Lecture for Week 8 (Secs. 4.1–4.2)

Exponentials and Inverse Functions

Let's get right to the point: The main reason for studying exponential functions is to solve problems like those in Sec. 4.5, where the growth of some "stuff" is proportional to the amount of stuff already present:

$$\frac{dP(t)}{dt} = kP(t). \qquad (*)$$

(Compound interest, population growth, radioactive decay all fall into this category.)

Suppose we knew a function whose derivative is itself:

$$\frac{d}{dx}\exp(x) = \exp(x).$$

Then $P(t) = C\exp(kt)$ would solve $(*)$ $[P' = kP]$, for any constant C.

Normalization: Suppose also that $\exp(0) = 1$. And suppose that we know $P(0)$. Then $P(t) = P(0)\exp(kt)$ is the correct solution of $(*)$ for our problem.

With this, we have solved our first nontrivial *differential equation*. The trouble with this approach to $\exp(x)$ is that it may not be obvious that such a function exists, or that there are not more than one of them satisfying $\exp(0) = 1$.

Here is a different approach (summarized from Stewart):

1. From elementary algebra we understand what a^x means, for any $a > 1$ and any rational number x.

2. We can define a^x for irrational x by continuity ("filling in the holes in the graph").

3. Define e as the number such that

$$\lim_{h\to 0} \frac{e^h - 1}{h} = 1.$$

4. Prove that $\frac{d}{dx}e^x = e^x$.

5. Observe that $a^0 = 1$ for all a, hence $e^0 = 1$.

So we can define $\exp(x)$ as e^x. You can think of e^x either as a number, e, raised to a power, or as a special function, $\exp(x)$, analogous to $\sin(x)$ and the other trig functions.

The numerical value of e is a famous "transcendental" number like π:

$$e = 2.7182818\cdots.$$

Algebraic properties of exponentials ("the laws of exponents")

$$e^{x+y} = e^x e^y.$$

$$(e^x)^y = e^{xy}.$$

$$e^{-x} = \frac{1}{e^x}.$$

$$e^0 = 1.$$

$$e^1 = e.$$

These laws also hold with a in place of e everywhere. Also,

$$(ab)^x = a^x b^x.$$

And we also have $\quad e^x > 0 \quad$ and

$$\lim_{x\to+\infty} e^x = +\infty,$$

$$\lim_{x\to-\infty} e^x = 0.$$

These also hold for a^x if $a > 1$; if $0 < a < 1$, the limits reverse, since $a^x = (1/a)^{-x}$.

Exercise 4.1.29

Differentiate $y = xe^{2x}$.

Exercise 4.1.51

If $f(x) = e^{-2x}$, find $f^{(8)}(x)$.

$$y = xe^{2x}.$$

Use the product rule, the chain rule, and the basic exponential derivative formula $\frac{d}{du}e^u = e^u$:

$$\begin{aligned} y' &= e^{2x} + x\,\frac{d}{dx}e^{2x} \\ &= e^{2x} + 2xe^{2x}. \end{aligned}$$

■

$$y = e^{-2x}.$$

What is its 8th derivative? Every time I differentiate, I just get a factor -2. So

$$f^{(8)}(x) = (-2)^8 e^{-2x} = 256e^{-2x}.$$

■

Exponential and logarithmic functions are inverses of each other, so at this point we digress to discuss inverse functions in general.

Remember the formula for the volume of a sphere:

$$V = \tfrac{4}{3}\,\pi r^3 \equiv f(r).$$

Solve it to get a formula for the radius as a function of the volume:

$$r = \sqrt[3]{\frac{3V}{4\pi}} \equiv g(v) \equiv f^{-1}(V).$$

The functions f and g are *inverse* to each other.

Note that f^{-1} does not mean $\dfrac{1}{f}$ in this context.

In a physical application like that, the variables have natural names (r and V), because they represent physical quantities. But in generic math, we usually write $y = f(x)$. Should we then write $x = f^{-1}(y) = g(y)$, or $y = f^{-1}(x) = g(x)$? Both Stewart and Maple insist on the latter, so that x is always the independent variable and y the dependent one. To avoid confusion, I try to use neutral letters, say $u = f(w)$ and $w = f^{-1}(u)$.

In obtaining an inverse for a given function, two complications may arise, the *domain problem* and the *branch problem*. You know that they both occur for the *square root* — the inverse of a very simple function, $u = w^2 \equiv f(w)$.

1. $f^{-1}(u)$ may not be defined for some values of u. *Example:* If $u < 0$, then u is not equal to w^2 for any real w. So the *domain* of the square root function contains only *nonnegative* numbers.

2. To make f^{-1} single-valued (as required by the definition of a function), we may need to exclude some values of w that satisfy $f(w) = u$. We must choose *just one* w for each u. This is called "choosing a *branch* of the inverse function". *Example:* We define $\sqrt{u}$ to be the *nonnegative* square root. The graph of $\sqrt{u}$ is the right-hand half of the graph of w^2, flipped over so that the u and w positive axes are interchanged.

A condition that assures that the branch problem does not arise is the *horizontal line test* — which says that the graph of the inverse will pass the vertical line test without our having to throw part of the graph away. The original function is then called *one-to-one*. (We don't have two points w mapping into the same u.)

The domain problem does not arise if the function is *onto* $\mathbf{R}$ — that is, every $u \in \mathbf{R}$ appears as $f(w)$ for some w (which will be $f^{-1}(u)$).

Exercise 4.2.13

Show that $f(x) = 4x + 7$ is one-to-one and find its inverse function.

We need to solve $y = 4x+7$. That is elementary:

$$x = \frac{1}{4}(y-7) = \frac{y}{4} - \frac{7}{4}\,.$$

(This makes sense for all y, so the function f is onto. And the solution for x is unique, so f is one-to-one. Alternatively, you could sketch the graph of f and see that every horizontal line crosses it exactly once.)

Therefore, we could write $f^{-1}(y) = \frac{y}{4} - \frac{7}{4}$;

but to match the textbook's notation we must switch the variables:

$$f^{-1}(x) = \frac{x}{4} - \frac{7}{4}\,. \qquad \blacksquare$$

Finally we get to the main point: What is the derivative of $f^{-1}(x)$? (We're assuming we know the derivative of f.)

This question could have been answered back in the section on implicit differentiation. To say that $w = f^{-1}(u) \equiv g(u)$ is to say that $u = f(w)$ (and that a branch has been chosen, if necessary). So

$$1 = \frac{du}{du} = f'(w)\,\frac{dw}{du} = f'(w)g'(u).$$

So

$$g'(u) = [f'(w)]^{-1} = \frac{1}{f'\big(f^{-1}(u)\big)}\,.$$

Back in our original example, we might want to write this relation as

$$\frac{dr}{dV} = \left(\frac{dV}{dr}\right)^{-1} .$$

(Here the exponent -1 *does* mean to take the reciprocal (“one over” the number).) Like the chain rule,

$$\frac{dV}{dr} = \frac{dV}{dC}\,\frac{dC}{dr} ,$$

the theorem looks trivial in the Leibniz notation. (But be careful where the functions are evaluated. We need $f'(f^{-1}(u))$, not $f'(u)$.)

Exercise 4.2.31

Suppose $g = f^{-1}$ and $f(4) = 5$, $f'(4) = \frac{2}{3}$. Find $g'(5)$.

We want $g'(5)$. Since g is the inverse of f, g' is the reciprocal of f'. So we look for $f'(5)$ in the given information and don't find it. (Even worse, it might be given but be irrelevant!) What is wrong? You have to remember to evaluate f' at the number $f^{-1}(5)$, which is 4.

$$g'(5) = \frac{1}{f'(g(5))} = \frac{1}{f'(4)} = \frac{3}{2}\,.$$

■

Exercise 4.2.25

Find $g'(1)$ if $g = f^{-1}$ and $f(x) = x^3 + x + 1$.

$$f'(x) = 3x^2 + 1.$$

We need to evaluate at a number x where $f(x) = 1$. That requires

$$0 = f(x) - 1 = x^3 + x = x(x^2 + 1).$$

Since the quadratic factor has no real roots, only $x = 0$ qualifies. (If we did have more than one root, f^{-1} would not exist!)

$$g'(1) = \frac{1}{f'(0)} = \frac{1}{1} = 1. \qquad \blacksquare$$