Lecture for Week 5 (Secs. 3.4–5)

Trig Functions and the Chain Rule

The important differentiation formulas for trigonometric functions are

$$\frac{d}{dx}\sin x = \cos x, \qquad \frac{d}{dx}\cos x = -\sin x.$$

Memorize them! To evaluate any other trig derivative, you just combine these with the product (and quotient) rule and chain rule and the definitions of the other trig functions, of which the most important is

$$\tan x = \frac{\sin x}{\cos x}\,.$$

Exercise 3.4.19

Prove that

$$\frac{d}{dx} \cot x = -\csc^2 x.$$

Exercise 3.4.23

Find the derivative of $y = \csc x \cot x$.

What is $\frac{d}{dx}\cot x$? Well, the definition of the cotangent is

$$\cot x = \frac{\cos x}{\sin x}\,.$$

So, by the quotient rule, its derivative is

$$\frac{\sin x(-\sin x) - \cos x(\cos x)}{\sin^2 x}$$

$$= -\frac{1}{\sin^2 x} \equiv -\csc^2 x$$

(since $\sin^2 x + \cos^2 x = 1$). ■

To differentiate $\csc x \cot x$ use the product rule:

$$\frac{dy}{dx} = \frac{d \csc x}{dx} \cot x + \csc x \frac{d \cot x}{dx} .$$

The second derivative is the one we just calculated, and the other one is found similarly (Ex. 3.4.17):

$$\frac{d}{dx} \csc x = -\csc x \cot x.$$

So

$$\frac{dy}{dx} = -\csc x \cot^2 x - \csc^3 x.$$

This could be rewritten using trig identities, but the other versions are no simpler. ■

Another method:

$$y = \csc x \cot x = \frac{\cos x}{\sin^2 x}.$$

Now use the quotient rule (and cancel some extra factors of $\sin x$ as the last step). ■

Exercise 5.7.11 (p. 353)

Find the antiderivatives of

$$h(x) = \sin x - 2\cos x.$$

We want to find the functions whose derivative is $h(x) = \sin x - 2\cos x$. If we know two functions whose derivatives are (respectively) $\sin x$ and $\cos x$, we're home free. But we do!

$$\frac{d(-\cos x)}{dx} = \sin x, \qquad \frac{d \sin x}{dx} = \cos x.$$

So we let $H(x) = -\cos x - 2\sin x$ and check that $H'(x) = h(x)$. The most general antiderivative of h is $H(x) + C$ where C is an arbitrary constant.

■

Now let's drop back to see where the trig derivatives came from. On pp. 180–181 we're offered 4 trigonometric limits, but they are not of equal profundity.

The first two just say that the sine and cosine functions are continuous at $\theta = 0$. (But you already knew that, didn't you?)

The third limit is the important one:

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1.$$

It can be proved from the inequalities

$$\sin\theta < \theta < \tan\theta \quad (\text{for } 0 < \theta < \pi/2),$$

which are made obvious by drawing some pictures.

It says that $\sin\theta$ “behaves like” θ when θ is small.

In contrast, the fourth limit formula,

$$\lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta} = 0,$$

says that $\cos\theta$ "behaves like" 1 and that the difference from 1 vanishes *faster* than θ as θ goes to 0.

In fact, later we will see that

$$\cos\theta \approx 1 - \frac{\theta^2}{2}\,.$$

Exercise 3.4.15

$$\lim_{x \to 0} \frac{\tan 3x}{3 \tan 2x} \,.$$

Split the function into a product of functions whose limits we know:

$$\frac{\tan 3x}{3\tan 2x} = \frac{1}{3}\,\frac{\sin 3x}{\cos 3x}\,\frac{\cos 2x}{\sin 2x}$$
$$= \frac{\sin 3x}{3x}\,\frac{2x}{\sin 2x}\,\frac{\cos 2x}{2\cos 3x}\,.$$

As $x \to 0$, $2x$ and $3x$ approach 0 as well. Therefore, the two sine quotients approach 1. Each cosine also goes to 1. So the limit is $\frac{1}{2}$. ∎

The *chain rule* is the most important and powerful theorem about derivatives. For a first look at it, let's approach the last example of last week's lecture in a different way:

Exercise 3.3.11 (revisited and shortened)

A stone is dropped into a lake, creating a circular ripple that travels outward at a speed of 60 cm/s. Find the rate at which the circle's area is increasing after 3 s .

In applied problems it's usually easier to use the "Leibniz notation", such as df/dx, instead of the "prime" notation for derivatives (which is essentially Newton's notation).

The area of a circle of radius r is

$$A = \pi r^2.$$

So

$$\frac{dA}{dr} = 2\pi r.$$

(Notice that $dA/dr = 2\pi r$ is the circumference. That makes sense, since when the radius changes by Δr, the region enclosed changes by a thin circular strip of length $2\pi r$ and width Δr, hence area $\Delta A = 2\pi r \, \Delta r$.)

From $A'(r) = 2\pi r$ we could compute the rate of change of area **with respect to radius** by plugging in the appropriate value of r (namely, $60 \times 3 = 180$). But the question asks for the rate **with respect to time** and tells us that

$$\frac{dr}{dt} = 60 \text{ cm/s}\,.$$

Common sense says that we should just multiply $A'(r)$ by 60, getting

$$\frac{dA}{dt} = \frac{dA}{dr}\,\frac{dr}{dt} = 2\pi(60)^2 t, \qquad t = 3.$$

Of course, this is the same result we got last week. ∎

Now consider a slight variation on the problem:

Exercise 3.3.11 (modified)

A stone is dropped into a lake, creating a circular ripple that travels outward at a speed of 60 cm/s. Find the rate at which the area is increasing when the radius is 180 cm.

To answer this question by the method of last week, using $A(t) = 3600\pi t^2$, we would need to calculate the *time* when $r = 180$. That's easy enough in this case ($t = 3$), but it carries the argument through an unnecessary loop through an inverse function. It is more natural and simpler to use this week's formula,

$$\frac{dA}{dt} = \frac{dA}{dr}\,\frac{dr}{dt}\,.$$

It gives $\displaystyle \frac{dA}{dt} = (2\pi r)60 = 21600\pi.$ ∎

In both versions of the exercise we dealt with a function of the type $A(r(t))$, where the output of one function is plugged in as the input to a different one. The *composite function* is sometimes denoted $A \circ r$ (or $(A \circ r)(t)$). The **chain rule** says that

$$(A \circ r)'(t) = A'(r(t))r'(t),$$

or

$$\frac{dA}{dt} = \frac{dA}{dr}\,\frac{dr}{dt}\,.$$

Now we can use the chain rule to differentiate particular functions.

Exercise 3.5.7

Differentiate

$$G(x) = (3x - 2)^{10}(5x^2 - x + 1)^{12}.$$

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It would be foolish to multiply out the powers when we can use the chain rule instead. Of course, the first step is a product rule.

$$G'(x) = 10(3x - 2)^9 \frac{d}{dx}(3x - 2)\,(5x^2 - x + 1)^{12}$$
$$+12(3x - 2)^{10}(5x^2 - x + 1)^{11} \frac{d}{dx}(5x^2 - x + 1)$$

$$= 30(3x-2)^9(5x^2-x+1)^{12} + 12(10x-1)(3x-2)^{10}(5x^2-x+1)^{11}.$$

(The book's answer combines some terms at the expense of factoring out a messy polynomial.)

■

Note that it is *not* smart to use the quotient rule on a problem like

$$\frac{d}{dx}\,\frac{x+1}{(x^2+1)^3}\,.$$

You'll find yourself cancelling extra factors of x^2+1. It's much better to use the product rule on

$$(x+1)(x^2+1)^{-3},$$

getting only 4 factors of $(x^2+1)^{-1}$ instead of 6.

Exercise 3.5.51

Find the tangent line to the graph of

$$y = \frac{8}{\sqrt{4 + 3x}}$$

at the point $(4, 2)$.

$$y = 8(4+3x)^{-1/2}.$$

$$\frac{dy}{dx} = -4(4+3x)^{-3/2}(3).$$

When $x = 4$, $y' = -12(16)^{-3/2} = -3/16$ (and $y = 8(16)^{-1/2} = 2$ as the problem claims). So the tangent is

$$y = 2 - \frac{3}{16}(x-4). \qquad \blacksquare$$

The book simplifies the equation

$$y = 2 - \frac{3}{16}(x - 4)$$

to $3x + 16y = 44$, but I think it is better to leave such equations in the form that emphasizes the dependence on Δx ($= x - 4$ in this case). The point of a tangent line is that is the best linear approximation to the curve in the region where Δx is small.

Exercise 3.5.59

Suppose $F(x) = f(g(x))$ and

$$g(3) = 6, \quad g'(3) = 4, \quad f'(3) = 2, \quad f'(6) = 7.$$

Find $F'(3)$.

$$\begin{aligned} F'(x) &= f'(g(x))g'(x) \\ &= f'(g(3))g'(3) = f'(6) \times 4 = 28. \end{aligned}$$

$f'(3)$ is irrelevant — a trap. If $y = g(x)$ and $z = f(y)$ are physical quantities, then $z = F(x)$, and in Leibniz notation we would write

$$\frac{dz}{dx} = \frac{dz}{dy}\,\frac{dy}{dx}\,.$$

Nothing wrong with that, except that it makes it easy to fall into the trap!