Lecture for Week 2 (Secs. 1.3 and 2.2–2.3)

Functions and Limits

First let's review what a *function* is. (See Sec. 1 of "Review and Preview".) The best way to think of a function is as an imaginary machine, or "black box", that takes in any of various objects, labeled x whenever there's no reason to call it something else, and processes it into a new object, labeled $y = f(x)$.

A function is not necessarily given by a formula.

Usually both x and y are numbers, and in that case we can easily think of a function as being the same thing as a *graph*. A curve, or any set of points in the x–y plane, defines a function, provided that no vertical line intersects the set more than once.
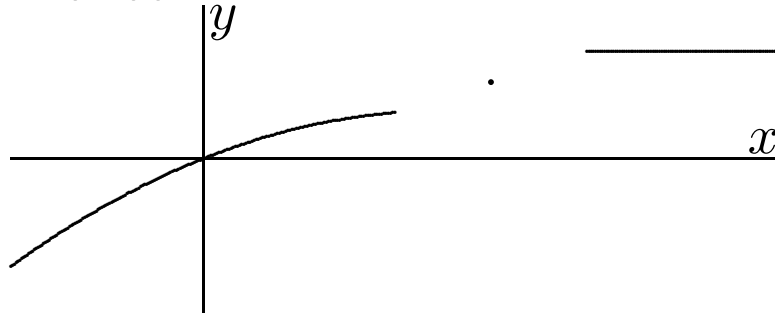
Sec. 1.3, however, deals with functions $\mathbf{r}(t)$ whose values are vectors, or points, not numbers.

The graph of such a function still exists, but it lies in a more abstract space (with three dimensions in this case, one for t and two for $\mathbf{r}$).

A point is not quite the same thing as a vector. A point is *represented* by a vector, $\mathbf{r} = \langle x, y \rangle$, when we choose an *origin* of coordinates in space. If you move the origin, the numbers x and y will change, but the numerical components of true vectors, such as velocity and force, will not change.

Exercise 1.3.7

Sketch the curve represented by the parametric equations

$$x = 3\cos\theta, \quad y = 2\sin\theta, \quad 0 \le \theta \le 2\pi,$$

and eliminate the parameter to find the Cartesian equation of the curve.

Note, no actual vector notation here, although one could have written $\mathbf{r}(\theta) = \langle 3\cos\theta, 2\sin\theta \rangle$.

Well, first I'd plot the points for

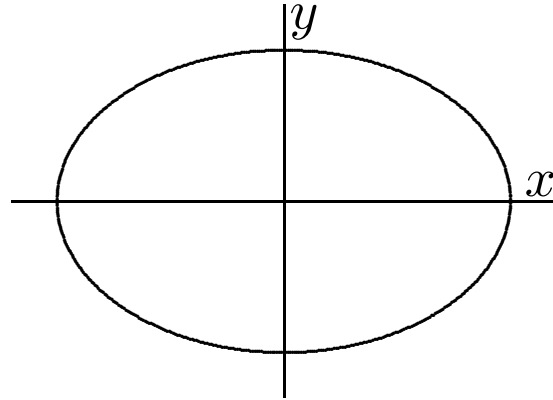$$\theta = 0, \frac{\pi}{4}, \frac{\pi}{2}, \ldots, 2\pi.$$

(The last point is the same as the first one.) Then I notice that

$$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{2}\right)^2 = \cos^2\theta + \sin^2\theta = 1,$$

so the curve is recognized as an ellipse. (It's the whole ellipse, since we observed that the path closes.)

Fortunately my plotting software has an ellipse command, so I didn't need to do the arithmetic for the first part.

y

x

■

Exercise 1.3.27

Find (a) a vector equation, (b) parametric equations, and (c) a Cartesian equation for the line passing through the points $(4, -1)$ and $(-2, 5)$.

(The exercise in the book doesn't ask for a Cartesian equation, but the next group of exercises does, so I'll do it here.)

First let's find the vector pointing from the first vector to the second:

$$\mathbf{v} \equiv \langle -2, 5 \rangle - \langle 4, -1 \rangle = \langle -6, 6 \rangle.$$

If we add *any multiple of* $\mathbf{v}$ to *any point on the line*, the result is a point on the line (and you get *all* the points that way). So an answer to (a) is

$$\mathbf{r}(t) = \langle 4, -1 \rangle + t\langle -6, 6 \rangle = (4 - 6t)\mathbf{i} + (-1 + 6t)\mathbf{j}.$$

$$\mathbf{r}(t) = \langle 4, -1 \rangle + t\langle -6, 6 \rangle = (4 - 6t)\mathbf{i} + (-1 + 6t)\mathbf{j}.$$

Notice that the question asked for "a" vector equation, not "the" vector equation. There are many other correct answers to (a), corresponding to different starting points on the line or different lengths and signs for $\mathbf{v}$.

For (b), just write the components separately:

$$x(t) = 4 - 6t, \qquad y(t) = 6t - 1.$$

To get a Cartesian equation, we need to eliminate t. In the present case that is easily done by adding the two parametric equations:

$$x = 4 - 6t, \quad y = 6t - 1 \implies$$

$$x + y = 3.$$

In more general situations, you would need to solve one equation for t and substitute the result into the other equation. ∎

Limit is the fundamental concept of calculus. Everything else is defined in terms of it:

- continuity
- derivative
- integral
- sum of infinite series

It took mathematicians 200 years (of calculus history) to arrive at a satisfactory definition of "limit". Not surprisingly, the result is not easy for beginners to absorb.

For that reason, Sec. 2.4, "The Precise Definition of a Limit", is not a required part of our syllabus. That doesn't mean that you are forbidden to read it! But you might find it more meaningful if you come back to it after gaining some experience with how limits are used and why they are important. Two natural places in the textbook from which to loop back here are

- after infinite sequences and series (Chap. 10);
- when functions of several variables arise (Sec. 12.2). (In that place Stewart simply states the multivariable generalization of the "precise definition" without fuss or apology.)

So, we have to make do with "intuitive" ideas of a limit. The graphical problems on p. 89 are a good place to start. However, they don't lend themselves to this projector presentation,, because I have no good way to reproduce the graphs. So we'll look at them in real time....

I'll come back to infinite limits and vertical asymptotes later.

Let's go on to Sec. 2.3. The key issue in that section is this: When you are presented with a formula defining a function, such as

$$f(x) = \frac{\sqrt{x^2 + 1}}{x - 3},$$

usually the limit of the function at a point is just the value of the function at that point, but *not always*.

$$\lim_{x \to 1} f(x) = \frac{\sqrt{1^1 + 1}}{1 - 3} = -\frac{\sqrt{2}}{2} = f(1).$$

$$\left(f(x) = \frac{\sqrt{x^2 + 1}}{x - 3} \right)$$

But $\lim_{x \to 3} f(x)$ does not exist; the function values get arbitrarily large near $x = 3$. (Even worse, they are positive on one side, negative on the other.)

So, the big question is, when can you get away with just sticking the number into the formula to find the limit?

$$\lim_{x \to a} f(x) = f(a) \,?$$

Textbooks give you a list of "limit laws" that state conditions that guarantee that the limit can be taken in the obvious way. Let's turn the question around and try to identify "danger signs" that label situations where the obvious way might go wrong.

In practice, the most common trouble is a *zero of the denominator.* (Notice that in the limit laws on p. 91, the last one, concerning division, is the only one that needs a caveat ("if $\lim_{x \to a} g(x) \neq 0$").)

Now two things can happen:

1. The limit of the numerator as $x \to a$ **is not** 0. Then we probably have some kind of "infinite limit" and a vertical asymptote.

2. The limit of the numerator as $x \to a$ **is** 0. Then we have to look carefully to see which factor goes to zero faster, the numerator or the denominator. Typically, they will vanish "at the same rate" so that the limit of the fraction is some finite, nonzero number. *This is the situation that is fundamental to the definition of the* derivative *in calculus.*

Here are some examples.

Exercise 2.3.5

$$\lim_{x \to -1} \frac{x - 2}{x^2 + 4x - 3}$$

Exercise 2.3.23

$$\lim_{h \to 0} \frac{(1 + h)^4 - 1}{h}$$

$$\lim_{x \to -1} \frac{x - 2}{x^2 + 4x - 3}$$

This first one is a typical example of a straightforward limit. When $x = -1$, the denominator equals $1 - 4 - 3 = -6$. Since this is not zero, and there is nothing else peculiar about this example, we can conclude that the limit is

$$\frac{-3}{-6} = \frac{1}{2}.$$

Oh, but the instruction in the book was "justify each step by indicating the appropriate Limit Law(s)." So a grader would not give me full credit for that answer! Let's go back and do the argument *carefully*.

First off, by Limit Law 5,

$$\lim_{x \to -1} \frac{x - 2}{x^2 + 4x - 3} = \frac{\lim_{x \to -1} (x - 2)}{\lim_{x \to -1} (x^2 + 4x - 3)}$$

provided that the limit of the denominator exists and is not 0. We won't know whether that's true

until later, so we mark it as unfinished business and forge ahead.

To take the limit of the numerator, use Law 2 (or Laws 1 and 3, with $c = -1$):

$$\lim_{x \to -1} (x - 2) = \lim_{x \to -1} x - \lim_{x \to -1} 2 = -1 - 2 = -3.$$

The last step uses the Laws 7 and 8.

Now attack the denominator in the same way:

$$\lim_{x \to -1} (x^2 + 4x - 3)$$

$$= \lim_{x \to -1} x^2 + \lim_{x \to -1} 4x + \lim_{x \to -1} (-3)$$

$$= (-1)^2 + 4(-1) - 3 = -6$$

(using Law 9 in addition to those previously mentioned). So the denominator is not 0, and our first step was justified.

Conclusion: The limit is $(-3)/(-6) = \frac{1}{2}$.

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$$\lim_{h \to 0} \frac{(1+h)^4 - 1}{h}$$

This one is harder, and very typical of the limits that arise in calculating derivatives from first principles.

If we try to use Law 5, we see immediately that the limit of the denominator is 0. So we have to do some work to see whether that zero somehow cancels out of the numerator.

The obvious thing to do next would be to multiply out the fourth-degree polynomial in the numerator. But to do that completely is a huge waste of effort. Let's be smarter.

Inside the front cover of the book is something called "Binomial Theorem". For the case $n = 4$ it says

$$(x + y)^4 = x^4 + 4x^3y + \cdots + y^4$$

(where I left out some complicated terms we won't need). We can use this with $x = 1$ and

$y = h$:

$(1 + h)^4 = 1 + 4h +$ terms containing powers of h.

The powers are h^2, h^3, h^4 — exponents greater than 1. According to the function formula, we're supposed to subtract 1 and then divide by h. That gives

$$4 + \text{terms containing } h, \, h^2, \, h^3.$$

Obviously the limit of this function as $h \to 0$ is 4. ∎

There are other situations (besides vanishing denominators) where a limit either doesn't exist at all, or exists but can't be calculated by simple application of the limit laws. Apart from piecewise defined functions (see Figures 2 and 4 on p. 97), it is hard to construct examples using the simple algebraic functions at our disposal now. Some more interesting pathologies will turn up later when we study trigonometric functions, for example.