler-Poincare equation, it is possible to systematically search for regular meshes. First note that if all faces have the same number of sides n and all vertices has the same valence m, we can obtain the following relationships:

$$nf = 2e \tag{2}$$

$$mv = 2e \tag{3}$$

If we plug in these relationships in Euler-Poincare equation, we obtain a simplified equation

$$\left(\frac{1}{n} + \frac{1}{m} - \frac{1}{2}\right) e = 1 - g. \tag{4}$$

The integer solutions of these equations for any given n,m,g give us an idea about regular manifold meshes for that triplet. But, note that having integer solutions to equation (4) alone does not prove existence of the regular mesh.

3 Regular Genus-0 Meshes

For genus-0 surfaces, the equations (2), (3) and (4) can be further simplified to the followings.

$$e = \frac{2nm}{2n - 2m + nm} \tag{5}$$

$$v = \frac{4m}{2n - 2m + nm} \tag{6}$$

$$f = \frac{4n}{2n - 2m + nm} \tag{7}$$

3.1 Manifold Polygons and Their Duals

The polygons can be defined as meshes in which all vertices are valence-2. If we plug in $m = 2$ to Equations (5), (2(and (3), we simply get $e = n$, $f = 2$ and $v = n$. In other words, in this case, there are exactly n edges and vertices, which is also the number of the sides of the faces; and there are only two faces; i.e. these are two-sided (manifold) polygons. Some examples of manifold polygons are shown in Figure 3

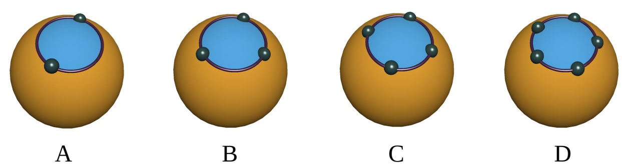

Figure 3: Polygon-manifolds: A. $(2,2,0)$, Two-gon; B. $(3,2,0)$, Triangle; C. $(4,2,0)$, Quadrilateral; D. $(5,2,0)$, Pentagon.

If we choose polygons as two-gons, i.e. $n = 2$, we simply get the dual of the manifold-polygons. In this case, there will be only two vertices, the number of edges and faces will be $e = f = m$.

3.2 Regular Platonic Meshes

Regular genus-0 meshes also include platonic meshes, which are generalized version of platonic polyhedra [Stewart 1991; Williams 1972; Wells 1991], and regular polygons. Regular or Platonic polyhedra are defined by a set of geometric conditions. However, we do not need any geometric condition to find mesh structures of regular polyhedra. For both n and m larger than 2, we obtain those mesh structures as $(3,3,0)$, $(3,4,0)$, $(3,5,0)$, $(4,3,0)$ and $(5,3,0)$ (See Figures 4 and 1).

3.3 Regular Genus-1 Meshes

For $g = 1$ the Euler-Poincare equation greatly simplifies

$$nm - 2n - 2m = 0 \tag{8}$$

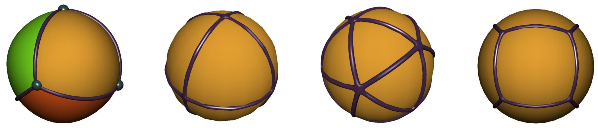

Figure 4: Regular genus-0 meshes with both n and m larger than 2(See Figure 1 for $(5,3,0)$, dodecahedron).

This equation has only three integer solutions: $(4,4,1)$, $(3,6,1)$ and $(6,3,1)$. These solutions correspond classical regular tessellations of an infinite plane [Grunbaum and Shephard 1987; Williams 1972].

For $(4,4,1)$ case, value of e should be an even number since $e = 2v = 2f$. Therefore, vertices, faces and edges of any regular mesh $(4,4,1)$ can be given as $f = k$, $v = k$ and $e = 2k$ where $k = 1,2,\ldots$. For $v = f = 1$ and $e = 2$, we get the simplest genus-1 mesh shown in Figure 5(A). If we apply a 4-conversion subdivision scheme such as Doo-Sabin [Doo and Sabin 1978; Sabin 2000], Catmull-Clark [Catmull and Clark 1978], Simplest [Peters and Reif 1997] or dual of Simplest [Zorin and Schröder 2002] to an $(4,4,1)$ mesh, we obtain another (denser) $(4,4,1)$ mesh.

For $(3,6,1)$ case, $2e = 6v = 3f$, so f should be an even number, v should be twice of f and $e = 3v$. Therefore, vertices, faces and edges of any regular mesh $(3,6,1)$ can be given as $v = k$, $f = 2k$ and $e = 3k$ where $k = 1,2,\ldots$. For $v = 1$, $f = 2$ and $e = 3$, we get the simplest $(3,6,1)$ genus-1 mesh. An example of regular mesh $(3,6,1)$ is shown in Figure 6(A). If we apply a triangle based subdivision scheme such as Loop [Loop 1987] or $\sqrt{3}$ [Kobbelt 2000] to an $(3,6,1)$ mesh, we obtain other (denser) $(3,6,1)$ meshes.

For $(6,3,1)$ case, $2e = 6f = 3v$, so v should be an even number, f should be twice of v and $e = 3f$. Therefore, vertices, faces and edges of any regular mesh $(6,3,1)$ can be given as $f = k$, $v = 2k$ and $e = 3k$ where $k = 1,2,\ldots$. For $f = 1$, $v = 2$ and $e = 3$, we get the simplest $(6,3,1)$ genus-1 mesh. An example of regular mesh $(6,3,1)$ is shown in Figure 7(A). If we apply a hexagonal based subdivision scheme such as dual of Loop [Prautzsch and Boehm 2000] or dual of $\sqrt{3}$ [Claes et al. 2002; Akleman and Srinivasan November 2002; Oswald and Schröder 2003] to an $(6,3,1)$ mesh, we obtain other (denser) $(6,3,1)$ meshes.

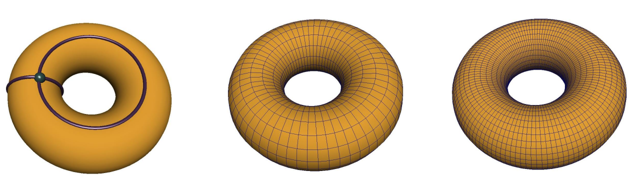

Figure 5: Regular $(4,4,1)$.

4 Regular Genus¿1 Meshes

For $g > 1$ if we rearrange the Euler-Poincare equation, we find the following equations for $e, v,$ and f.

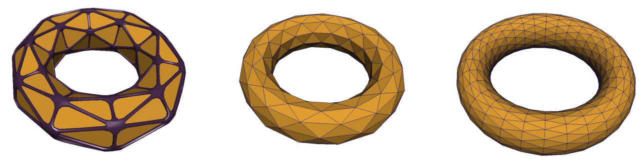

Figure 6: Examples of regular $(3,6,1)$*meshes*.

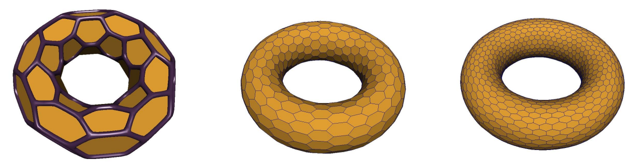

Figure 7: Examples of regular $(6,3,1)$ meshes.

$$e = \frac{2nm}{nm - 2n - 2m}(g-1) \quad (9)$$

$$f = \frac{4m}{nm - 2n - 2m}(g-1) \quad (10)$$

$$v = \frac{4n}{nm - 2n - 2m}(g-1) \quad (11)$$

Integral solutions to these equations makes necessary conditions for regular meshes.

Theorem 4.1 *There exist infinitely many integer solutions to equations (9), (10) and (11).*

PROOF.

- Let there exist a value of $g = g_1$ such that equations (9), (10) and (11) give integer solution with $e = e_1$, $v = v_1$ and $f = f_1$. Then for any $g_k = k(g_1 - 1) + 1$ where $k = 1,2,\ldots$ there exist integer solutions $e = ke_1$, $v = kv_1$ and $f = kf_1$.
- Let $g_1 = nm - 2n - 2m + 1$, then the integer solutions to equations (9), (10) and (11) are $e = 2nm$, $v = 4n$ and $f = 4m$.

$\square$

Theorem 4.2 *If for given n, m, and $g > 1$, a regular mesh exists for (n,m,g), then we can construct regular meshes $(n,m,k(g-1)+1)$ for any $k \geq 1$ from (n,m,g).*

PROOF. Suppose that M is an (n,m,g) regular mesh, where $g > 1$, and that S_M is the corresponding 2-manifold. We perform the following topological operations on S_M:

- Cut a handle along a circle C that does not pass through any vertex of M (note that this is always possible since the mesh M has only finitely many vertices on the surface S_M). This leaves a 2-manifold S' with two "holes" H_1 and H_2 (see Figure 8);
- make k copies of the above structure S': $S'_1, \ldots, S'_k$;
- arrange the k copies of the structure $S'_1, \ldots, S'_k$ as a ring, and paste the hole H_1 in S'_i with the hole H_2 in S'_{i+1} for all i, $1 \leq i \leq k$ (here we have let $S'_{k+1} = S'_1$) and let the related edges in M crossing the boundaries of the holes H_1 and H_2 aligned properly (see Figure 9).

Figure 8: Cutting a handle of S_M along a circle C that does not pass through any vertex of M.

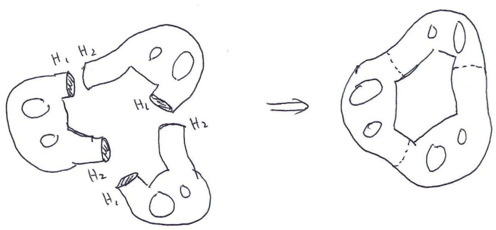

Figure 9: Reassembling surfaces by pasting the holes.

To compute the genus of the resulting surface, note that if we cover the two holds in the structure S', we get a surface of genus $g-1$. Assembling the k copies of the structure S' into a ring increases the genus by 1. Therefore, the resulting surface has genus $k(g-1)+1$. Moreover, since no vertex is involved in these operations, all vertex valences remain as m. Finally, since all edges are aligned through the boundaries of the holes H_1 and H_2, it is easy to verify that all face sizes remain as n. Therefore, the resulting mesh is a $(n,m,k(g-1)+1)$ regular mesh.

Based on this theorem, genus-2 regular meshes are the most interesting ones since if a regular mesh $(n,m,2)$ exist, then (n,m,g) also exists regardless of genus g. □

Now, all we need to show is the existence of at least one constructible (n,m,g) for any given n,m. Unfortunately, classical topological graph theory does not give us good lower bounds [**?**]. The following two theorems are adjusted from well-known topological graph theory results.

Theorem 4.3 *For any given $m>6$, there exists an efficiently constructible regular mesh $(3,m,\lceil (m-3)(m-4)/12\rceil)$.*

PROOF. In order to construct an $(3,m,g)$ regular mesh, consider the complete graph K_{m+1} (i.e., K_{m+1} is the graph of $m+1$ vertices in which each pair of vertices are connected by an edge). By the famous construction for solving the Heawood Conjecture [Ringel and Youngs 1968], the complete graph K_{m+1} has a triangulated embedding on the surface of genus $g = \rceil (m-3)(m-4)/12\rceil$. This gives a $(3,m,g)$ regular mesh on surface of genus g. Note that this is the smallest mesh since the minimum graph whose vertex valence is m is the graph K_{m+1} of $m+1$ vertices.

We point out that the construction of such a $(3,m,g)$ can be done efficiently. We refer interested readers to [Gross and Tucker 1987] for more details.

Because of the duality, efficiently constructible regular mesh $(m,3,g)$ also exists. □

Theorem 4.4 *For any given $m>6$, there exists an efficiently constructible regular mesh $(4,m,\lceil (m-2)^2/4\rceil)$.*

PROOF. The proof is similar to that for the previous theorem. In order to construct a $(4,m,g)$ regular mesh, consider the complete bipartite graph $K_{m,m}$ (i.e., $K_{m,m}$ is a graph of $2m$ vertices, partitioned into two groups V_1 and V_2 of m vertices each such that every vertex in V_1 is adjacent to every vertex in V_2). The vertex valence of the graph $K_{m,m}$ is m. Now by the construction described in [Ringel and Youngs 1968], the graph $K_{m,m}$ has an embedding on the surface of genus $g = \rceil (m-2)^2/4\rceil$ in which every face is size 4. This gives a $(4,m,g)$ regular mesh on surface of genus g. Again the construction of such a $(4,m,g)$ regular mesh can be done efficiently [Gross and Tucker 1987].

Because of the duality, efficiently constructible regular mesh $(m,4,g)$ also exists.

□

Although the last two theorems showed the existence of infinitely many regular meshes that consist of triangular and quadrilateral faces. Using these theorems we miss most of such regular meshes that consist of triangular and quadrilateral faces. The following two theorems give much better lower bounds for such regular meshes.

Theorem 4.5 *For any given $m>4$, there exists an efficiently constructible regular mesh $(4,m,m-3)$.*

PROOF. For $n=4$, if we choose $g=m-3$, then the integer solutions to equations (9), (10) and (11) become $e=4m$, $v=8$ and $f=2m$.

Now, let us assume that there exists an operation that increases the genus by 1, vertex valence by 1, the number of quadrilaterals by 2 and the number of edges by 4. The operation will not change the number of vertices. If we apply this operation to a $(4,3,0)$ (hexahedron/cube) k times where $k=0,1,...$, we get a regular mesh $(4,3+k,k)$. In other words, we get $(4,m,m-3)$ for $m=k+3$.

Fortunately, such an operation exists. Let f_0 and f_1 be two quadrilaterals that do not share any vertex. We will call these two f_0 and f_1 a distinct-pair. For example, a cube has three such pairs and any pair covers all vertices of the cube. The operation simply connects these two faces with a handle by inserting four edges as shown in Figure 10. After the operation, the initial two faces f_0 and f_1 cease to exist and four new quadrilaterals are created; i.e. the number of quadrilaterals increase by 2. The newly created quadrilaterals consist of two distinct-pairs. This is also the number of distinct-pairs that are increased by each application of the operations. □

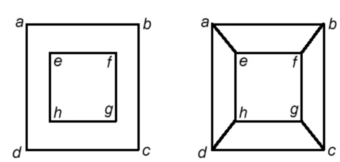

Figure 10: The procedure to create $(4,m,m-4)$. Initial two quadrilaterals are $f_0 = (a,b,c,d)$ and $f_1 = (e,h,g,f)$. After four edge insertions, f_0 and f_1 disappears and four quadrilaterals, (a,b,f,e), (b,c,g,f), (c,d,f,h) and (d,a,h,e), are created.

Figure 11 shows a regular mesh, $(4,5,2)$, that is obtained by applying the operation (see Figure 10) twice to a cube. Note that each operation added one handle to the initial cubical mesh.

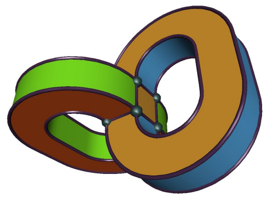

Figure 11: A view of $(4,5,2)$ that is obtained by applying the operation given in Figure 10 twice to a cube.

However, the theorem does not guarantee to get the regular mesh with smallest g. For instance, for $n=4$ and $m=6$, using the theorem we obtain $(4,6,3)$, however, the simplest regular mesh with $n=4$ and $m=6$ is $(4,6,2)$ as shown in Figure 12 (Also see [Coxeter 1965]). Note that the duals of these regular meshes, $(5,4,2)$ and $(6,4,2)$, also exist and constructible.

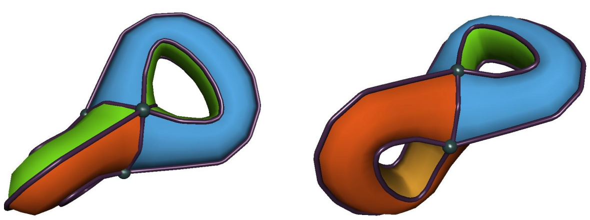

Figure 12: Two views of $(4,6,2)$ that cannot be obtained by using the operation presented in Theorem 4.5.

The concept behind to create a lower bound for triangular regular meshes is similar but it requires to solve a slightly more complicated problem.

Theorem 4.6 *For any given $m>6$, there exists an efficiently constructible regular mesh $(3,m,m-5)$.*

PROOF.

For $n=3$, if we choose $g=m-5$, then the integer solutions to equations (9), (10) and (11) becomes $e=6m$, $v=12$ and $f=4m$.

Now, let f_0 and f_1 be two triangles that do not share any vertex. We will call these two f_0 and f_1 again a distinct-pair. The operation simply connects these two faces with a handle by inserting 6 edges as shown in Figure 13. After the operation, the initial two faces f_0 and f_1 cease to exist and 6 new triangles are created; i.e. the number of triangles increases by 4. The newly created triangles consist of more distinct-pairs, i.e. the number of distinct-pairs increases by each application of the operations.

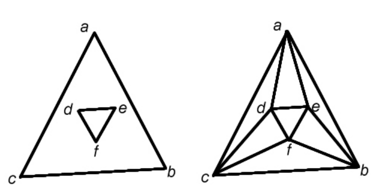

Figure 13: The procedure to create $(3,m,m-5)$. Initial two triangles are $f_0=(a,b,c)$ and $f_1=(d,e,f)$. After 6 edge insertions, f_0 and f_1 disappear and 6 new triangles are created.

Let us now apply this operation to an icosahedron $(3,5,0)$. An icosahedron has two distinct-pairs, and therefore, the operation must be applied twice to cover all vertices. Then, the genus increases by 2 and vertex valence also increases by 2. The number of vertices stays the same. The number of faces increases by 4. As a result, after k iterations the regular mesh will be $(3,5+2k,2k)$. Note that here genus has always to be an even number. In other words, we get $(3,m,m-5)$ where m is an odd number.

It is also possible to get regular meshes with even vertex valences and odd genera by starting from a regular mesh $(3,6,1)$ with 12 vertices. This toroidal mesh also has two distinct-pairs, and therefore, the operation must again be applied twice to cover all vertices. Similarly, the genus and vertex valence increases by 2. The number of vertices stays the same. The number of faces increases by 4. As a result, after k iterations the regular mesh will be $(3,6+2k,2k+1)$. Note that in this case genus is always an odd number. In other words, we get $(3,m,m-5)$ where m is an even number. This concludes that we can get $(3,m,m-5)$for both even and odd m values.

$\square$

Again, above theorem can identify $(3,7,2)$ and its dual $(7,3,2)$, for $g=2$. However, 3 more triangulated regular meshes exist for $g=2$. which are $(3,8,2)$; $(3,9,2)$; $(3,12,2)$ [Brahana 1926]. Note that the duals of these regular meshes, $(8,3,2)$, $(9,3,2)$ and $(12,3,2)$, also exist and constructible.

Conjecture 4.1 *For any given $m>3$, there exists an efficiently constructible regular mesh $(5,m,3m-9)$.*

Conjecture 4.2 *For any given $m>3$, there exists an efficiently constructible regular mesh $(6,m,m-2)$.*

Note that these conjectures also do not provide tight lower bounds. We have already known that the regular meshes $(5,5,2)$ and $(6,6,2)$ exist and constructible [Brahana 1926]. For $(5,5,2)$ see Figure 14. $(6,6,2)$ can be obtained from a truncated tetrahedron. A truncated tetrahedron consists of 4 hexagons and 4 triangles. All vertices are valence-3. By simply pasting triangular faces, we create a genus-2 object with hexagonal faces and each vertex becomes valence-6.

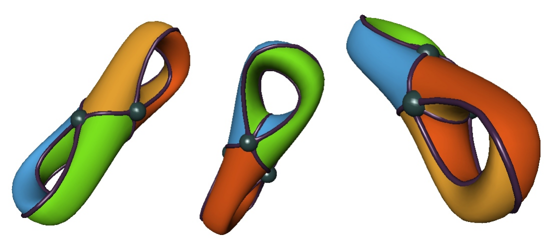

Figure 14: Three views of $(5,5,2)$.

5 Conclusion

This paper presents our preliminary results on regular meshes. Our work shows that there exist infinitely many regular meshes for $g>1$. Moreover, we have constructive proofs that describe how to create high genus regular meshes that consist of triangles and quadrilaterals $(3,m,g)$ and $(4,m,g)$.

Regular meshes seem to be an extremely fertile area. For instance, there are many interesting regular meshes such as $(4g,4g,g)$, which we did not discuss in this paper. In fact, $(4g,4g,g)$ is widely used in topology to cut a genus-g surface to a 4g-gon polygon. Ferguson et al. used those regular meshes to create genus-more-than-1 surfaces [Ferguson et al. 1992].

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