Lecture for Week 11 (Secs. 5.1–3)

Analysis of Functions

(We used to call this topic "curve sketching", before students could sketch curves by typing formulas into their calculators. It is still important to understand what derivatives tell us about the qualitative and geometrical behavior of functions and their graphs.)

Here are the things to look for when analyzing a function (or sketching its graph).

Domain; any points of discontinuity
Intercepts
Symmetry
Asymptotes
Intervals of increase and decrease
Local extrema (maxima and minima)
Inflection points and intervals of concavity

I will approach this subject by doing examples, each of which illustrates the full set of concepts and theorems involved, rather than giving a theoretical discussion of each concept in turn as the book needs to do.

Example 1

Discuss $f(x) = -x^3 + 3x$ (with respect to whatever concepts apply from the list on the previous slide).

$$y = f(x) = -x^3 + 3x = -x(x^2 - 3).$$

It's a polynomial, so it's defined and continuous everywhere, with no asymptotes. All the powers are odd, so the function is odd [$f(-x) = -f(x)$], which implies that the graph is symmetric through the origin. The horizontal intercepts occur when $0 = y = x(3 - x^2)$, or $x = 0$ and $x = \pm\sqrt{3} \approx \pm 1.7$. The vertical intercept is just $f(0) = 0$.

So far I haven't mentioned derivatives. Let's calculate them now:

$$f'(x) = -3x^2 + 3 = -3(x^2 - 1) = -3(x - 1)(x + 1).$$

$$f''(x) = -6x.$$

The *critical numbers* are the places where f' equals 0 (or doesn't exist, but that can't happen in this example). Here the critical numbers are ± 1. The only zero of the *second* derivative is $x = 0$; this is a *potential inflection point*.

It is useful to make a table listing the signs of f, f', and f'' on each of the intervals into which the real line is divided by the horizontal intercepts, critical numbers, potential inflection points, and vertical asymptotes. To see the signs easily, it's good to write the functions in *factored form*:

$$f(x) = -x(x - \sqrt{3})(x + \sqrt{3}),$$

$$f'(x) = -3(x - 1)(x + 1), \qquad f''(x) = -6x.$$

A function changes sign where a factor does:

x		$-\sqrt{3}$		-1		0		1		$\sqrt{3}$	
f	$+$		$-$		$-$		$+$		$+$		$-$
f'	$-$		$-$		$+$		$+$		$-$		$-$
f''	$+$		$+$		$+$		$-$		$-$		$-$

We see that f is decreasing as x varies from $-\infty$ to -1, increasing from -1 to $+1$, then decreasing again. Calculate $f(-1) = -2$, $f(1) = 2$, to see where to plot the minimum and maximum points.

We also see that f'' changes from positive to negative at $x = 0$, so that is indeed an inflection point. The graph is concave up on the left side and down on the right.

With this information you can easily sketch the graph freehand (much more easily than I can type it). ■

Example 2

Discuss $f(x) = \dfrac{2x^2}{x^2 - x - 2}$.

$$f(x) = \frac{2x^2}{(x+1)(x-2)} \,.$$

The most obvious feature is the vertical asymptotes at $x = -1$ and $x = 2$. (Those are the only two points where f is undefined or discontinuous.) There is also a horizontal asymptote, $y = 2$. There are no obvious symmetries.

Use the quotient rule and simplify to find

$$f'(x) = \frac{-2x(x+4)}{(x+1)^2(x-2)^2} .$$

So the critical numbers (not counting the asymptotes) are $x = 0$ and $x = -4$.

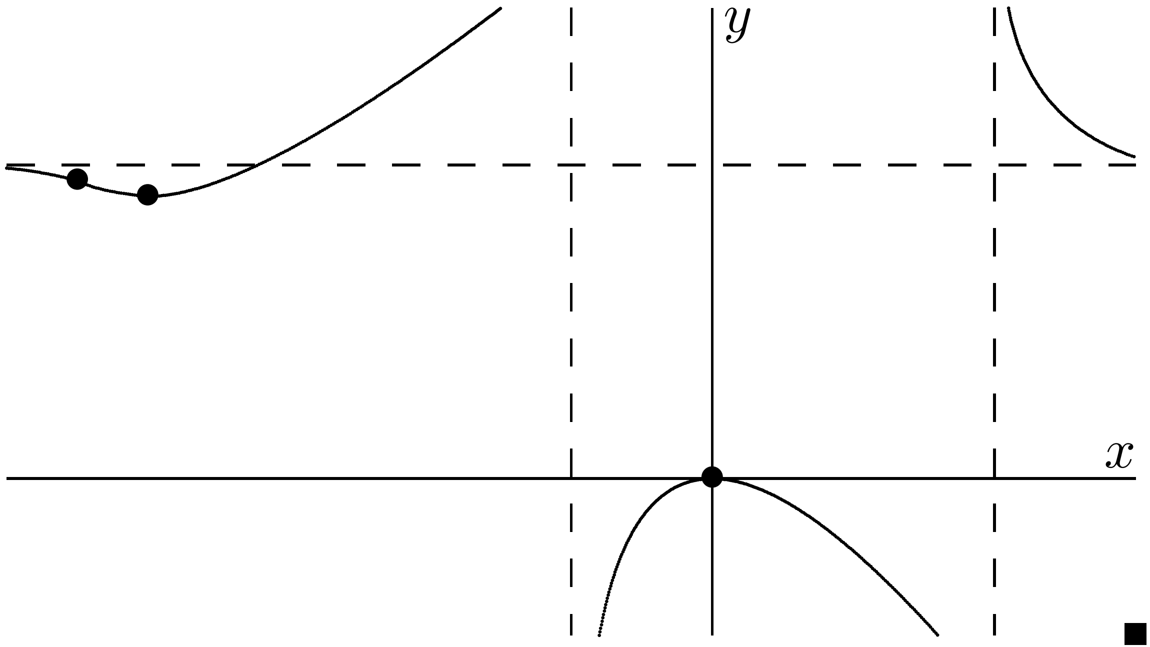
f'' is a mess in this problem, so let's see how much we can do without it. Again make a table:

x		-4		-1		0		2	
f	$+$		$+$		$-$		$-$		$+$
f'	$-$		$+$		$+$		$-$		$-$

Minimum: $f(-4) = 16/9 \approx 1.8$.
Maximum: $f(0) = 0$.

The function is increasing on the interval $[-4, -1]$ and on the interval $[-1, 0]$. At $x = -1$ (a vertical asymptote) it jumps from $+\infty$ to $-\infty$. (Therefore, it is *not correct* to say that it is increasing from $x = -4$ to $x = 0$, even though our table has only plus signs for f' in that range.) Similarly, there is a jump from $-\infty$ to $+\infty$ at the other asymptote.

Clearly, the graph must be mostly concave down between the asymptotes and mostly concave up outside them. However, because the minimum value of 1.8 is *below* the horizontal asymptote at 2, the concavity must be downward as $x \to -\infty$. Therefore, there must be an inflection point somewhere to the left of $x = -4$. There may be other inflection points (an even number in each of the three intervals delimited by the vertical asymptotes), but probably not.


y
x

Example 3

Discuss $f(x) = x^{2/3}(x - y)^2 + 2.$

$$f(x) = x^{2/3}(x-7)^2 + 2.$$

This function is positive everywhere. It is continuous, but the derivative will be discontinuous at 0.

$$f'(x) = \tfrac{2}{3}x^{-1/3}(x-7)^2 + 2x^{2/3}(x-7).$$

Remember that our strategy is to write f' as a product (factor it) so its zeros will be obvious.

Use $x^{2/3} = x^{-1/3}x^{+1}$:

$$\begin{aligned} f'(x) &= 2(x-7)\,\frac{x-7+3x}{3x^{1/3}} \\ &= \tfrac{2}{3}x^{-1/3}(4x-7)(x-7). \end{aligned}$$

So the critical numbers are $x = 7$ and $x = \frac{7}{4}$ (the zeros of f') and $x = 0$ (a *cusp*: f' is not continuous there but f is).

To get the second derivative it is useful to multiply f' out again:

$$f'(x) = \tfrac{2}{3}x^{-1/3}(4x^2 - 35x + 49).$$

$$\begin{aligned}
&f''(x) \\
&= -\tfrac{2}{9}x^{-4/3}(4x^2 - 35x + 49) + \tfrac{2}{3}x^{-1/3}(8x - 35) \\
&= \tfrac{2}{9}x^{-4/3}(20x^2 - 70x - 49).
\end{aligned}$$

The roots of f'' are

$$x = \frac{70 \pm \sqrt{70^2 + 80 \times 49}}{40} = \frac{70 \pm 42\sqrt{5}}{40}.$$

How did I do that arithmetic? Break things into prime factors:

$$\begin{aligned} 70^2 + 80 \times 49 &= 7^2 2^2 5^2 + 7^2 2^4 5 = 7^2 2^2 5(5+4) \\ &= 7^2 2^2 3^2 5 = (42)^2 5. \end{aligned}$$

Conclusion: $f''(x) = 0$ at two places, $x \approx -0.6$ and $x \approx 4.1$. Furthermore, f'' is undefined at $x = 0$, so that also is a potential inflection point.

We need to find the sign of f' in the intervals between critical numbers. Note that at each such number $(0, \frac{7}{4}, 7)$, one of the three factors $(x^{-1/3}, 4x - 7, x - 7)$ changes from negative to positive. Thus f is increasing on $(0, \frac{7}{4})$ and $(7, \infty)$ and decreasing on $(-\infty, 0)$ and $(\frac{7}{4}, 7)$. It therefore has minima at 0 and 7 and a maximum at $\frac{7}{4}$.

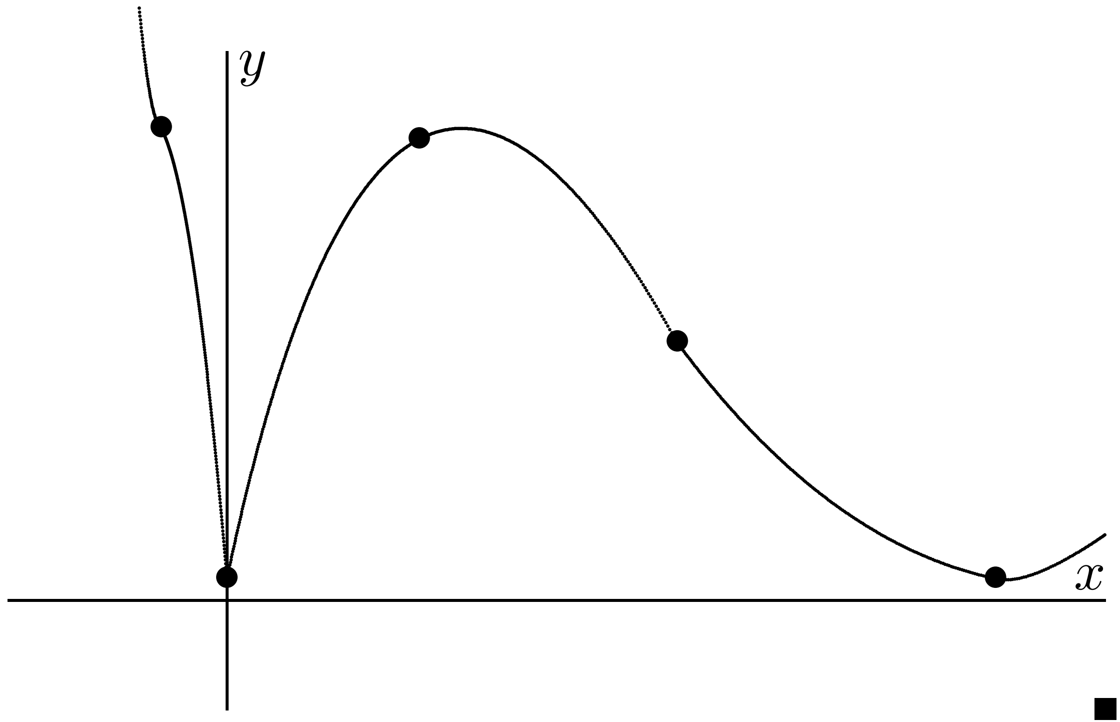
Now note that f'' has the form $x^{-4/3}(ax^2 + bx + c)$ with $a > 0$, so f'' is negative precisely in the interval between its two roots. (Since 4 is even, $x^{-4/3} > 0$.) So f is concave up on $(-\infty, -0.6)$ and $(4.1, \infty)$ and concave down on $(-0.6, 0)$ and $(0, 4.1)$.

Put the slope and concavity information together: At $x = 0$ the graph has a sharp point (cusp). The graph reverses itself there from decreasing to increasing, but on both sides the concavity is *down*, not up as it would be at a typical, smooth minimum.

Let's plot the points corresponding to the critical numbers and the possible inflections:

$$f(-0.6) \approx 43, \quad f(0) = 2, \quad f(\tfrac{7}{4}) \approx 42,$$
$$f(4.1) \approx 23.5, \quad f(7) = 2.$$

For this example I did not make a table like the ones on slides 7 and 12. Of course, you should do that if it helps you. You can even make a table with a line for each factor in $f'(x)$, for instance, if that helps you count the minus signs. (In blackboard lectures I do that.)



y
x