Lecture for Week 6 (Secs. 3.6–9)

Derivative Miscellany I

Implicit differentiation

We want to answer questions like this:

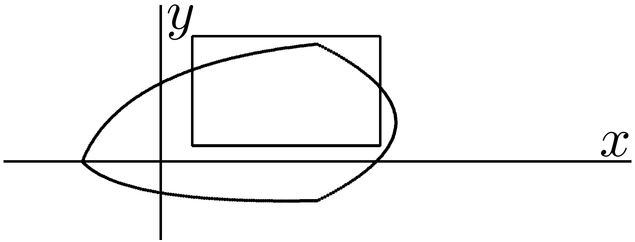
1. What is the derivative of $\tan^{-1} x$?

2. What is $\dfrac{dy}{dx}$ if

$$x^3 + y^3 + xy^2 + x^2y - 25x - 25y = 0\,?$$

$$x^3 + y^3 + xy^2 + x^2y - 25x - 25y = 0.$$

Here we don't know how to solve for y as a function of x, but we expect that the formula defines a function "implicitly" if we consider a small enough "window" on the graph (to pass the "vertical line test").



Temporarily assuming this is so, we differentiate the equation with respect to x, remembering that y is a function of x.

$$\begin{aligned}
0 &= \frac{d}{dx}(x^3 + y^3 + xy^2 + x^2y - 25x - 25y) \\
&= 3x^2 + 3y^2y' + y^2 + 2xyy' + 2xy + x^2y' - 25 - 25y' \\
&= (3x^2 + y^2 + 2xy - 25) + y'(3y^2 + 2xy + x^2 - 25).
\end{aligned}$$

$$y' = \frac{25 - 3x^2 - y^2 - 2xy}{3y^2 + 2xy + x^2 - 25}.$$

To use this formula, you need to know a point (x, y) on the curve. You can check that $(3, 4)$ does satisfy

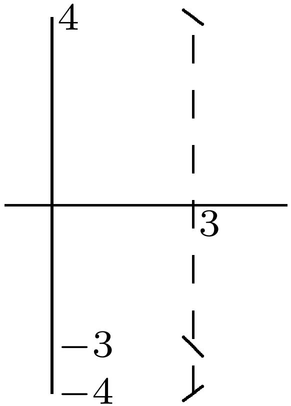
$$x^3 + y^3 + xy^2 + x^2y - 25x - 25y = 0.$$

Plug those numbers into

$$y' = \frac{25 - 3x^2 - y^2 - 2xy}{3y^2 + 2xy + x^2 - 25}$$

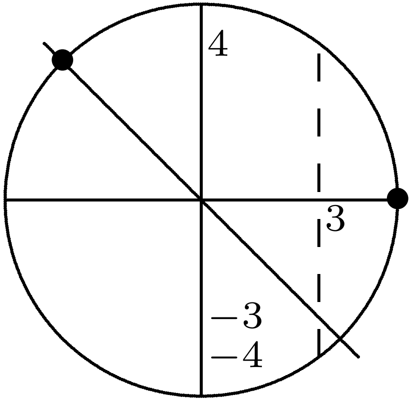
to get $y' = -\dfrac{3}{4}$.

But $(x, y) = (3, -3)$ also satisfies the equation, and it gives $y' = -1$. And $(3, -4)$ satisfies the equation and gives $y' = +\frac{3}{4}$. *Three different functions are defined near* $x = 3$ by our equation, and each has a different slope.



The curve in this problem is the union of a circle and a line:

$$0 = x^3 + y^3 + xy^2 + x^2y - 25x - 25y$$
$$= (x^2 + y^2 - 25)(x + y).$$



We can clearly see the three points of intersection with the line $x = 3$. Two other interesting points are:

1. $x = 5$, $y = 0$ (vertical tangent): The denominator of the formula for y equals 0, but the numerator does not.

2. $x = -5/\sqrt{2}$, $y = 5/\sqrt{2}$ (intersection): Both numerator and denominator vanish, because the slope is finite but not unique.

An inportant application of implicit differentiation is to find formulas for derivatives of inverse functions, such as $u = \tan^{-1} v$. This equation just means $v = \tan u$, together with the "branch condition" that $-\frac{\pi}{2} < u < \frac{\pi}{2}$ (without which u would not be uniquely defined). So

$$1 = \frac{dv}{dv} = \frac{d}{dv} \tan u = \left(\frac{d}{du} \tan u \right) \frac{du}{dv} \, .$$

But

$$\frac{d}{du} \tan u = \sec^2 u = 1 + \tan^2 u = 1 + v^2.$$

Putting those two equations together, we get

$$\frac{d}{dv}\tan^{-1} v = \frac{du}{dv} = \left(\frac{d}{du}\tan u\right)^{-1} = \frac{1}{1+v^2}\,.$$

Generally speaking, the derivative of an inverse trig function is an algebraic function! We will see more of this in Sec. 4.2.

Exercise 3.6.39

Show that the curve families

$$y = cx^2, \qquad x^2 + 2y^2 = k$$

are orthogonal trajectories of each other.

(That means that every curve in one family (each curve labeled by c) is orthogonal to every curve in the other family (labeled by k).

For the curves $y = cx^2$ we have $\dfrac{dy}{dx} = 2cx$. (No implicit differentiation was needed in this case.) For the curves $x^2 + 2y^2 = k$ we have

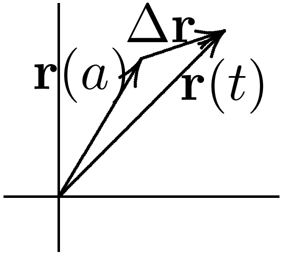
$$2x + 4y\,\frac{dy}{dx} = 0 \;\Rightarrow\; \frac{dy}{dx} = -\,\frac{x}{2y}\,.$$

If the curves are orthogonal, the product of the slopes must be -1 (and vice versa). Well, the product is

$$(2cx)\left(-\,\frac{x}{2y}\right) = -\,\frac{cx^2}{y} = -1. \qquad \blacksquare$$

Derivatives of vector functions

No big surprise here: Conceptually, we are subtracting the "arrows" for two nearby values of the parameter, then dividing by the parameter difference and taking the limit.



$$\mathbf{r}'(a) = \lim_{t\to a} \frac{\mathbf{r}(t) - \mathbf{r}(a)}{t - a} \equiv \lim_{h\to 0} \frac{\Delta\mathbf{r}}{h}$$

And calculationally, since our basis vectors do not depend on t, we just differentiate each component:

$$\frac{d}{dt}\left[t^2\,\hat{\mathbf{i}} + 3t\,\hat{\mathbf{j}} + 5\,\hat{\mathbf{k}}\right] = 2t\,\hat{\mathbf{i}} + 3\,\hat{\mathbf{j}}.$$

Second (and higher) derivatives

This is fairly obvious, too: The second derivative is the derivative of the first derivative.

$$\begin{aligned} s(t) = At^2 + Bt + C \;&\Rightarrow\; s'(t) = 2At + B \\ &\Rightarrow\; s''(t) = 2A. \end{aligned}$$

(This was essentially Exercise 3.8.37.)

The most important application of second derivatives is *acceleration*, the derivative of velocity, which is the derivative of position.

Exercise 3.8.49

A satellite completes one orbit of Earth at an altitude 1000 km every 1 h 46 min. Find the velocity, speed, and acceleration at each time. (Earth radius = 6600 km.)

The period is $1\frac{46}{60} = 1.767$ hr. Therefore, the angular speed is $2\pi/1.767 = 3.557$ radians per hour. The radius of the circle is 7600, so the speed in the orbit is $7600 \times 3.557 = 27030$ km/h at all times. To represent the velocity we must choose a coordinate system; say that the satellite crosses the x axis when $t = 0$ and moves counter-clockwise (so it crosses the y axis after a quarter period). Then

$$\mathbf{v}(t) = 27030\langle -\sin(3.557t), \cos(3.557t)\rangle.$$

(When $t = 0$, $\mathbf{v}$ is in the positive y direction; after a quarter period, it is in the negative x direction.) The acceleration is the derivative of that,

$$\mathbf{a}(t) = 27030 \times 3.557\langle -\cos(3.557t), -\sin(3.557t)\rangle.$$

Finally, let's find the position function. Its derivative must be $\mathbf{v}$, so a good first guess is

$$\mathbf{r}(t) = \frac{27030}{3.557}\langle \cos(3.557t), \sin(3.557t)\rangle.$$

To this we could add any constant vector, but a quick check shows that $\mathbf{r}(0)$ is in the positive x direction as we wanted, and this orbit is centered at the origin as it should be. So this is the right answer. Notice that $\mathbf{a}$ points in the direction opposite to $\mathbf{r}$ (i.e., toward the center of the orbit), as always for uniform circular motion. ■

Slopes and tangents of parametric curves

What is the slope of a curve defined by parametric equations

$$x = x(t), \qquad y = y(t)\,?$$

If we had y as a function of x, we would just calculate $\frac{dy}{dx}$. But we can find the slope without eliminating t from the equations. It may come as no surprise that the answer is obtained by "dividing numerator and denominator by dt":

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} .$$

The valid proof of this formula is simply an application of the chain rule:

$$\frac{dy}{dt} = \frac{dy}{dx}\,\frac{dx}{dt} .$$

But, you should be shouting, what if the denominator is 0? If $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} \neq 0$, the curve is vertical at that point, so the slope is properly undefined. If both derivatives are 0, we need to consider another parametrization to get an answer; the moving point has slowed to a standstill at the time of interest, so the parametric derivatives give no information.

To apply the formula, you may need to do some work to determine the correct value of t to plug in.

Exercise 3.9.19

At what point does the curve

$$x = t(t^2 - 3), \qquad y = 3(t^2 - 3)$$

cross itself? Find equations of both tangents at that point.

If the curve crosses itself, there must be two values of t that yield the same x and y, so

$$t_1({t_1}^2-3) = t_2({t_2}^2-3) \quad \text{and} \quad 3({t_1}^2-3) = 3({t_2}^2-3).$$

From the second equation, $t_1 = \pm t_2$, and so from the first one, either $t_1 = +t_2$ or $t_1 = \pm\sqrt{3} = -t_2$. Only the second possibility is of interest to us. Let's define t_1 to be the positive root.

Now calculate the derivatives:

$$x'(t) = (t^2 - 3) + t(2t) = 3t^2 - 3, \qquad y'(t) = 6t.$$

So the slope is

$$\frac{dy}{dx} = \frac{6t}{3t^2 - 3}.$$

Substituting $t = \pm\sqrt{3}$, we get

$$\frac{dy}{dx} = \frac{\pm 6\sqrt{3}}{6} = \pm\sqrt{3}.$$

(Unlike the implicit differentiation example earlier, there is no $\frac{0}{0}$ ambiguity, because the two local curve segments correspond to different values of t, each with a uniquely defined slope.)

To find the tangent lines we need to know the point, which is easily found from the original formulas:

$$(x, y) = (0, 0).$$

Then in Cartesian terms, the tangent lines are

$$y = \pm\sqrt{3}\,x.$$

In parametric terms, they are

$$x = (t - \sqrt{3}), \qquad y = \sqrt{3}(t - \sqrt{3});$$
$$x = (t + \sqrt{3}), \qquad y = -\sqrt{3}(t + \sqrt{3}). \qquad \blacksquare$$