Problem 1: (5 points)

Write Maxwell's equations in the integral form.

$$\oint \vec{E} \cdot d\vec{S} = \frac{Q_{encl}}{\varepsilon_0}$$

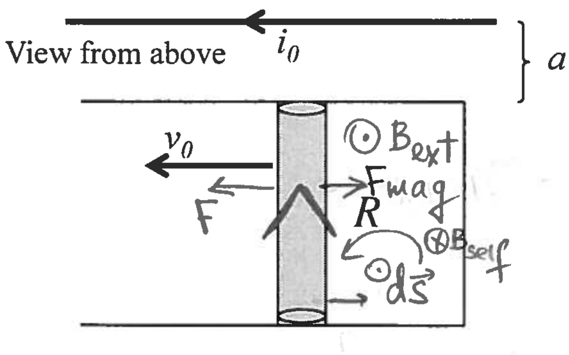
$$\oint \vec{B} \cdot d\vec{S} = 0$$

$$\oint \vec{E} \cdot d\vec{r} = -\frac{\partial}{\partial t} \int \vec{B} \cdot d\vec{S}$$

$$\oint \vec{B} \cdot d\vec{r} = \mu_0 \left(i + \varepsilon_0 \frac{\partial}{\partial t} \int \vec{E} \cdot d\vec{S} \right)$$

Problem 2: (20 points)

A rod of length l is forced to move at constant speed v_0 along horizontal rails connected by wire at the right. The rod has resistance R; the rest of the loop has negligible resistance. A wire, carrying a constant current i_0, is parallel to the tracks, in the same plane at distance a from the loop.



a) Find the direction of the current in the loop.
Explain your answer within this box:

Area icreases, Flux increases, $B_{self} \uparrow\downarrow B_{ext}$, $B_{self} \otimes$, i is cw

b) Find the current through the rod. Ignore self-inductance.

$$\oint \vec{E}\cdot d\vec{r} = -\frac{\partial}{\partial t}\int \vec{B}\cdot d\vec{s}; \quad \oint \vec{B}\cdot d\vec{r} = \mu_0 i; \quad B\, 2\pi r = \mu_0 i_0; \quad \vec{B} = \frac{\mu_0 i_0}{2\pi r}\odot$$

$$\Phi = \int_a^{a+l} \frac{\mu_0 i_0}{2\pi r} x\, dr = \frac{\mu_0 i_0}{2\pi} x \ln\frac{a+l}{a}$$

$$\frac{d\Phi}{dt} = \frac{\mu_0 i_0}{2\pi} \ln\frac{a+l}{a} \overbrace{\frac{dx}{dt}}^{=v_0} = \frac{\mu_0 i_0 v_0}{2\pi} \ln\frac{a+l}{a}$$

$$-iR = -\frac{\mu_0 v_0 i_0}{2\pi}\ln\frac{a+l}{a}$$

$$\boxed{i = \frac{\mu_0 v_0 i_0}{2\pi R}\ln\frac{a+l}{a}} \quad \text{cw}$$

c) What is the magnitude and direction of the force that must be applied to the rod to make it move at constant speed?

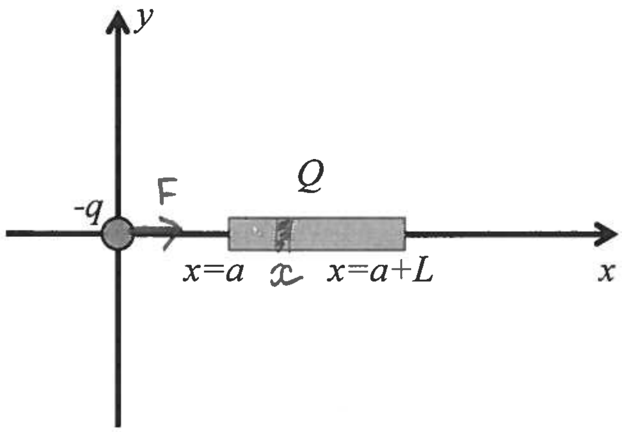
$$d\vec{F}_{mag} = i\, d\vec{s}\times\vec{B}; \quad \vec{F}_{mag} = i\int_a^{a+l}\frac{\mu_0 i_0}{2\pi r}dr = \frac{i\mu_0 i_0}{2\pi}\ln\frac{a+l}{a} \text{ (to the right)}$$

$$= \frac{\mu_0^2 v_0 i_0^2}{(2\pi)^2 R}\left(\ln\frac{a+l}{a}\right)^2 \text{ to the right}$$

$$\vec{F} = \frac{\mu_0^2 v_0 i_0^2}{(2\pi)^2 R}\left(\ln\frac{a+l}{a}\right)^2 \text{ to the left}$$

Problem 3: (15 points)

A charge $-q$ is at the origin and an amount of charge Q is uniformly distributed along x-axis from $x = a$ to $x = a + L$. Find the force on the charge at the origin.



$$dF_x = \frac{1}{4\pi\varepsilon_0}\frac{dQ\,q}{x^2}$$

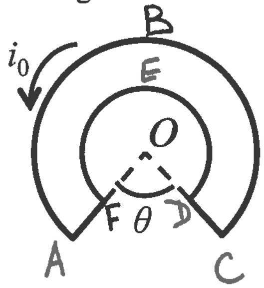
$$dQ = \frac{Q}{L}\,dx$$

$$F_x = \frac{1}{4\pi\varepsilon_0 L}\int_a^{a+L}\frac{Qq}{x^2}\,dx = -\frac{Qq}{4\pi\varepsilon_0 L}\,\frac{1}{x}\Big|_a^{a+L} =$$

$$= \frac{1}{4\pi\varepsilon_0}\,\frac{Qq}{L}\left(\frac{1}{a}-\frac{1}{a+L}\right)$$

Problem 4: (18 points)

a) A closed loop carries current i_0. The loop consists of two radial straight wires and two concentric circular arcs of radii R_1 and R_2, $R_1 > R_2$. The angle θ is given. Find the magnitude and direction of magnetic field created by this current at point O. (see the figure below).



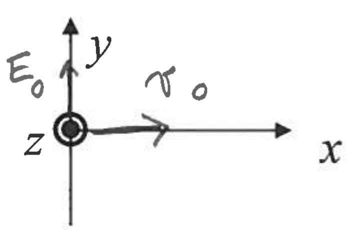
$$d\vec{B} = \frac{\mu_0 i}{4\pi}\frac{d\vec{s}\times\vec{r}}{r^3}; \quad B_{AF} = B_{CD} = 0 \ (d\vec{s}\times\vec{r} = 0)$$

$$\vec{B}_{ABC} = \frac{\mu_0 i_0}{4\pi}\frac{R_1}{R_1^2}(2\pi-\theta)\odot = \frac{\mu_0 i_0}{4\pi R_1}(2\pi-\theta)\odot$$

$$\vec{B}_{DEF} = \frac{\mu_0 i_0}{4\pi R_2}(2\pi-\theta)\otimes$$

$$\vec{B}_{tot} = \frac{\mu_0 i_0}{4\pi}(2\pi-\theta)\left(\frac{1}{R_2}-\frac{1}{R_1}\right)\otimes$$

b) A particle with a **positive** charge q is moving in the positive x direction with velocity of magnitude v_0. If there were an electric field in the positive y direction with magnitude E_0, and the velocity of the particle is unchanged, find the magnitude and direction of magnetic field.



$\uparrow qE$

$\oplus$

$\downarrow qvB$

$$qE = qvB$$

$$\vec{B} = \frac{E_0}{v_0}\vec{i}_z$$

c) A particle with a **negative** charge q is moving in the positive x direction with velocity of magnitude v_0. If there were an electric field in the positive y direction with magnitude E_0, and the velocity of the particle is unchanged, find the magnitude and direction of magnetic field.

$\uparrow qvB$

$\ominus$

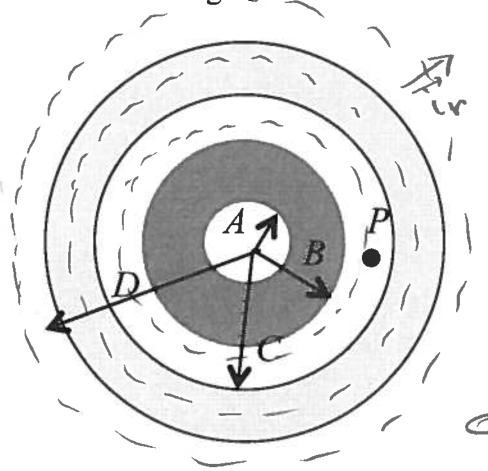
$\downarrow qE$

the same

$$\vec{B} = \frac{E_0}{v_0}\vec{i}_z$$

Problem 5: (20 points)

A spherical conducting shell, inner radius A and outer radius B, is charged with charge Q. It is surrounded by an insulating spherical shell of inner radius C and outer radius D. The insulating shell has a uniform charge density ρ.



a) Find the charge per unit area at $r = A$ and $r = B$.

$r = A \quad \sigma = 0$

$r = B \quad \sigma = \frac{Q}{4\pi B^2}$

$$\oint \vec{E} \cdot d\vec{S} = \frac{Q_{encl}}{\varepsilon_0}$$

b) Find the electric field at

i) $r < A$

$E = 0 \quad (Q_{encl} = 0)$

ii) $A < r < B$

$E = 0$ (conductor)

iii) $B < r < C$

$$E\, 4\pi r^2 = \frac{Q}{\varepsilon_0} ; \quad \vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{Q}{r^2} \vec{i}_r$$

iv) $C < r < D$

$$E\, 4\pi r^2 = \frac{1}{\varepsilon_0} \left(Q + \rho \frac{4}{3}\pi (r^3 - c^3)\right)$$

$$E = \frac{1}{4\pi\varepsilon_0 r^2} \left(Q + \rho \frac{4}{3}\pi (r^3 - c^3)\right) \vec{i}_r$$

v) $r > D$

$$E\, 4\pi r^2 = \frac{1}{4\pi\varepsilon_0} \left(Q + \rho \frac{4}{3}\pi (D^3 - c^3)\right); \quad \vec{E} = \frac{1}{4\pi\varepsilon_0 r^2} \left(Q + \rho \frac{4}{3}\pi (D^3 - c^3)\right) \vec{i}_r$$

c) Find the difference in electric potential between the center and point $r = P$, $V(P)$-$V(0)$.

$$V(P) - V(0) = -\left[\int_0^A 0 + \int_A^B 0 + \int_B^P \frac{Q}{4\pi\varepsilon_0} \frac{dr}{r^2}\right] = \frac{Q}{4\pi\varepsilon_0} \left(\frac{1}{P} - \frac{1}{B}\right)$$

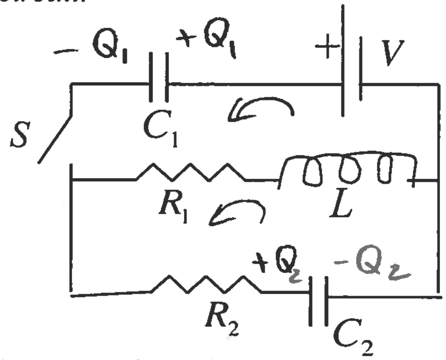
d) If the conducting shell has Q = 0 and the insulating shell, inner radius C and outer radius D, has a nonuniform volume charge density $\rho = cr^2$ where c is a known positive constant, find the electric field at $C < r < D$.

$$E\,4\pi r^2 = \frac{1}{\varepsilon_0}\int_C^r \rho\, 4\pi r^2 dr = \frac{1}{\varepsilon_0}\int_C^r c r^2 4\pi r^2 dr = \frac{1}{\varepsilon_0} 4\pi c\left(\frac{r^5}{5} - \frac{C^5}{5}\right)$$

$$\vec{E} = \frac{c}{5\varepsilon_0}\left(r^3 - \frac{C^5}{r^2}\right)\vec{i}_r$$

Problem 6: (20 points)

a) In a circuit below, R_1, R_2, C_1, C_2, L, and V are given. The switch S was closed for a long time so that the steady state was reached. Find the current in each resistor and the charge on the capacitor. *The problem will not be graded without a direction of current and charges on the capacitor indicated on the circuit.*



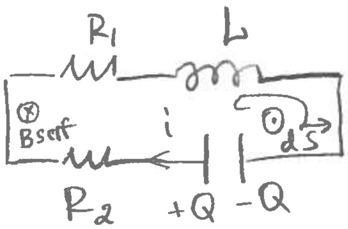
$$\oint \vec{E}\cdot d\vec{r} = 0$$

$$i = 0$$

$$\frac{Q_2}{C_2} = 0$$

$$Q_1 = C_1 V$$

b) At $t = 0$ the switch is open. Assume that the charge on the capacitor 2 is Q_0. Starting from some famous law, derive the equation that describes charge Q on the capacitor as a function of time. *The problem will not be graded without a direction of current and charges on the capacitor indicated on the circuit.*



$$\oint \vec{E}\cdot d\vec{r} = -\frac{\partial}{\partial t}\int \vec{B}\cdot d\vec{S}$$

$$\Phi = -Li \qquad \frac{d\Phi}{dt} = -L\frac{di}{dt} \qquad i = -\frac{dQ}{dt}$$

$$-i(R_1+R_2) + \frac{Q}{C_2} = L\frac{di}{dt} = -L\frac{d^2Q}{dt^2}$$

$$\boxed{L\frac{d^2Q}{dt^2} + (R_1+R_2)\frac{dQ}{dt} + \frac{1}{C_2}Q = 0}$$

c) Neglect L. Solve for the charge on the capacitor as a function of time.

$$(R_1+R_2)\frac{dQ}{dt} + \frac{1}{C_2}Q = 0$$

$$Q(t) = \alpha e^{-\beta t}$$

$$(R_1+R_2)\alpha(-\beta)e^{-\beta t} + \frac{1}{C_2}\alpha e^{-\beta t} = 0$$

$$\beta = \frac{1}{C_2(R_1+R_2)}$$

$$Q(t=0) = \alpha = Q_0 \qquad Q(t) = Q_0 e^{-\frac{t}{(R_1+R_2)C_2}}$$

d) In part b), neglect R_1 and R_2. Solve for the charge on the capacitor as a function of time.

$$L\frac{d^2Q}{dt^2} + \frac{1}{C_2}Q = 0 \; ; \quad \frac{d^2Q}{dt^2} + \frac{1}{LC_2}Q = 0$$

$$Q(t) = A\cos(\omega t) + B\sin(\omega t)$$

$$Q(t=0) = A = Q_0$$

$$i(t) = -\frac{dQ}{dt} = A\omega\sin(\omega t) - B\omega\cos(\omega t)$$

$$i(t=0) = -B\omega = 0$$

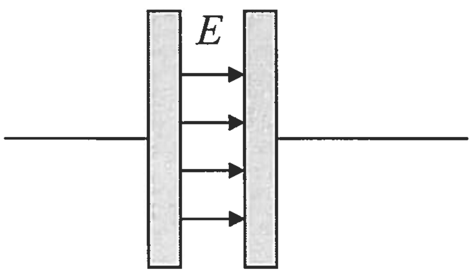
$$B = 0$$

$$Q(t) = Q_0\cos(\omega t)$$

$$\omega = \sqrt{\frac{1}{LC_2}}$$

Problem 7: (7 points)

a) Find the displacement current if the electric field between parallel plate capacitor is $E = E_0 \sin(\omega t + \gamma)$. The area of the plates is A.



$$i_D = \varepsilon_0 \frac{\partial}{\partial t} \int \vec{E} \cdot d\vec{s} =$$

$$= \varepsilon_0 \frac{\partial}{\partial t} \left(E_0 \sin(\omega t + \gamma) \right) A =$$

$$= \varepsilon_0 E_0 \omega \cos(\omega t + \gamma) A$$

b) Which of the following equations is the wave equation?

$$\frac{\partial E_y}{\partial x} = C \frac{\partial E_y}{\partial y} \qquad \frac{\partial^2 E_y}{\partial x^2} = \mu_0 \varepsilon_0 \frac{\partial^2 E_y}{\partial t^2} \qquad \frac{\partial E_y}{\partial x} = -\frac{\partial B_z}{\partial t}$$