Lecture for Week 9 (Secs. 4.3–4.4)

Logarithms

The *natural logarithm* function is the inverse of the exponential function:

$$y = \ln x \iff x = e^y .$$

Since e^y is positive for all real y, the domain of ln consists of the positive numbers only. The function values are negative for $x < 1$ and positive for $x > 1$.

In "pure" math, $\ln x$ is often written $\log x$. Engineers and scientists prefer to keep "log" for $\log_{10} x$, discussed below.

Algebraic properties of logarithms
("the laws of logarithms")

$$\ln(xy) = \ln x + \ln y.$$

$$\ln(x^y) = y \ln x.$$

$$\ln(1/x) = -\ln x.$$

$$\ln 1 = 0.$$

$$\ln e = 1.$$

The derivative of the logarithm is surprisingly simple:

$$\frac{d}{dx}\ln x = \frac{1}{x}\,.$$

Proof: Apply the inverse function formula from last week, or, equivalently, apply implicit differentiation to $x = e^y$. ■

This formula is the most important reason for studying logarithms in calculus (just as the most important property of the sine function is that it satisfies $f''(x) = -f(x)$, not its use to calculate the height of a flagpole as you thought in high school).

Exercise 4.4.13

Differentiate $\quad \ln \dfrac{a-x}{a+x}$.

$$g(x) = \ln \frac{a-x}{a+x}\,.$$

Although we could attack this directly with the chain and quotient rules, we get a simpler answer by first using one of the "laws of logarithms":

$$g(x) = \ln(a-x) - \ln(a+x).$$

$$g'(x) = -\frac{1}{a-x} - \frac{1}{a+x}\,.$$

So far I've discussed "logarithms to base e", and all my exponential calculus examples last week involved exponential functions with base e. But I did start last week by defining exponential functions with arbitrary bases a. Similarly, the *logarithm to base a* is defined by

$$y = \log_a x \iff x = a^y.$$

In words, $\log_a x$ is the power to which a must be raised to yield x. (Obviously, $\ln x = \log_e x$.)

Exponentials and logarithms to arbitrary bases have many practical applications. However, the most important fact to memorize about these functions is that most of their formulas are not worth memorizing! For calculus purposes, you can always get rid of these functions by reducing them to base e.

$$a^x = e^{x \ln a}\,; \qquad \log_a x = \frac{\ln x}{\ln a}\,.$$

These two formulas, plus all those involving just e and ln, are all you really need to know.

Exercise

Prove those two crucial formulas!

Exercise (similar to 4.3.3)

Find $\log_5 3125$.

$$
\begin{aligned}
e^{x \ln a} &= (e^{\ln a})^x \\
&= a^x,
\end{aligned}
$$

by a law of exponents and the definition of ln.
Then, by that formula,

$$
\begin{aligned}
a^{\ln x / \ln a} &= e^{(\ln x / \ln a) \ln a} \\
&= e^{\ln x} = x,
\end{aligned}
$$

so by definition $\log_a x = \ln x / \ln a$. ■

Now, what is $\log_5 3125$? This one of the rare (I hope) occasions when you should ignore my advice. Don't try to reduce the problem to natural (base-e) logs. When both numbers are integers, that's a dead giveaway (in homework or tests) that the argument is an exact power of the base. Start multiplying 5 by itself:

$$5,\ 25,\ 125,\ 625,\ 3125\,!$$

$$\log_5 3125 = 5. \qquad \blacksquare$$

Exercise 4.4.35

Differentiate $x^{\sin x}$.

Exercise 4.4.63

Differentiate $\dfrac{e^x\sqrt{x^5+2}}{(x+1)^4(x^2+3)^2}$ in a slick way.

$$y = x^{\sin x} = e^{\sin x \ln x}.$$

$$\begin{aligned}\frac{dy}{dx} &= e^{\sin x \ln x} \frac{d}{dx}(\sin x \ln x) \\ &= x^{\sin x}\left(\cos x \ln x + \frac{\sin x}{x}\right). \qquad \blacksquare\end{aligned}$$

The other problem looks like a quotient-rule monstrosity that no teacher would give except as a punishment. But, it can be made more pleasant by employing *logarithmic differentiation*: Notice that for any function,

$$\frac{d}{dx} \ln f(x) = \frac{f'(x)}{f(x)} \,.$$

When f involves lots of products and powers and maybe a quotient, $\ln f$ will be simpler, because of the logarithm laws:

$$f(x) = \frac{e^x\sqrt{x^5+2}}{(x+1)^4(x^2+3)^2}\,.$$

$$\ln f(x) = x + \tfrac{1}{2}\ln(x^5+2) - 4\ln(x+1) - 2\ln(x^2+3).$$

$$\frac{d}{dx}\ln f(x) = 1 + \tfrac{1}{2}\frac{5x^4}{x^5+2} - \frac{4}{x+1} - \frac{4x}{x^2+3}\,.$$

Now comes the hard part: Remember to multiply by $f(x)$ to get the derivative you want, $f'(x)$. (Typing out the result is something no student would require except as a punishment.) ■