Abstract

Most frequently used subdivision schemes such as Catmull-Clark create regular regions after several applications. This paper shows that all semi-regular regions can be created by subdivision schemes and each semi-regular region type can be created with one application of a particular subdivision scheme to a particular regular region. Using this property of subdivision schemes it is easy to cover any given surface with semi-regular tiles by applying one semi-regularity creating subdivision after several applications of a regularity creating subdivision.

1. Introduction and Background

Most widely used subdivision schemes such as Catmull-Clark, Doo-Sabin or Loop are regularity creating; i.e. they create regular regions after several applications of the scheme. A topologically regular region is a region where all vertices have the same valence and all faces have the same number of sides. The Schlafli symbol \((n, m)\) is used to characterize regular region, where \(n\) is the number of the sides in each face and \(m\) is the valence of vertices. For instance, \((6, 3)\) means regular regions that consist of hexagons with 3-valent vertices [20].

There exists only 3 types of regular regions. These are \((3, 6)\), \((4, 4)\) and \((6, 3)\) [20]. Only genus-1 surfaces can be covered using only regular regions. The other type of surfaces cannot be covered using only regular regions. However, using subdivision schemes they can be covered by patches of regular regions. For instance, Figure 1 shows examples of how a sphere can be covered by patches of regular regions. These regular regions can be created by more than one type of subdivision schemes. (Note that these regions are only topologically regular, they are not regular geometrically. Geometric entities such as the angles and edge lengths are not the same and do not have to be the same.)

- Regular region \((3, 6)\) can be created by both Kobbelt’s \(\sqrt{3}\) subdivision [10] and Loop subdivision [11]. In \(\sqrt{3}\) subdivision, after one iteration all polygons become triangles. Each \(k\)-gon produces a \(2k\) valence vertex; each \(k\)-valence vertex continues to be \(k\)-valent. In other words, if \(\sqrt{3}\) is applied to a \((3, 6)\) regular region, all triangles and all valent-6 vertices create valent-6 vertices. In Loop subdivision, each \(k\)-gon produces a \(k\)-gon and \(k\) triangles; each \(k\)-valence vertex continues to be \(k\)-valence and each edge creates a 6-valence vertex. In other words, if Loop is applied to a \((3, 6)\) regular region, all 6-valence vertices and all edges create valent-6 vertices.

- Regular region \((6, 3)\) can be created by the dual of Loop subdivision [13, 16, 14], and the dual of \(\sqrt{3}\) subdivision [9, 3, 14]. Dual of a subdivision scheme consists of successive applications of three operators to a mesh: (1) Dual; (2) Subdivision; (3) Dual. Since \((6, 3)\) is the dual mesh of \((3, 6)\); it is clear that duals of \((3, 6)\) creating subdivision schemes create \((6, 3)\) regular regions.

- Since \((4, 4)\) is self-dual, dual of any subdivision scheme that creates regular region \((4, 4)\) can also create regular region \((4, 4)\). For instance, Vertex Insertion, such as the remeshing scheme of Catmull-Clark [6] subdivision, is a \((4, 4)\) regularity creating scheme. It makes all faces quadrilateral and creates \((4, 4)\) regular regions. Its dual, the corner cutting scheme, also creates \((4, 4)\) regular regions. An example of corner cutting is the remeshing algorithm of Doo-Sabin subdivision [8]. After one application of corner cutting the valence of all the vertices becomes 4. Corner cutting is a very useful remeshing operator that can create interesting structures such as the one shown in Figure ??.
Simplest and its dual also create (4,4) regular regions.

Figure 1. Examples of Regular Regions on the surface of Sphere. (SC) Spherical cube is obtained by applying Catmull-Clark to a cube and then mapping vertices of subdivided surface to a unit sphere. (GD) is a Geodesic dome or Pseudo-Icosahedron. It is created by applying Loop subdivision to an icosahedron and then mapping vertices of subdivided surface to a unit sphere. (BF) is a Buckminster-fullerene or generalized soccer-ball. It is created by applying Dual of $\sqrt{3}$ to a Dodecahedron and then mapping vertices of the subdivided surface to a unit sphere.

If we relax the conditions of regularity, we obtain semi-regular regions. In semi-regular regions, either all faces or all vertices have exactly the same topological structure, but not both. For instance, a semi-regular region consists of exactly 3,4,3,4. The dual of such a semi-regular region consists of exactly 5-valent vertices. Although, polygons around the vertex do not have same number of sides, the number of sides of polygons around the vertex follows the same cyclic pattern: triangle, triangle, quadrilateral, triangle and quadrilateral; i.e. 3,3,4,3,4. This sequence uniquely defines the structure of a semi-regular region. In fact, these sequences give Schlafli symbols for semi-regular regions (instead of comma, we use dots to separate numbers). For instance, the above pentagon example will be given as (3.3.4.3.4).

In a plane, there are only 7 distinct semi-regular regions and their duals [20]. These are given by the following Schlafli symbols.

1. (4.8.8) : Primary region consists of triangles. Dual region consists of 3-valent vertices.
2. (3.12.12) : Primary region consists of triangles. Dual region consists of 3-valent vertices.
3. (12.6.4) : Primary region consists of triangles. Dual region consists of 3-valent vertices.
4. (6.4.3.4) : Primary region consists of quadrilateral. Dual region consists of 4-valent vertices.
5. (6.3.6.3) : Primary region consists of quadrilateral. Dual region consists of 4-valent vertices.
6. (3.3.4.3.4) : Primary region consists of pentagons. Dual region consists of 5-valent vertices.
7. (3.3.3.3.6) : Primary region consists of pentagons. Dual region consists of 5-valent vertices.

To create all of these semi-regular tilings, we need to introduce only two new (previously-unpublished) remeshing schemes and one of them is simply the dual of the Simplest scheme.

2. Semi-Regularity Creating Remeshing Operations

We have developed a system that provides (global) operators that are applied to the whole mesh, or what we call remeshing operators. Under the remeshing category, we only consider global operators that do not change the topology of the mesh. These operators do not increase or decrease the genus of the surface nor do they connect or disconnect surface components.

The most important remeshing scheme in our system is the dual operator [20], which is essential for creating dual meshes. Dual is also very useful operator for smoothing. If a dual operation is applied even number of times, in most cases it will create a smooth mesh. Zorin and Schröder recently showed that the dual operation allows creation of higher-order quadrilateral subdivision surfaces that provide higher-degree continuity [21]. The same idea is also used to derive the rules for the schemes that create regular (6,3) regions [9,14,3].

All remeshing operators we provide (except dual) increase the number of vertices. Our remeshing operators are not probabilistic, i.e., they always give the same result for
the given initial mesh. Although, we consider the subdivision schemes as remeshing schemes, a remeshing scheme does not have to be a subdivision – remeshing schemes may not necessarily be applied successively and may not smooth the mesh.

Providing a large variety of remeshing operators is particularly useful to making mesh structures ornamental or decorative. By applying the provided remeshing operators to regular regions all major semi-regular regions can be obtained. To illustrate the creation of particular semi-regular regions we apply remeshing operators to three initial spherical shaped meshes shown in Figure 1. Among these three, spherical cube (SC) includes \((4,4)\) regular regions, and geodesic dome (GD) [19] includes \((3,6)\) regular regions, and finally Buckminsterfullerene (BF) [19] includes \((6,3)\) regular regions.

The remeshing operators that can create semi-regular regions are conversion operators. These operators can change any given mesh into a mesh that consist of the same type of faces (e.g. all triangles or all pentagons) or vertices (e.g., all valent-3). Operators are also classified into two additional categories: (1) Primary operators that create the same type of faces, and (2) dual operators that create the same type of vertices. Although the dual of an operator can simply be obtained as dual + operator + dual, in most cases we have implemented dual operators separately to achieve faster computational speed.

### 2.1. Triangulization and Valent-3 Conversion

- **Vertex Truncation and its dual.** Polyhedral vertex truncation is a widely used operator in polyhedral modeling to obtain Archimedean solids [20]. We have recently extended [2] polyhedral vertex truncation by adopting Chaikin’s algorithm [17] to vertex truncation. Vertex truncation can be used to create semi-regular regions 4.8.8 and 3.12.12 as shown in Figures 2 and 3. Vertex truncation is a valent-3 conversion scheme. Dual of Vertex truncation, a triangulation scheme, is also implemented and duals of 4.8.8 and 3.12.12 created by these operators are also shown in Figures 2 and 3.

- **1264 and its dual.** We have developed these two remeshing operators to create semi-regular region 12.6.4 when applied to the regular region \((3,6)\) as shown in Figure 4.

### 2.2. Quadrilateralization and Valent-4 Conversion

- **Vertex Insertion and its dual, Corner Cutting.** Vertex Insertion is remeshing scheme of Catmull-Clark [6] subdivision. The system provides both Catmull-Clark and linear vertex insertion. This subdivision creates \((4,4)\) regular regions. But, if it is applied once to a mesh with a \((6,3)\) region, it creates semi-regular regions 6.4.3.4 as shown in Figure 5. The corner cutting scheme is the dual of vertex insertion. An example of corner cutting is remeshing algorithm of Doo-Sabin subdivision [8]. After one application of corner cutting the valence of all the vertices becomes 4. Corner cutting is a very useful remeshing operator that can create interesting structures such as the one shown in Figure 5.

- **Simplest and its dual.** The remeshing scheme of simplest subdivision [15] converts any mesh to a mesh with only 4-valent vertices. Simplest creates 6.3.6.3
Figure 4. (A) An example of primary 12.4.6 obtained by applying 1264 operator to the geodesic dome. (B) An example of dual 12.4.6 obtained by applying 1264 operator to the Buckminsterfullerene.

Figure 5. (A) An example of primary 6.4.3.4 obtained by applying vertex insertion operator to the Buckminsterfullerene. (B) An example of dual 6.4.3.4 obtained by applying corner cutting operator to the Buckminsterfullerene.

Figure 6. (A) An example of primary 6.3.6.3 obtained by applying dual of simplest operator to the Buckminsterfullerene. (B) An example of dual 6.3.6.3 obtained by applying simplest operator to the Buckminsterfullerene.

Figure 7. (A) An example of primary 3.3.3.3.6 obtained by applying pentagonalization operator to the subdivided cube. (B) An example of dual of 3.3.3.3.6 obtained by applying dual of pentagonalization operator to the subdivided cube.

2.3. Pentagonalization and Valent-5 Conversion

We have recently shown that it is possible to convert any 2-manifold to a pentagonal mesh [5]. We also showed that there is no conversion method that can convert any given semi-regular regions when it is applied to (6,3) regular regions as shown in Figure 6. We have also implemented the dual of simplest. In the program, We call it stellate with edge removal. Dual of simplest is not a known subdivision scheme and it is not useful as a smoothing scheme. However, it allows the creation of appealing 6.3.6.3 semi-regular regions as shown in Figure 6.

- Pentagonalization and its dual. This is the only known pentagonalization algorithm, which we have introduced recently This scheme can create two types of semi-regular regions: 3.4.3.4 and 3.3.3.3.6. We have also implemented its dual which converts every vertex of a mesh to valent-5 vertex. (see Figures 7 and 8)
3. Conclusion and Future Work

This is a short paper that shows power of mixed usage of subdivision and remeshing schemes. Creating semi-regular regions is important especially in shape modeling applications where tiling is important. Note that we use sphere just as an example. The approach presented here can work for any genus and any smooth surface. An example is shown in Figure 9. Using more than one subdivision can also help to create interesting tessellations that are not semi-regular such as the one shown in Figure 11.

One important application of creating semi-regular tiles is for opening holes as seen in Figure 12. We make use of the fact that each semi-regular tiling creates distinct groups of faces on which we can apply any polygonal operator. For instance, meshes shown in Figure 12 are created by wire modeling [12]. The mesh shown in Figure ?? is created by rind modeling [4].

An interesting question is how to create Coxeter’s (4, 6), (6, 6) and (6, 4) regular regions [7]. We have recently shown that [1], these regular mesh structures exists in any genus larger than 2. Therefore, it can be possible to develop genus-changing subdivision schemes to create such structures. For an example of genus-changing subdivision schemes see [18].

References


Figure 9. A series of Genus-3 surfaces covered with semi-regular regions.

Figure 10. A sample of high genus shapes that are created from surfaces with semi-regular regions using wire modeling.
Figure 11. A Genus-3 surface covered with interesting but not a semi-regular region. This was created applying simplest twice to the bottom mesh in Figure 9.

Figure 12. A high genus shape that are created from surfaces with semi-regular regions using rind modeling.